Our Mathematical Senses

The Geometry Vision

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Lecture-58

Video 11C: homogenous proof that parallel lines converge

Okay, so now let's put homogeneous coordinates to use and prove something that we've proved before, but we'll prove it in a slightly more precise way this time, and a way that generalizes a bit more easily. So using homogeneous coordinates, let's revisit the question, why do parallel lines converge? And let's just bring back our old setup where we're looking at railway tracks in a ground plane and seeing them on a picture plane. But let's get rid of the I and let's instead put in a coordinate system. Let's think of this as R3 and let's let the origin be our center of perspective. So this is the origin, 0, 0, 0, here. And let's let the ground plane be the plane z equals minus 1, and let's let y equals 1 be the picture plane.

So this is the x coordinate, this is the y coordinate, this is the positive z coordinate. So that's why this is z equals negative 1, the ground plane. The origin is up here and the picture plane is y equals positive 1. So that's our framework.

And using homogeneous coordinates, it's easy to locate image points on the picture plane. So here's a point 1, 2, minus 1, and it's a point on this side rail. So the railway tracks are on this ground plane z equals minus 1. You can look at this side rail here, and this is one point on that side rail, 1, 2, minus 1. And if we write that point in homogeneous coordinates, this is a way of now referring not just to the point, but to this line through the origin containing that point.

We're actually looking at all scalar multiples of this vector now. And suppose we want to know the image point on the picture plane of this point in space, this point 1, 2, minus 1, which is a point in space on the side rail. Well, the image point also lies on this sight line. It also lies on this line through the origin. And how do we get it? Well, it lies on the picture plane y equals 1.

So we want this y coordinate to be equal to 1. Can we make it equal to 1 through scalar multiplication? Can we multiply this by a scalar to make this 1? Well, of course, we can multiply it by 1 half. Multiplying by 1 half, we get a new equivalent representation of that same line in space. The homogeneous coordinates, we're writing it as 1 half, 1, minus 1 half. We just multiplied each coordinate by 1 half to get this.

But that's really using this point as a representative of this line. So I'm not doing any rocket science here. There's nothing that we couldn't do earlier without homogeneous coordinates. But we'll see in a second that homogeneous coordinates make this a little bit more slick. So the point in the picture plane, the image point of this point here in the picture plane, is this point, 1 half, 1, minus 1 half.

And similarly, we can look at this point here, 1, 3, minus 1. It's a little further along the side rail. And its image point in the picture plane, well, we just want to get a 1 here for our y coordinates. We divide everything by 3, and we get this representation of that line. This vector is now representing the line.

And its coordinates are minus 1 third, 1, minus 1 third. So that's the image point from this line. And we can actually keep doing this. We can take an arbitrary point on the side rail, 1, t, minus 1. And we can look at its image in the picture plane as we look.

And we can do that as we look further and further and further along. So its image in the picture plane is 1 over t, 1, minus 1 over t. And that's true for arbitrary t. So it's true for this point. It's true for this point when we increase t a little bit.

We can make t even bigger. And we'll keep getting points on the picture plane given by this expression, 1 over t, 1, minus 1 over t. So what happens to this image point in the picture plane as t goes to infinity? Well, now this is something that we can explicitly calculate in coordinates. So what happens to this expression as t goes to infinity? Well, 1 over t goes to 0. Minus 1 over t also goes to 0.

So this expression goes to 0, 1, 0 as t goes to infinity. So the image in the picture plane approaches this point, 0, 1, 0. And on the other hand, we can look at the other side rail. And this point, minus 1, 2, minus 1 right here. So I've just earlier, now the x coordinate is negative.

That's the only change. So we have minus 1, 2, minus 1. That has this image point. Minus 3, minus 1, 3, minus 1 has this image point. Minus 1t minus 1 has this image point.

And as we let t go to infinity, what does this go to? Well, minus 1t minus 1, that's equal to, in homogeneous coordinates, minus 1 over t, 1, minus 1 over t, dividing everything through by t. t is clearly non-zero. It's clearly positive here from the picture. So this goes to 0.

This goes to 0. So the whole thing goes to 0, 1, 0. We reach the same vanishing point as t goes to infinity. So I want to give a slightly more general question. Forget these railway tracks now. Let's just take a general line in space, which is parameterized by p plus t, v.

Or, writing it out a little bit more carefully with the x, y, and z coordinates, p1, p2, p3 plus t times v1, v2, v3. What is the vanishing point of the line in the picture plane parameterized by this expression here, as we vary t? So here is the line. Here's my point p, rather, p1, p2, p3. Here's my vector v, v1, v2, v3. And p plus t, v is going to range over all of the points in this line here.

It's going to parameterize this line here as t varies over the real numbers. So we're getting this whole line. So what is this line going to look like in the picture plane, and what is its vanishing point going to be in the picture plane? Remember, every line has a vanishing point, unless it's parallel to the picture plane. That's the first thing we proved in this class. So assuming this is not parallel to the picture plane, what is its vanishing point going to be? Well, let's imagine looking further and further down the line, just like we did the first time we did this exercise.

But now we have coordinates to make it all precise. So what is our sight line at time t going to be? Well, here's our sight line at some point in time, looking at the line, at this point on the line. And it's just p1 plus t, v1, p2 plus t, v2, and p3 plus t, v3. That's what this vector is. And as we increase t, we'll look a little further down, increase t a little more, and look even further down.

And we can keep increasing t and looking further down. So what's happening here? Well, we'll just let t become bigger and bigger, and let t go off to infinity. And eventually, we'll reach a limiting sight line. So the sight line at time t is given by this expression, these homogeneous coordinates. And we can rewrite that to give a representation or representative in the picture plane.

We've made this y-coordinate 1 by dividing everything through by p2 plus t, v2. So we get this p1 plus t, v1 over p2 plus t, v2 as our x-coordinate, p3 plus t, v3 over p2 plus t, v2 as our y-coordinate, and 1 as our z-coordinate, and 1 is our y-coordinate. So the vanishing point at time t, oops, that's a typo. I meant to say the image point at time t is just this expression here. To find the vanishing point, we have to let t go to infinity.

So as t goes to infinity, what do we get? Well, this expression here, the constants p1 and p2 don't matter as t goes to infinity. All that matters is these coefficients, and we get v1 over v2. Similarly for this third expression, the p3 and the p2 no longer matter as t gets infinitely large, and we just get v3 over v2. So this goes to the following line, the following homogeneous coordinate as t goes to infinity. So in other words, in the picture plane, the line is going to have a vanishing point, v1 over v2, 1, v3 over v2.

That's the point in the picture plane which is the vanishing point as we follow this line with our sight line off to infinity. And the important thing to notice here is that the vanishing point is independent of the line's position in space. It's independent of p. We could have taken any other p, and it would disappear by the time we actually calculate our vanishing point. And that agrees with the fact that any line that's parallel to this one will converge to the same vanishing point.

So that position in space doesn't matter. All that matters is the direction of the line given by the vector v. So really all that matters is the parallel family of the line. Now, and using homogeneous coordinates, we can see this all very explicitly. So that's the point of the picture plane.

And the other point is that the line is going to have a vanishing point. So that's the point of the picture plane. And the other point is that the line is going to have a vanishing point. So that's the point of the picture plane. And the other point is that the line is going to have a vanishing point.

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