Our Mathematical Senses

The Geometry Vision

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Lecture-60

Video 12A: affine charts

Now, I can't blame you if you're finding it confusing that projective points are actually lines through the origin, and projective lines are actually planes through the origin. If when projective points look like lines and projective lines look like planes, how do we go about visualizing RP2? It seems kind of confusing. This is where affine charts are going to be helpful. I wanted to find a map, phi z, from RP2, RP2 remember is a set of all lines through the origin, to the plane z equals 1 in R3, to that specific plane z equals 1. The way this map works, it sends an element of RP2, in other words it sends a line through the origin in R3, to its intersection with the plane z equals 1. In other words, it sends, in homogeneous coordinates, I'm referring to a line through the origin this way, x colon y colon z, and phi z is going to send that to its intersection with the plane z equals 1.

We can find that intersection just by dividing every coordinate by z, to get the representative sitting on the plane z equals 1. Pictorially, what are we doing? We're just taking a line through the origin, one of these lines through the origin, and just seeing where it intersects the plane z equals 1. Then this line through the origin, this element of RP2, gets mapped to this point in z equals 1. This line through the origin, which is an element of RP2, gets sent to this point in z equals 1.

This line through the origin in RP2, which is an element of RP2, gets sent to this point in z equals 1. This map is called an affine chart. Under this map, a chart is just another word for a map, basically, for us. So that terminology, chart, doesn't mean anything very special here. This map is called an affine chart.

Under this map, projective points look like points, and projective lines actually look like lines. You can see this line through the origin is a projective point, and indeed, the image under the map is a point. It looks like a point. And the image of a projective line, a plane through the origin, well, that plane through the origin will intersect z equals 1 in a line. But unfortunately, the affine chart doesn't actually capture all of RP2.

We're missing the projective line z equals 0. We're missing this projective line down here, z equals 0. None of the projective points in that line, none of the lines through the origin sitting on this plane z equals 0, are going to intersect the plane z equals 1. So we really have to, I'll just draw one here, for example, take the y-axis, that line. This line is not going to intersect the plane z equals 1.

So phi z is not going to be defined on that line, or any other line sitting in z equals 0. So let's change our definition. Phi z is not defined on all of RP2. It's defined on RP2 minus the plane z equals 0. And now this map is well-defined.

So this is our map. It sends x, y, z to x over z, y over z, 1. It sends lines through the origin to points in the plane z equals 1. Now under this affine chart, the plane z equals 1 can actually be thought of as P2, as the extended Euclidean plane that we've been working with for a long time now. And really, if we try and understand RP2 from this chart alone, then it's a lot like staring at R2, because z equals 1, the plane z equals 1 is really just a copy of R2.

And we're really just then staring at R2 and imagining abstract points at infinity, one for each family of parallel lines. But we can also look at the full RP2. And suddenly the formerly abstract points at infinity that we've always been imagining lying out there can now be visualized as lines through the origin in the xy plane z equals 0. So we can kind of think of this affine chart image, this z equals 1, plus all of these extra lines through the origin in the plane z equals 0, together making up RP2.