Our Mathematical Senses

The Geometry Vision

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Lecture-64

Video 12E: proving the fundamental theorem of PGL(3,R)

So let's return to the fundamental theorem of PGL3R and try and prove it. So this proof will involve some linear algebra. Now the fundamental theorem of PGL3R states that if we take a, b, c, and d, and a' b' c' and d' to be two different ordered sets of four points in RP2, in which no three are collinear in each of the sets. Then there's a unique element of PGL3R taking a to a', b to b', c to c', and d to d'. So remember, a, b, c, and d are points in RP2, meaning they're actually lines through the origin in R3. Similarly, a', b', c', and d' are each a line, each of them is a line through the origin in R3.

So that's what we're really dealing with. So for our proof, the first thing that will be useful to do is take some vector representations of these eight different lines through the origin. So in particular, let's let v1' be a vector that lies on the line a. I'll write it this way.

v2' is a vector lying on the line b. We could also write it in this notation, where I said that a is equal to the set of scalar multiples of v1'. It's the line spanned by v1'. b is the line spanned by v2'. c is the line spanned by v3'.

And d is the line spanned by v4'. So if we choose these vectors in such a way that they each lie on their respective lines, then we can also say that the lines are spanned by the vectors. Similarly, a' is spanned by w1', b' is spanned by w2', c' is spanned by w3', and d' is spanned by w4'. So we're choosing eight vectors lying on those eight lines through the origin. Notice that this set of vectors, v1, v2, v3, all prime, and w1', w2', w3', each of these forms a basis for R3.

How do we know it forms a basis? Well, it comes from this restriction here. In each of these sets in ABCD, we know that no three are collinear. In particular, a, b, c, and d are not collinear points in RP2. What does that mean? It means they do not lie on a projective line in RP2. A projective line in RP2 is just a plane through the origin.

So a, b, c, and d are lines through the origin, but they don't lie on a common plane through the origin. In other words, if we take vector representations of them, they will actually span all of RP2. They won't just span a plane in RP2. So each of these are full-fledged bases for R3, which means we can write v4' as a linear combination of v1', v2', and v3' for scalars alpha1, alpha2, and alpha3. And we can write w4' as a linear combination of the w'', w1', w2', w3' for these scalars beta1, beta2, and beta3.

Now let's just define new vectors. Let's forget that we had our prime vectors. Now let's define our prime vectors are just a stepping stone for getting to these real vectors. We want some better vector representatives of a, b, c, and d. So let's let vi be alpha i vi prime for each i.

That's in other words, v1 is equal to alpha1 v1, v2 is equal to alpha2 v2. We're defining these. Sorry. Alpha1 v1 prime, v2 is alpha2 v2 prime, and v3 is alpha3 v3 prime. Why would we care about doing that? We've just scaled these a little bit here and there.

But let's let v4 just be the same as v4 prime, which is just v1 plus v2 plus v3. No scalars. It's just the sum of those three vectors now. Let's do the same thing for the w's. Let wi equal beta i wi prime for each i.

Then w4, we'll just let that be w4 prime. It's the same thing. And that's just the sum of the three initial w's, w1, and w2, and w3. Now let's let t be an element of the general linear group. Let's let it be a linear transformation, which takes v1 to w1, v2 to w2, and v3 to w3.

Then since it's a linear transformation, it's going to take v4. Remember v4 is just v1 plus v2 plus v3. Since it's a linear transformation, it's linear. This is t of v1 plus t of v2 plus t of v3. But that's just equal to w1 plus w2 plus w3, which is just w4.

t of v4 is equal to w4. In other words, t is taking v1, v2, and v3, and v4 to w1, w2, w3, and w4, which means its corresponding element in PGL3R, so I'm denoting the corresponding element in PGL3R as R times t, the set of scalar multiples of the transformation t. That's an element of PGL3R. And that's going to take the line a to the line a prime. It's going to take the line b to the line b prime, the line c to the line c prime, and the line d to the line d prime.

So that takes care of the existence. This is our element in PGL3R, which takes these four points to these four points in Rp2, and I'll leave it as an exercise to prove that this is a unique element in PGL3R that does the trick. So that's an exercise for you to try. As a hint, you can look at the previous uniqueness proofs we've done and see if you can adapt one of those or use a similar line of argument. So good luck with that.