PROBABILITY THEORY FOR DATA SCIENCE

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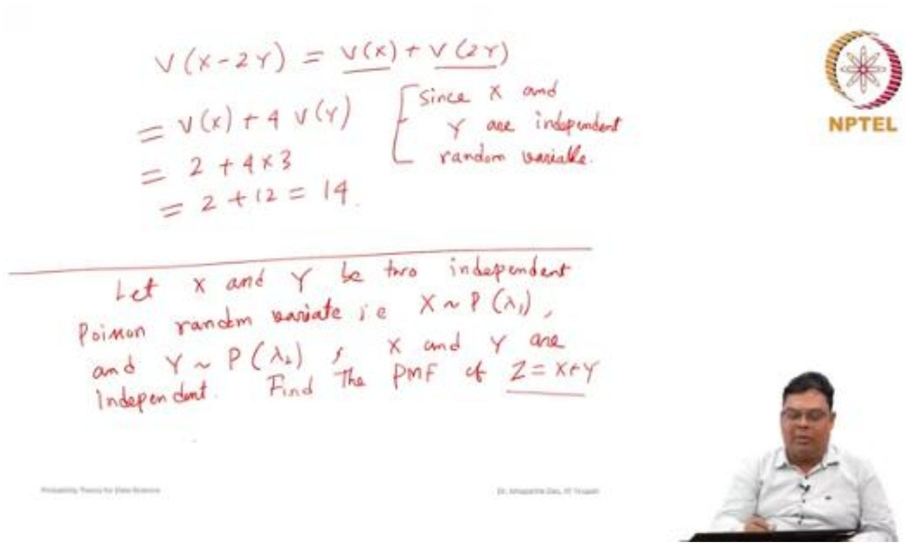
Week - 05

Lecture - 22

Numerical Examples of Poisson Distribution and Uniform Distribution

Now, suppose it is given that X and Y are two independent Poisson random variables. This means that X follows a Poisson distribution with parameter λ_1, and Y follows a Poisson distribution with parameter λ_2. Also, X and Y are independent, which we are assuming here. Now, we need to find the probability mass function (PMF) of Z, where Z = X + Y. For example, let X represent the number of accidents in a city, say Chennai, and Y represent the number of accidents in another city, say Mumbai.

If you want to find the total number of accidents in both cities, we need to determine the probability mass function for this combined number of accidents. So, now we have to find the probability mass function (PMF) of Z, which is the sum of X and Y. The probability that Z = z is the probability that Z = z. The range of X, the number of accidents for a Poisson random variable, is 0, 1, 2, …, and the range of Y is also 0, 1, 2, …. So, the range of Z, which is the sum of the number of accidents, will also be 0, 1, 2, ….



We don't know yet what the distribution will be, so we need to find the probability mass function (PMF). Now, if we consider the probability that Z = 0, this is the probability that X + Y = 0. For this to happen, both X and Y must be 0 because X can take a minimum value of 0. Since X and Y are independent random variables, we can multiply their individual probabilities. This is a property of independent events where the probability of the intersection of two events is the product of their probabilities.

So, the probability that Z = 0 is the probability that X = 0 multiplied by the probability that Y = 0. The probability that X = 0 is $e^{(-\lambda_1)}$, and the probability that Y = 0 is $e^{(-\lambda_2)}$. So, the probability that Z = 0 is $e^{(-(\lambda_1 + \lambda_2))}$. Next, let's look at the probability that Z = 1. This is the probability that X + Y = 1.

The possible combinations for X and Y that sum to 1 are X = 1 and Y = 0, or X = 0 and Y = 1. These are mutually exclusive events, so we can add the probabilities. Since X and Y are independent, the probability that X = 1 and Y = 0 is the probability that X = 1 multiplied by the probability that Y = 0. The probability that X = 1 is $e^{(-\lambda_1)} * \lambda_1$, and the probability that Y = 0 is $e^{(-\lambda_2)}$. Similarly, the probability that X = 0 and Y = 1 is the probability that X = 0, which is $e^{(-\lambda_1)}$, multiplied by the probability that Y = 1, which is $\lambda_2 * e^{(-\lambda_2)}$.

So, the total probability that Z = 1 is the sum of these two probabilities. This is how we find the probability mass function of Z, where Z = X + Y. So, here you can see that what we're getting finally is the probability that Z = 1, which is:

$e^{(-(\lambda_1 + \lambda_2))} * (\lambda_1 + \lambda_2)$

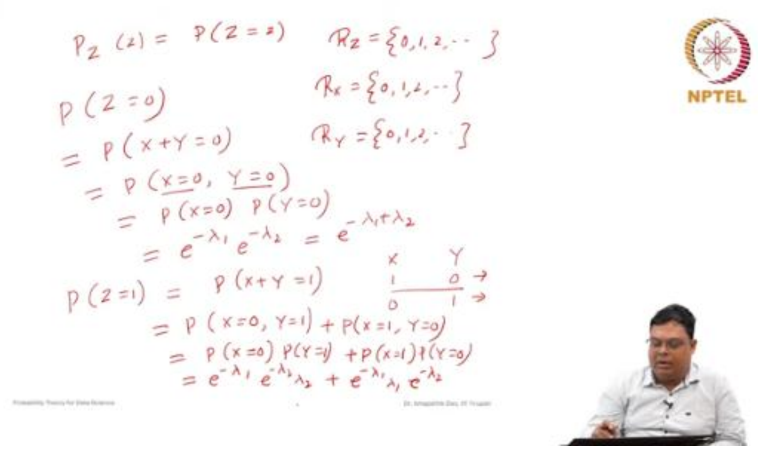
This simplifies to:

$e^{(-(\lambda_1 + \lambda_2))} * (\lambda_1 + \lambda_2)$

If you take common $e^{(-(\lambda_1 + \lambda_2))}$, you get:

$(\lambda_1 + \lambda_2) * e^{(-(\lambda_1 + \lambda_2))}$

which is the probability of Z = 1.



By computing this, we got a pattern for how this probability works. So, in general, to find the probability that Z = z (where Z can be 0, 1, 2, 3, etc.), we calculate the probability that X + Y = Z. To do this, we consider the possibilities for X and Y. If X = 0, then Y = Z; if X = 1, then Y = Z - 1, and so on, up to Z. When X = Z, Y = 0. We can express this probability as the summation of the probability that X = r, with r ranging from 0 to Z, and Y = Z - r.

This involves considering all the cases, such as X = 0 and Y = Z, X = 1 and Y = Z - 1, and so on. This is written as the summation from r = 0 to Z of the probability that X = r, and Y = Z - r. Since X and Y are independent, we multiply their probabilities. This becomes:

$e^{(-\lambda_1)} * (\lambda_1^r / r!) * e^{(-\lambda_2)} * (\lambda_2^{(Z-r)} / (Z-r)!)$

Simplifying this, we get:

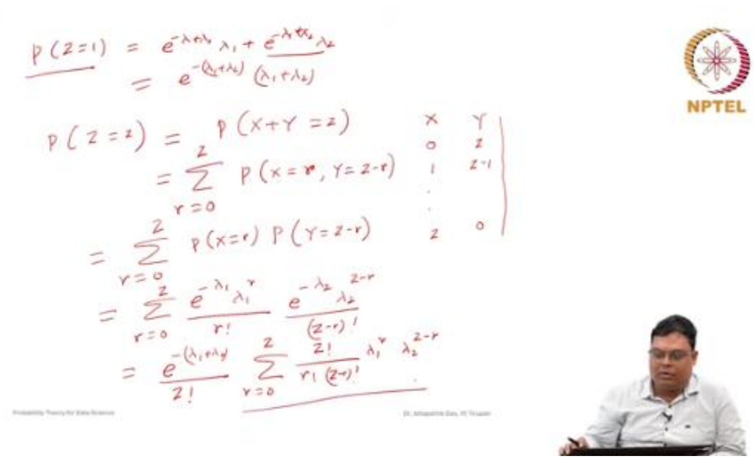
$e^{(-(\lambda_1 + \lambda_2))} * (\lambda_1^r * \lambda_2^{(Z-r)} / r! * (Z-r)!)$

Since r + (Z - r) = Z, we divide and multiply by Z! in the summation, resulting in the term Z! / (r! * (Z - r)!), which is a standard form for this kind of calculation. So, we know that this binomial expression we discussed earlier, $(a + b)^m$, is nothing but the summation from k = 0 to m of (m choose k) * a^k * $b^{(m - k)}$. This is the binomial expansion we already discussed earlier. So, in the previous things, you saw that the probability of Z = z, what we found was:

$e^{(-(\lambda_1 + \lambda_2))} * ((\lambda_1 + \lambda_2)^z / z!)$

Then, this summation, from r = 0 to z, can be written as:

z! / (r! * (z - r)!) * (z choose r)

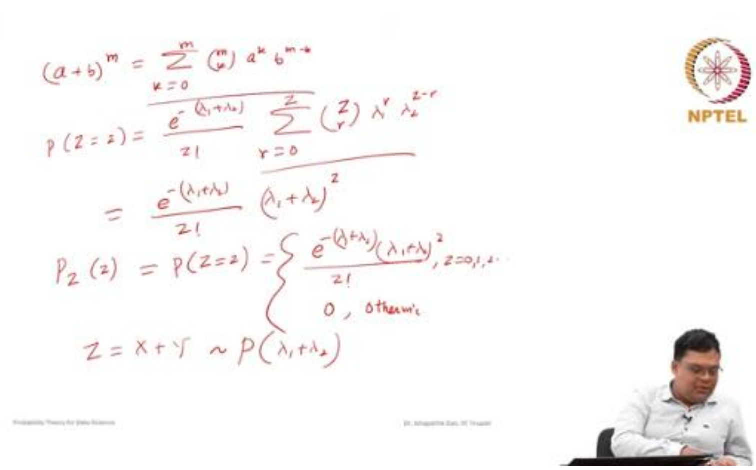


So, then you get λ_{1r} and λ_2^{z-r}. Now, if you compare this with the binomial expansion, you can see that this is $e^{(-(\lambda_1 + \lambda_2))} / z!$, and this looks similar to the binomial expansion form where $a = \lambda_1$ and $b = \lambda_2$. So, we can write the probability mass function of Z, $P_Z(z)$, as:

$$e^{(-(\lambda_1 + \lambda_2))} * ((\lambda_1 + \lambda_2)^z) / z!$$

where z = 0, 1, 2, ... Otherwise, it's equal to 0. So, this has the same form as the probability mass function of a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Hence, Z, which is equal to X + Y, follows a Poisson distribution with parameter $\lambda_1 + \lambda_2$. So, if a numerical problem gives independent Poisson random variables, you can use this concept to solve it. So, here let's consider a numerical problem. Suppose the number of telephone calls coming into a telephone exchange between 10 a.m. and 11 a.m. is X_1, which follows a Poisson distribution with parameter 2. Similarly, the number of calls arriving between 11 a.m. and 12 noon is X_2, which follows a Poisson distribution with parameter 6.



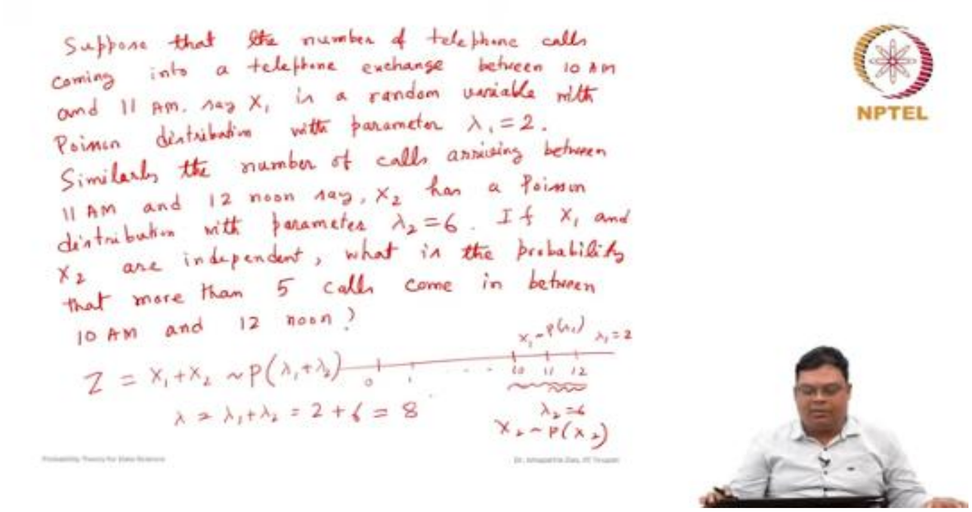
If X_1 and X_2 are independent, what is the probability that more than 5 calls come in between 10 a.m. and 12 noon? To break it down: between 10 and 11 a.m., the number of calls, X_1, follows a Poisson distribution with parameter 2. Between 11 and 12, X_2 follows a Poisson distribution with parameter 6. To find the total number of calls between 10 a.m. and 12 noon, we sum X_1 and X_2. Since X_1 and X_2 are independent, the sum, $Z = X_1 + X_2$, follows a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

So, $\lambda = 2 + 6 = 8$. So, we know that Z has a Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2 = 6 + 2 = 8$. So, the probability that Z = z is given by:

$e^{(-\lambda)} * (\lambda^z) / z!$

where z = 0, 1, 2, ... and the probability is 0 otherwise. Here, $\lambda = 8$, so the probability mass function becomes:

$e^{(-8)} * (8^z) / z!$



Now, if we want to find the probability that more than 5 calls come in between 10 a.m. and 12 noon, this is the probability that Z > 5. So, we calculate this as:

$1 - P(Z \leq 5)$

To find the probability that $Z \leq 5$, we sum the probabilities for z from 0 to 5. This becomes the sum of:

$e^{(-8)} * (8^z) / z!$

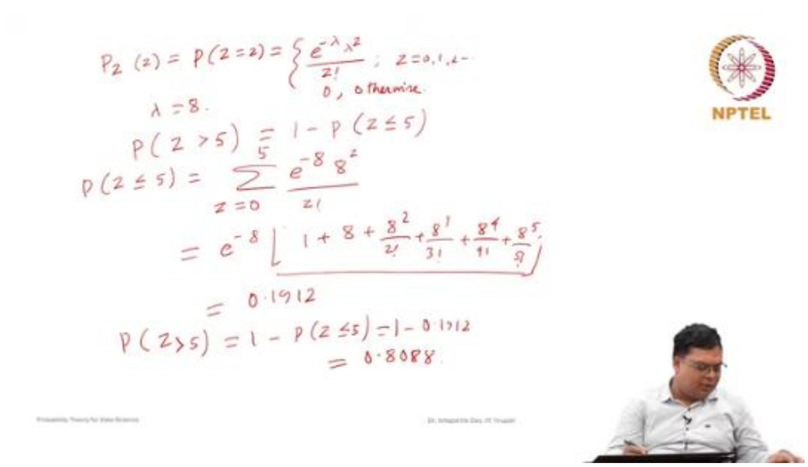
for $z = 0$ to 5. So, this is:

$e^{(-8)} * (8^0 / 0!) + (8^1 / 1!) + (8^2 / 2!) + (8^3 / 3!) + (8^4 / 4!) + (8^5 / 5!)$

If you calculate this, you get approximately 0.1912. So, the probability that $Z > 5$ is:

$1 - 0.1912 = 0.8088$

This is how you can use the sum of Poisson random variables. I hope you understand this now. Now, let's discuss another problem related to Poisson distributions.



Suppose X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively. The problem asks to find the probability that $X = Y$. To solve this, we can use the fact that X and Y are independent. The probability that $X = Y$ means both X and Y must take the same value, say r. This is expressed as the sum from $r = 0$ to ∞ of $P(X = r$ and $Y = r)$.

Since X and Y are independent, this is the product of their individual probabilities. For a Poisson random variable, the probability mass function is:

$e^{(-\lambda_1)} * (\lambda_{1r} / r!)$ for X, and

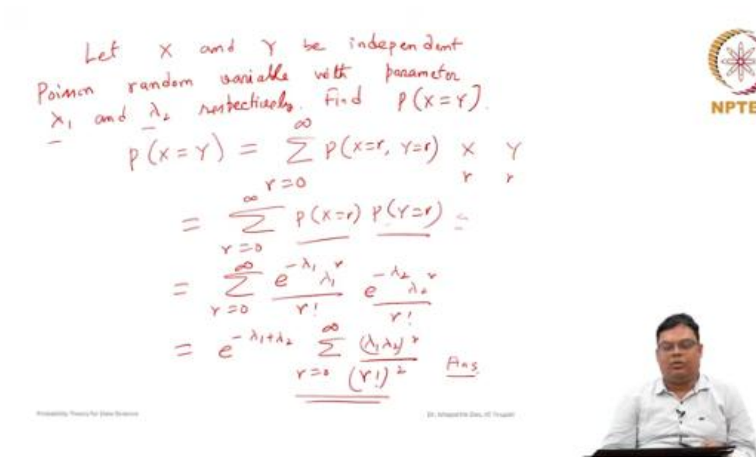
$e^{(-\lambda_2)} * (\lambda_{2r} / r!)$ for Y.

So, the probability that $X = Y$ is the sum from $r = 0$ to ∞ of:

$e^{(-(\lambda_1 + \lambda_2))} * ((\lambda_1 * \lambda_2)_r / (r!)^2)$.

This expression doesn’t simplify further, but it represents a convergent series, as the terms are less than or equal to 1. Thus, we can conclude that the series converges, although we can’t directly calculate the exact value of the convergence here.

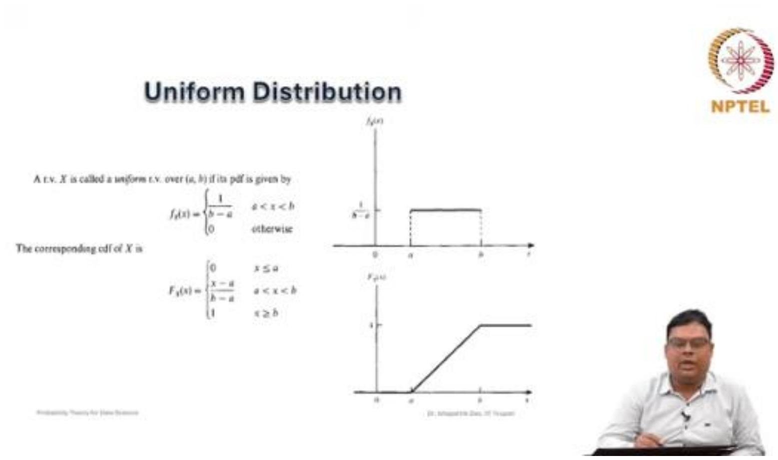
This is the final answer in its current form. I hope you are following along. We’ve now covered Bernoulli distribution, binomial distribution, and Poisson distribution, including how to compute their means, variances, probability mass functions, cumulative distribution functions, graphical representations, and also solved some numerical problems. I hope you have understood and are following along. Next, we will discuss some other distributions, specifically continuous random variables.



We will begin with uniform distribution. Now, we will discuss some important continuous distributions, starting with the uniform distribution. A random variable X is called a uniform random variable over an interval [a, b], where a and b are real numbers and a < b, if its probability density function (PDF) is given by:

f(x) = 1 / (b - a) for a ≤ x ≤ b, and f(x) = 0 otherwise.

Since it is a uniform distribution, the density is constant between a and b, meaning the probability of X taking any value within that interval is the same. For a continuous random variable, this can be written using either a half-open or closed interval.



The constant value in the PDF, denoted as c, must meet the requirements of a probability density function, which means it must be ≥ 0. Additionally, the integral of the PDF over the entire real line must equal 1. Since the probability is zero outside the interval [a, b], we only need to consider the integral from a to b. This calculation gives the integral of 1 / (b - a) over the interval [a, b], which equals 1, confirming that the value of c is 1 / (b - a). Therefore, the probability density function of X is:

f(x) = 1 / (b - a) for a ≤ x ≤ b, and f(x) = 0 otherwise.

The graph of this probability density function will be a rectangle with height 1 / (b - a) between a and b, and zero outside of that range. Now, suppose we want to find the cumulative distribution function (CDF) of this random variable. How can we find the CDF?