Our Mathematical Senses

The Geometry Vision

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Lecture-54

Video 10F: proving the '3 fixed points' theorem

So, let's see some consequences of the cross-ratio injectivity lemma. And first an easy consequence, relatively speaking. So I'm actually going to give it as an exercise. Can you use the cross-ratio injectivity lemma to prove that if any four points have a cross-ratio other than negative one-third, then they cannot possibly be evenly spaced no matter what perspective you view them from? That's a claim I made earlier. Can we now justify that claim using the cross-ratio injectivity lemma? So I'll leave that as an exercise. And as a second consequence, which I'll do in more detail, we can finally give a complete proof of the three fixed points theorem.

Remember, we gave a sketch of a proof earlier that involved linear fractional functions. But we'll give a very different proof now using the cross-ratio injectivity lemma. And this is an important consequence because we've been using other consequences of the three fixed points theorem all over the place. So finally, everything will be justified once we can prove this theorem.

So remember the three fixed points theorem, what it states. It states that if a projectivity gamma from a line L to itself fixes three distinct points, then it fixes everything. Then it's the identity map on L. So for simplicity, let's assume that L is the x-axis. And in particular, we have coordinates on L.

So we can consider the cross-ratio of four points on L. So let's suppose that gamma fixes the points B, C, and D on the line L. And let's let x be any fourth point on L. Since the cross-ratio is a projective invariant, we have that the cross-ratio xCBD is equal to the cross-ratio gamma of x, gamma of C, gamma of B, gamma of D. That's what it means to be a projective invariant.

You can take the projective image under gamma of those four points and then calculate the cross-ratio of those. It'll be the same as the cross-ratio you would have gotten for the original four points. And this is in fact, well, we know that B, C, and D are fixed points of gamma. So this is just the cross-ratio of gamma of x with C, B, and D. But now, by the cross-ratio injectivity lemma, what can we say? We have that this cross-ratio and this cross-ratio are equal.

So by the cross-ratio injectivity lemma, the thing in the first parameter over here, namely gamma of x, has to be equal to the thing in the first parameter over here, namely x. So x is equal to gamma of x. Since gamma fixes every point x, so in particular, gamma is now fixing x. If gamma of x is equal to x, gamma also fixes x. But this was true for an arbitrary x.

So gamma fixes every single x on the line L. And therefore, it's the identity map on L. And one final remark. In the previous lecture, we proved the fundamental theorem of projective geometry as a corollary of the three fixed points theorem. So this gives you a sense of how fundamental the three fixed points theorem really is.

And it's also why in many more axiomatic treatments of projective geometry, like where, for example, you can define everything, develop everything over an arbitrary field, not necessarily real numbers. It's often just taken as an axiom. So rather than it's not something you prove in a more general treatment, it's actually taken as an axiom, or some equivalent statement is taken as an axiom. Thank you.