Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-62

Video 12C: the fundamental theorem of PGL(3,R)

So, we want to visualize the action of PGL3R, that matrix group we just constructed. We want to visualize how it transforms RP2. But before we get to that, there's a highly relevant theorem that's worth mentioning, which is the following, the fundamental theorem of PGL3R. And it states that if we let A, B, C, and D, and A', B', C', and D', be two different ordered sets of four points in RP2, where in each set no three are collinear. Then there exists a unique element of PGL3R, taking A to A', B to B', C to C', and D to D'. So, remember, this is RP2, so a point in RP2 is a line through the origin.

So A, B, C, and D are all different lines through the origin. A', B', C', D' are all different lines through the origin. And there exists a unique element of PGL3R, taking these four onto these four. So that's the fundamental theorem of PGL3R.

But we're not going to prove it right now. We'll prove it in the final video. So we will prove it in this class, but not right now. But I want to just think about what it's saying. What it's saying is that there's a unique element of PGL3R taking any four points to any other four points.

And that feels like a familiar statement. This feels like something we may have seen before. And the reason is that this is almost the exact same statement as the fundamental theorem of projective geometry. So is there any connection that suggests there might be a connection between the elements of PGL3R and the projectivities from an extended plane to itself? Because remember, the fundamental theorem of projective geometry was almost the same statement except we were looking at an extended plane in P3, not looking at RP2. So is there a connection between PGL3R and the set of projectivities from an extended plane to itself? And indeed there is.

The synthetic analytic equivalence theorem states that projectivities from an extended affine plane z equals 1 to itself are in one-to-one correspondence with the elements of the matrix group PGL3R. So projectivities from an extended plane to itself are in one-to-one correspondence with elements of PGL3R. There's very much a connection. And this theorem in a way bridges the synthetic and analytic approaches. RP2 and PGL3R represents the analytic approach.

And on the other hand, the extended plane and projectivities represented the synthetic approach. So an important remark here, there's nothing at all special about the plane z equals 1. I'm just using that for the statement of the theorem. We were looking at projectivities from that particular extended plane to itself, because that'll make things a little easier to relate. But you can look at projectivities from any fixed extended plane to itself.

And the same theorem will hold. So let's see some examples to help us visualize how elements of PGL3R transform this extended affine plane z equals 1. Okay. Thanks.