PROBABILITY THEORY FOR DATA SCIENCE

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Week - 04

Lecture - 20

Poisson Distribution

Now, the question is how many bombs are required to completely destroy the target. If we throw n bombs, this random variable can be represented by X_1, X_2, ..., X_n, where each bomb X_i has a 50% chance of striking the target, which is equal to 1/2 for i = 1 to n. As we discussed earlier, these are all Bernoulli distributions; they are independently and identically distributed random variables, meaning they all have the same probability mass function. Now, we know that $X_1 + X_2 + ... + X_n$ follows a binomial distribution with parameters n and p = 1/2.

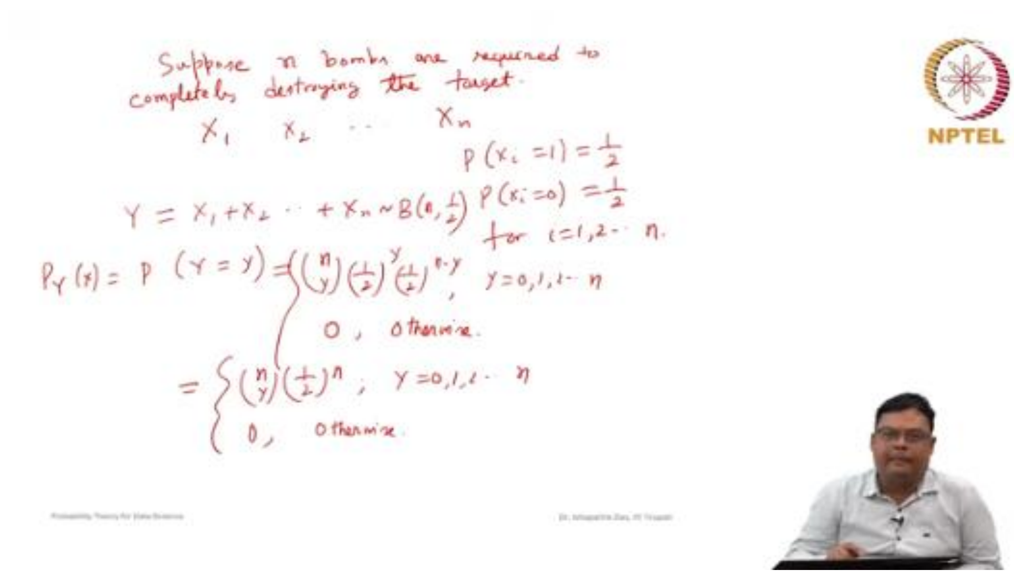
For destroying the target, we need two direct hits, so that is why we have this Y, which is a binomial distribution. The probability that Y = y represents how many bombs actually hit the target out of n, and this probability is given by:

$P(Y = y) = nCy * p^y * (1 - p)^{(n - y)}$

where p = 1/2. So, y can be equal to 0, 1, ..., n, and it is equal to 0 otherwise. Now, this can be simplified as:

$P(Y = y) = nCy * (1/2)^n$

because whenever the probability is 1/2, we can simplify in this way. To completely destroy the target, at least two direct hits are required. The question asks how many bombs must be dropped to give a 99% chance or better of completely destroying the target.



We need at least two hits, as stated in the information provided. The probability of getting at least two direct hits means that the random variable Y, representing the number of hits, should be greater than or equal to 2. This means $Y \geq 2$. Since we do not know n, the total number of bombs dropped, we express this as the probability that $Y = 2$ plus the probability that $Y = 3$, and so forth up to n. The sum of the probability mass function should equal 1, so we can write this as:

$1 - P(Y = 0) - P(Y = 1)$

We can simplify this expression to:

$1 - P(Y < 2)$

which gives us:

$1 - P(Y = 0) + P(Y = 1)$

We are asked to compute these probabilities, and they must be greater than or equal to 99%. Specifically, we need to find the probabilities that $Y = 0$ and $Y = 1$. The probability that $Y = 0$ can be found from the expression:

$P(Y = 0) = nC0 \times (1/2)^n$

and the probability that $Y = 1$ is:

$P(Y = 1) = nC1 \times (1/2)^n$

Hence, we need to determine n such that the probability of $Y \geq 2$ is at least 99%. This implies that:

$1 - P(Y < 2) \geq 0.99$

or:

$1 - P(Y < 2) \leq 0.01$

This sets the stage for finding the number of bombs needed. This probability is the probability of $Y < 2$, which implies:

$P(Y \leq 1) \leq 1/100$

So, if $Y < 2$, we found:

$(1/2)^n \times (1 + n) \leq 1/100$

This implies that:

$2^n \geq 100 + 100n$

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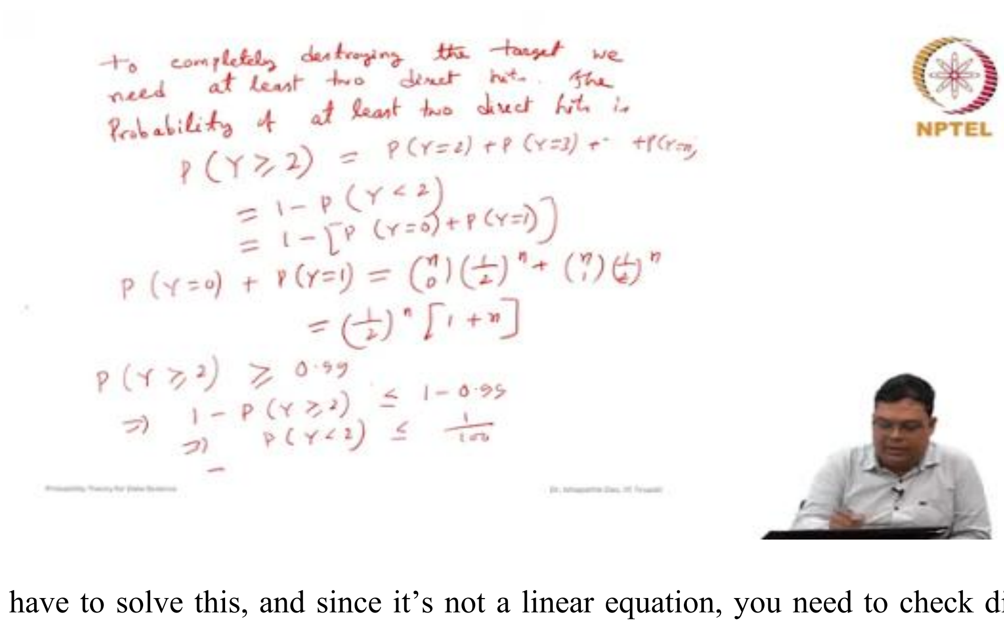
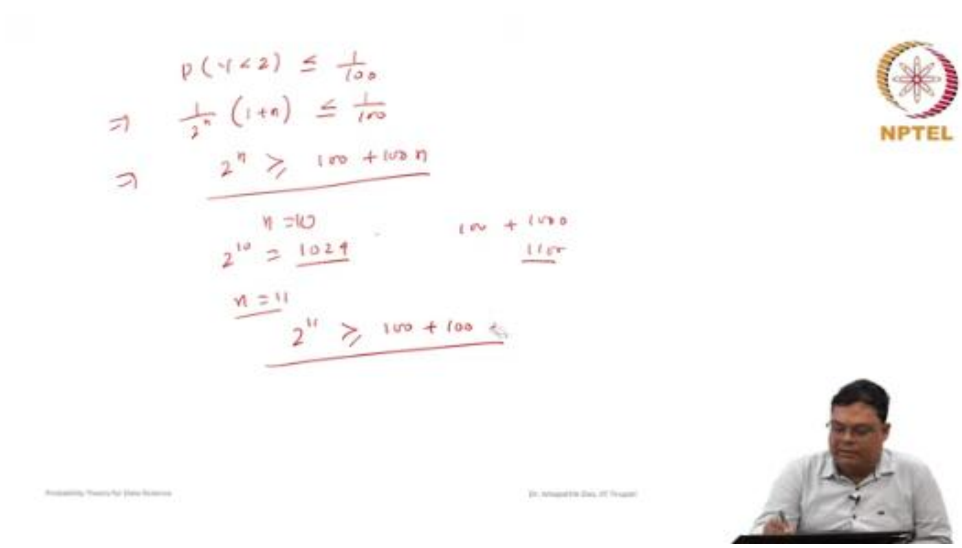


You have to solve this, and since it's not a linear equation, you need to check different values of n. For example, if $n = 1$, then $2^1 = 2$, which doesn't satisfy the condition. If you take $n = 10$, then $2^{10} = 1024$, and you compare this to $100 + 100 \times 10 = 1100$. This still doesn't satisfy the inequality. However, if you try n = 11, 2^11 ≥ 100 + 100n, which confirms that the minimum number of bombs needed is at least 11.

So, if n is anything greater than or equal to 11, then n = 11. So, when n ≥ 11, this satisfies the relation. In that case, we will mention that at least n = 11 bombs should be dropped to achieve at least two hits, with a minimum 99 percent probability. Two direct hits are required to destroy the target completely, so at least 11 bombs must be dropped to give a 99 percent chance or better of completely destroying the target. I hope you have understood how to use the binomial distribution for solving this kind of numerical problem.



Now, we will discuss the Poisson distribution, another important distribution. We have completed our discussion on Bernoulli distribution and binomial distribution, and now we will start discussing some other important distributions. The range of the random variable for Bernoulli distribution is 0 or 1, and for binomial distribution, the range is 0 to n, which are finite numbers. Now, we will discuss another distribution that is also a discrete random variable. The range of this random variable can be infinite, but it will be countably infinite.

A random variable X is called a Poisson random variate with parameter λ > 0 if its probability mass function is given by a specific relation. A random variable X is said to have a Poisson distribution with one parameter, which is λ, a real number greater than 0 and less than infinity. The probability mass function of X is given by:

$P(X = x) = e^{-\lambda} \times \lambda^x / x!$

where $x = 0, 1, 2, \ldots$, and it equals 0 otherwise. You know that e^x has an infinite series expansion:

$e^x = 1 + x/1! + x^2/2! + x^3/3! + \ldots$

which is valid for any real number. This means it is always greater than or equal to 0, confirming that this is a probability mass function.

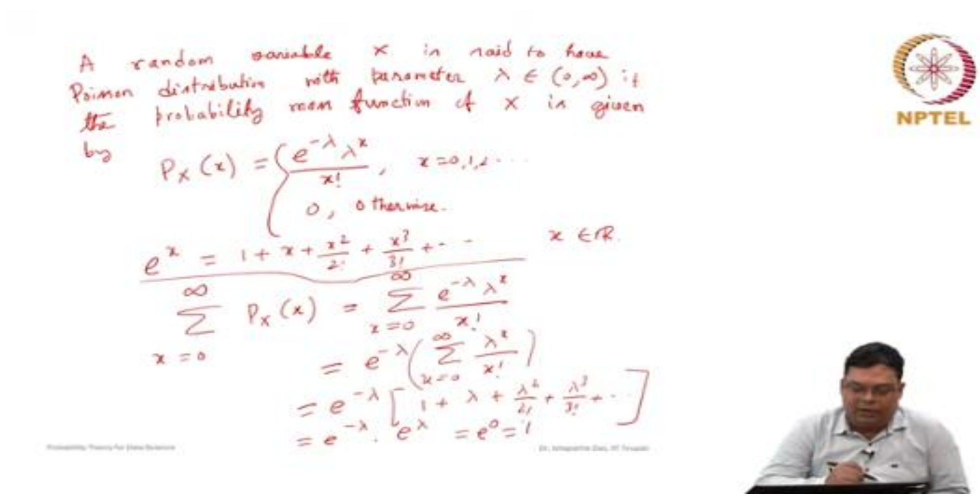
If you take the sum of this probability mass function from $x = 0$ to ∞, you have:

Σ (from $x = 0$ to ∞) $e^{-\lambda} \times \lambda^x / x!$

By factoring out $e^{-\lambda}$, you can express this as:

$e^{-\lambda} \times \Sigma$ (from $x = 0$ to ∞) $\lambda^x / x!$

This summation resembles the expansion of e^{λ}. Thus, $e^{-\lambda} \times e^{\lambda} = e^0 = 1$. So, that is why it is a probability mass function.



Now, what are the values? If you take, suppose x is equal to probability that x is equal to x, this is nothing but:

$P(X = x) = e^{-\lambda} * \lambda^x / x!$

whenever $x = 0, 1, 2, \ldots$; this is equal to 0 otherwise. So, now, if you take $x = 0$, $P(X = 0)$ is nothing but the probability that $x = 0$. This is:

$P(X = 0) = e^{-\lambda}$

So, λ is any value; λ can be, suppose, 3, then it will be e^{-3}. So, λ is any value on the real line. It is some number you can consider from 0 to infinity, which is a parameter; depending upon the value of λ, it will be a different value.

$P(X = 1)$ will be nothing but the probability that $x = 1$. So:

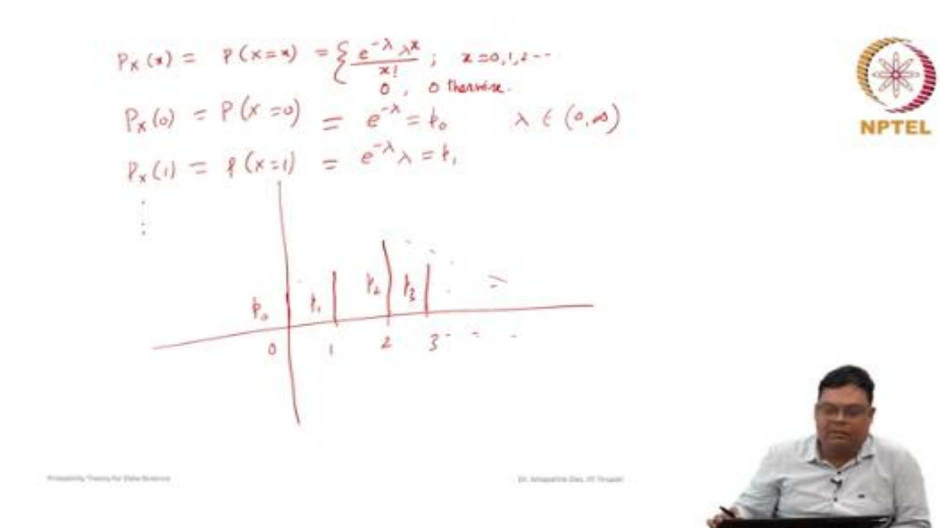
$P(X = 1) = e^{-\lambda} * \lambda^1 / 1! = \lambda$

So, similarly, you can denote this as P0 for $P(X = 0)$, and P1 for $P(X = 1)$, and so on.

Now, if you draw this curve to understand it, this is random variability at 0, 1, 2, and so on. At $x = 0$, this is P0; at $x = 1$, this is P1; at $x = 2$, this is P2, and so on. So, whenever you observe different values of λ, you will see that this will be the probability.

It is called a mode also, where it is maximum. The maximum probability will be around λ, which is the mode. According to this curve, this λ may be close to 2 because it is maximum. It increases and then decreases, like that. This curve looks like it increases and then decreases.

So, that is why it is a mode here and it is a unimodal distribution. Now, how can you find the cumulative distribution function (CDF) for this Poisson distribution? The CDF of a Poisson random variate is given by the probability that $x \leq x$. We have already discussed how to find the CDF from the probability mass function. The minimum value is 0, so it will be 0 when $x < 0$.



Whenever $X \geq 0$ and $X < 1$, this will be P(0). We have already discussed this in the context of binomial distribution cases. For values where $1 \leq X < 2$, this will be P(0) + P(1), and it will continue in this way. For any i, where $i \geq X$ and $i \leq i + 1$, this is nothing but P(0) + P(1) + ... + P(i). This series will converge to 1 at infinity, as it has an infinite range.

If you try to represent this graphically, the CDF looks like this: on the x-axis, you have values 0, 1, 2, 3, and so on. It starts at 0 here, then moves to some value, and continues to increase. Suppose it reaches 1; it will converge at infinity. This means that as $X \to \infty$, the cumulative distribution function approaches 1 asymptotically. This indicates that f(X) will approach 1 as $X \to \infty$.

It will not reach a finite value of exactly 1, but there will always be some probability remaining close to 1. So, this random variable has a countably infinite range. Now, we will discuss how we can find the mean and variance, which are important measures in the Poisson distribution. To find the mean of this random variable, denoted as μ, we use the expected value of X, defined as the summation of x_k multiplied by $P_x(x_k)$. The possible values are given by the summation from X = 0 to ∞ of $x \times P_x(x_k)$, which equals $e^{-\lambda}$.

This is the probability mass function:

$$P_x(x) = e^{-\lambda} (\lambda^x) / x!$$

Simplifying this, we start with the summation; for x = 0, this is 0, so we only consider the summation from x = 1 to ∞ of $x \times e^{-\lambda} (\lambda^x) / x!$. We can express x as $x \times (x - 1)!$, leading to cancellation and leaving us with $e^{-\lambda}$, which is independent of x. Now, we need to solve this, yielding:

$$e^{-\lambda} \times \sum_{x=1}^{\infty} (\lambda^x) / (x - 1)!$$

We can rewrite this as:

$$\lambda \times \sum_{x=1}^{\infty} (\lambda^{(x - 1)}) / (x - 1)!$$

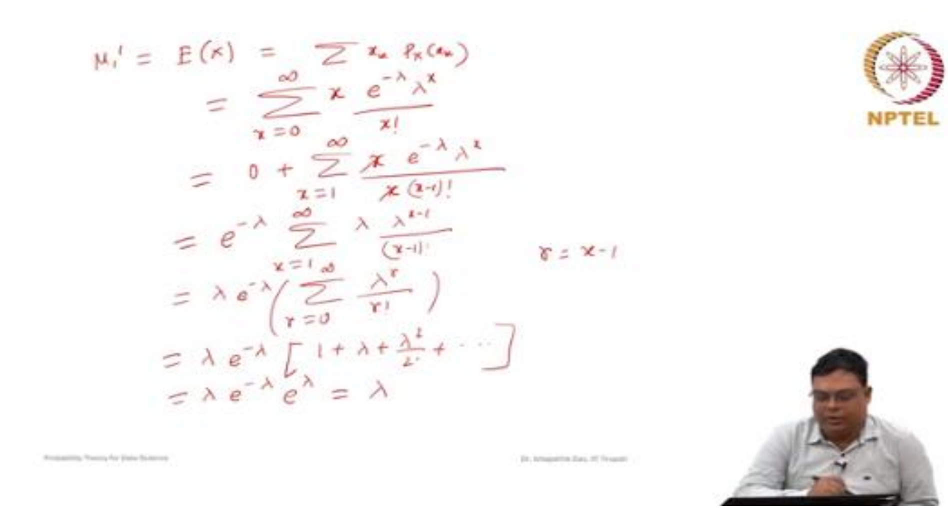
Letting r = x - 1, we have:

$$\sum_{r=0}^{\infty} (\lambda^r) / r!$$

This is an infinite series that converges to e^{λ}. Therefore, we find that:

$$\lambda \times e^{-\lambda} \times e^{\lambda} = \lambda$$

The mean of the Poisson distribution is nothing but the parameter λ. Now, we will find what the variance is. For finding the variance, we need the second-order moments because the variance of X, or σ^2, can be found by definition. The variance of X can be defined as $Var(X) = E(X^2) - (E(X))^2$.



In its simplified form, this is $\mu_2 - \mu_1^2$. Now, what is μ_2? μ_2 is the expected value of X^2. By definition, the expected value of X^2 is the summation of $x^2 \times P_x(x_k)$ over k. Here, x^2 takes values from 0 to ∞, so we have the summation from $x = 0$ to ∞ of $x^2 \times P_x(x)$.

The probability mass function $P_x(x)$ for this Poisson distribution is $e^{-\lambda} \times (\lambda^x) / x!$. Therefore, this is $e^{-\lambda} \times (\lambda^x) / x!$. Similarly, if we take $x = 0$, this contributes 0, and we sum from $x = 1$ to ∞, yielding $x^2 \times e^{-\lambda} \times (\lambda^x)$. We can express this as $x \times (x - 1)!$ in the denominator. So, one x cancels out, but another x remains, and further cancellation is not possible here.

We cannot simplify this in the same way because of the remaining x in the denominator. So, we can get $x - 1$.

$$\sigma_x^2 = V(X) = E[(X-\mu_1')^2] = \mu_2' - (\mu_1')^2$$

$$\mu_2' = E(X^2) = \sum_k x_k^2 \, P_X(x_k)$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda} \lambda^x}{x!} = 0 + \sum_{x=1}^{\infty} \frac{x^2 e^{-\lambda} \lambda^x}{x(x-1)!}$$