Our Mathematical Senses

The Geometry Vision

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Lecture-53

Video 10E: injectivity of the cross ratio

So, I want to turn our attention now to another really important property of the cross ratio, known as the injectivity of the cross ratio. And I want to prove the cross ratio injectivity lemma, which states the following. So let's let a, b, c, and d be distinct points on a line L. And let's let x be another point on L, with the property that the cross ratio x cbd is equal to the cross ratio a cbd. In that case, the claim is x is actually equal to a. So for simplicity, let's take L to be the x axis, and let's let a, b, c, and d be real numbers.

And let's let lambda be the cross ratio a cbd. Lambda is just a real number, it's that particular cross ratio. So here's our real line a, b, c, d, and lambda is the cross ratio of a, b, c, and d. And here's a point x, which I'll allow x to vary along the line.

And for various values of x, we'll consider the cross ratio x cbd. So the question I want to ask is, for how many values of x will x cbd be equal to lambda? For how many values of x, as we let x vary along the real numbers, are we going to get a cross ratio lambda that agrees with the cross ratio of a with b, c, and d? That's the question. Now let's unpack this cross ratio. x cbd, we can rewrite it by definition, by the definition of the cross ratio, as xb over bc, all over xd over dc. And that's equal to xb times dc over bc times xd.

I'm just flipping this denominator and multiplying these two fractions together. And that we can rewrite. xb is just b minus x. dc is c minus d. bc is c minus b.

And xd is d minus x. So we can rewrite it like that. We can get this expression for the cross ratio number. Remember, bc and d are all real numbers. x is also a real number, it's somewhere on the real line.

So this is just an expression involving a bunch of constants and x. So the question is, for how many values of x will this equation hold? And looking at this equation a little more carefully, well, if we multiply it out, if we put this denominator over to the other side, we get b minus x times c minus d is equal to lambda times c minus b times d minus x. But c minus d is just a real number, lambda is just a real number, c minus b is a real number. This is just a linear equation in the variable x. So the only solution is x equals a.

It's only going to have one solution. You can see that a is a solution because lambda is this cross ratio. We defined it to be the cross ratio with a. So there's one solution and that solution is equal to a. And that takes care of the proof.