Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-55

Video 10G: bonus: the cross ratio as a function

So, the last thing I want to talk about is another way of thinking about the cross ratio, not just as a number, but as a function. So in particular, by fixing B, C, and D on the X axis and letting a fourth point X vary along the axis, we can think of the cross ratio X, C, B, D as a function of X. And the natural question is, what is that function? What is it calculating? What is it telling us? So let's look at a special case first. Let's let B equal zero, C equal one, and D equal infinity. So B is zero, C is one, and D is infinity. And in this case, if I just write out the definition, X, C, B, D, the cross ratio, is X, B over B, C all over X, D over D, C.

What is that equal to? Well, X, B is just zero minus X, that's negative X. B, C is one, sorry, B, C is just one, one minus zero, that's over X, D, that's D minus X. I won't write the infinity yet, but rather I'll take a limit in a second. And finally, D, C is one minus D.

So now let's see what is that as D, D is actually infinity, so let's let D go to infinity and see what number we get. First let's simplify this a little. So this is just negative X times one minus D over D minus X. I'm flipping the denominator. And now let's take the limit as D goes to infinity.

So what is this expression as D goes to infinity? Let me just write it out here maybe. We have one minus D over D minus X. And I'm taking the limit as D goes to infinity. Well, as D goes to infinity, X is fixed, so X and one are not going to matter. As D goes to infinity, these Ds are just going to cancel out and this is going to leave us with negative one.

So this expression is going to go to negative X over negative one, which is just equal to X as D goes to infinity. So this function, this cross-ratio function, when B is equal to zero, C is equal to one and D is infinity, is just the very special function X equals X. It's the identity function. The cross-ratio of these guys is just equal to X for any value X. So it's the identity function on the real line.

But on the other hand, let's look at the more general case where B, C, and D can be any points, any numbers. If we fix general distinct values for B, C, and D, then what is the cross-ratio function X, C, B, D actually calculating? What is it actually telling us? So to see that, let's let gamma be the unique projectivity from L to itself that takes B, C, and D to zero, one, and infinity. By the fundamental theorem in one dimension, there's a unique gamma that does that, that takes B to zero, C to one, and D to infinity. That gamma is going to take X somewhere to some gamma of X. Let's just calculate, let's just look at this expression.

The cross-ratio X, C, B, D by the invariance of the cross-ratio is just equal to gamma of X, gamma of C, gamma of B, gamma of D, that cross-ratio. And gamma of C is one, gamma of B is zero, gamma of D is infinity. This is the cross-ratio of gamma of X with one, zero, and infinity. And we've seen that that's just the identity function on this first parameter. So that's just gamma of X.

The cross-ratio of X with an arbitrary B, C, and D can be thought of as just the value of X under this unique projectivity gamma that takes B to zero, C to one, and D to infinity. Let's just write it out a bit more explicitly. So we can write out this cross-ratio with an explicit algebraic expression for this cross-ratio. By definition, X, C, B, D is just X, B over B, C divided by X, D over D, C. And we can write X, B is just B minus X.

B, C is just C minus B. I'm flipping this part so that I can multiply by this denominator. So D, C is just C minus D, that's what I've written here. And X, D is just D minus X. These are all real numbers.

So that's what I've written here. So this is the expression I get for the cross-ratio of X with an arbitrary B, C, and D. And we can see that indeed this is a linear fractional function that sends B to zero C to D and D to infinity. And one way to see that is just to write it.

.. So it's easy to see that B goes to zero. If we substitute B in for X, we get a zero here. On the other hand, if we substitute D in for X, we get a zero in the denominator. So this expression goes to infinity. But what happens if we substitute C in for X? We get B minus C times C minus D on top.

On the bottom we get C minus B times D minus C. And if we multiply the numerator by negative one twice, we end up... We can actually get C minus B, multiplying this first term by negative one.

Second term we can multiply by negative one to get D minus C. The denominator stays the same, and we see that these cancel out. So this is just one. So in some way, what we're calculating under this cross-ratio function is just the value of X under this linear fractional function, which sends B to zero, C to one, and D to infinity. So that's another handy way of thinking of the cross-ratio.

We can think of reframing our line, our real line, by sending B, C, and D to zero, one, and infinity. There's a unique way to do that by the fundamental theorem of projective geometry. So that's a unique kind of way of framing our three points, zero, one, and infinity. And in that framing, where does X go to? Well, it goes to this gamma of X, which is just its cross-ratio with B, C, and D. So that's another way of thinking of the cross-ratio, another way of interpreting it.

And we can visualize the cross-ratio explicitly by explicitly constructing the projectivity gamma, which sends B to zero, C to one, and D to infinity. So here's my line with B, C, and D. I've drawn another line, a vertical line here, through zero. And I've put a point, a center of perspective here, vertically below D. So as a result, this is going to project D to infinity, because it's parallel to this new line.

And I've positioned it along this line in such a way that it projects C to the point one. And I've also arranged it so these points intersect at zero. So it's going to send zero to zero. And we can see that this perspective sends X to a gamma of X over here. So for this B, C, and D, the cross-ratio of X with those is this gamma of X.

We get to explicitly see its cross-ratio as this point gamma of X on this line. So it is possible to physically visualize the cross-ratio this way. But as you can see, it's also quite complicated. In some sense, there's no getting around the fact that the cross-ratio is slightly bizarre when we try and really understand it geometrically and visualize it. But that said, this slightly more algebraic expression can be very useful.

It's just the image of X under the unique linear fractional function, the unique projectivity, that sends B to zero, C to one, and D to infinity.