PROBABILITY THEORY FOR DATA SCIENCE

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Week - 06

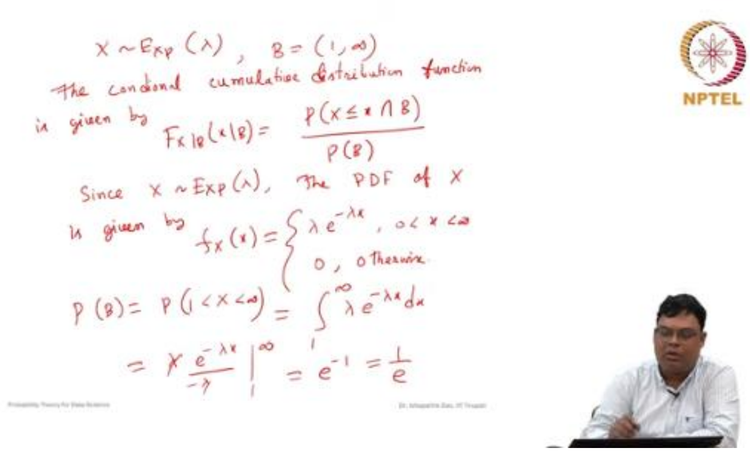
Lecture - 30

Examples of Conditional Distribution Function and Bivariate Random Variable

X has an exponential distribution with parameter λ, and B is the event taking values between 1 and ∞. Now, we need to find the conditional probability density function. First, we will determine the conditional cumulative distribution function (CDF). The conditional cumulative distribution function is given by $P(X \leq x \mid B) = P(X \leq x \cap B) / P(B)$.

First, we need to find P(B). Since X is an exponential random variable, we know the probability density function of X. The probability density function of X is given by $f_X(x) = \lambda * e^{-\lambda x}$ for $x > 0$ and 0 otherwise. Now, to find P(B), we calculate the probability that the random variable takes values between 1 and ∞. Since B is between 1 and ∞, this is the integral from 1 to ∞ of $\lambda * e^{-\lambda x}$. Since this is the probability density function, we are integrating over B. The result of the integration is $\int_1^{\infty} \lambda * e^{-\lambda x}\, dx$, which evaluates to $[-e^{-\lambda x}]$ from 1 to ∞.

At ∞, the value is 0, and λ cancels out. At 1, the result is $e^{-\lambda * 1} = 1/e$. Therefore, we have found P(B). Now, what is the probability of $X \leq x$, given the intersection with B? Note that X is a positive random variable, meaning it takes only positive values.

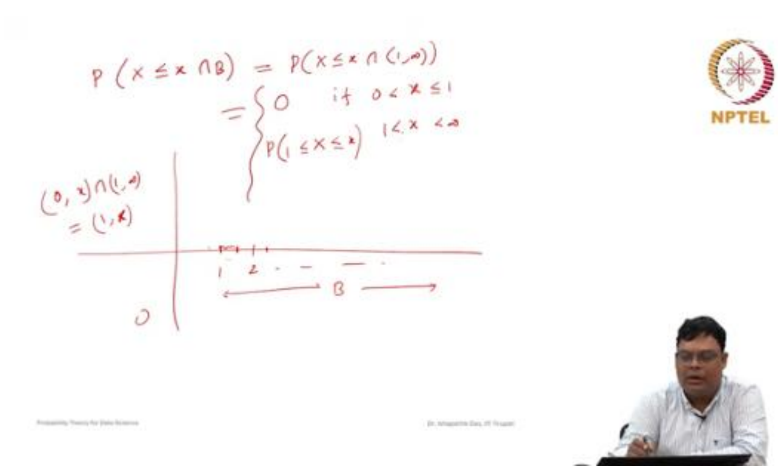


Therefore, X is a positive random variable, $X > 0$. Now, B is the set from 1 to ∞. So, $B = [1, \infty)$, and X can take any value. This probability will be 0 if $X < 1$, because when $X \in (0, 1)$, the intersection is empty. We can express this in another step.

This is the probability $P(X \leq x \mid X \cap B)$. Since $B = [1, \infty)$, this probability is equal to 0 if $X \leq 1$ and $X > 0$. If $X \in (0, 1)$, the intersection is the empty set, so the probability will be 0. For example, if $X = 0.5$, the intersection between X and B will be empty. Therefore, for any $X \in (0, 1)$, the intersection is the null set, and $P = 0$.

Now, what will be the case when $X > 1$ and $X < \infty$? We can compute this value. So, if $X > 1$, what will the intersection be? The intersection is when $X \in [1, x]$, and the interval from 1 to ∞. This is the intersection of the interval $[1, x]$ with $[1, \infty)$.

This gives us the interval $[1, x]$, where $x > 1$. Therefore, the intersection is $P(1 \leq X \leq x)$. This probability is simply the integral from 1 to x of $f_X(t)\,dt$, where $x \in (1, \infty)$. Let's find this probability first. I hope you have understood it. It is now clear that whenever $x > 1$, the intersection will be the interval $[1, x]$, because it is the intersection of $[1, \infty)$ and $[0, x]$.

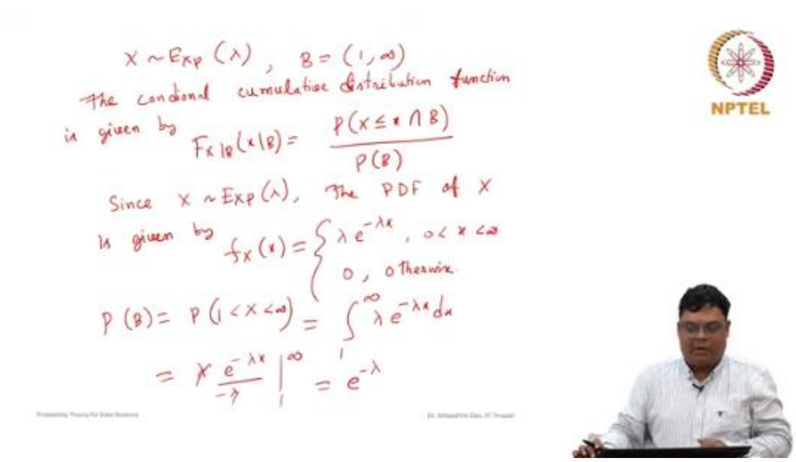


So, now we will erase these steps and find the probability. This probability is nothing but $P(1 \leq X \leq x)$. This probability is the integral from 1 to x, not from 1 to ∞, of $f_X(x)\,dx$. Just take a different variable, t, and integrate with respect to t. This is simply the integral from 1 to x of $\lambda * e^{-\lambda t}\,dt$.

So, this simplifies to $\lambda * e^{(-\lambda t)}$, evaluated from 1 to x. So, basically, the λ cancels out, and this becomes $e^{(-\lambda * 1)}$. There's a minus here, so that's why it becomes just -1. It's the result of $-\lambda * x$. One minute, I think $t = 1$, but because there's a λ here, it's $\lambda * 1$.

So, this is $e^{(-\lambda)}$, and then $e^{(-\lambda * x)}$. So, I think it's fine now. This is the probability. I just remembered that I made a mistake earlier. $e^{(-\lambda x)}$ goes to 0 as $x \to \infty$, and when $x = 1$, it becomes $e^{(-\lambda)}$.

Sorry about that. So, this is $e^{(-\lambda)}$. This is the probability of B. Finally, we will find the conditional probability cumulative distribution function. Hence, the conditional CDF is given by this, which is simply $f_X(x \mid B)$.

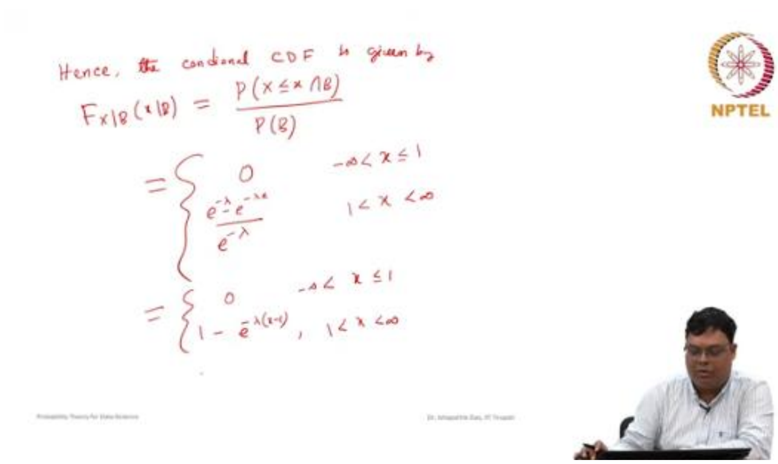


What did we find? So, we found that this is simply $P(X \leq x \cap B) / P(B)$, which we have already found. This is true whenever $x \leq 1$. Sorry, we have already determined that this probability is for $x \leq 1$. So, even if you take a negative number, it will always be 0. Whatever the number is, it will be 0. That's why we can write $-\infty$, as you need to define the entire real line. So, whenever x is from $-\infty$ to 1, this probability is 0. The upper part is 0. However, when $x > 1$ and $x < \infty$, the probability of B will be $e^{(-\lambda)}$, which we just checked.

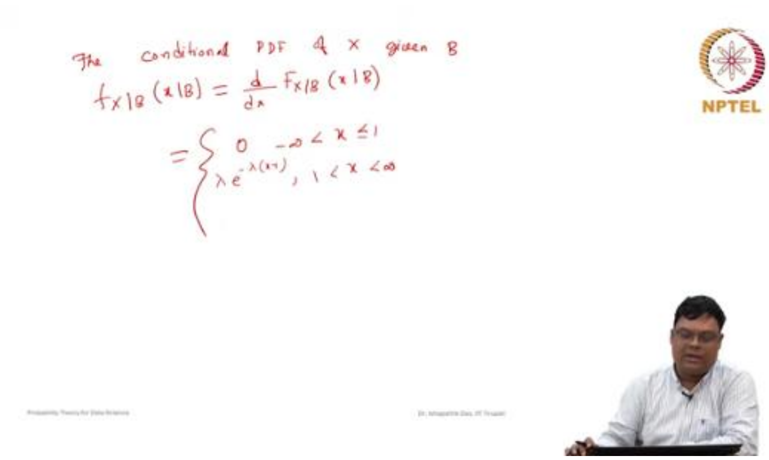
The probability of B is $e^{(-\lambda)}$. What is the probability of the upper part? So, we just found this probability. It is the same as this. This value is $e^{(-\lambda)} - e^{(-\lambda x)}$.

So, it is e^(-λ) * (e^(-λx) - 1). Finally, what did we get? So, this is 0 whenever x ≤ 1, and then it is 1 - e^(-λ) * (e^(-λx) - 1) whenever x > 1 and x < ∞. This is the conditional cumulative distribution function (CDF) for X given B.

Now, the question asks what the conditional probability density function (PDF) of X given B will be. So, what do we need to do now? We have to just differentiate it with respect to x. So, the conditional probability density function of X given B is nothing but the derivative of the conditional cumulative distribution function given B. So, we can see that this distribution function is 0 when x ≤ 1 or x < -∞. Now, if you take the derivative with respect to x, this is nothing but λ * e^(-λx).

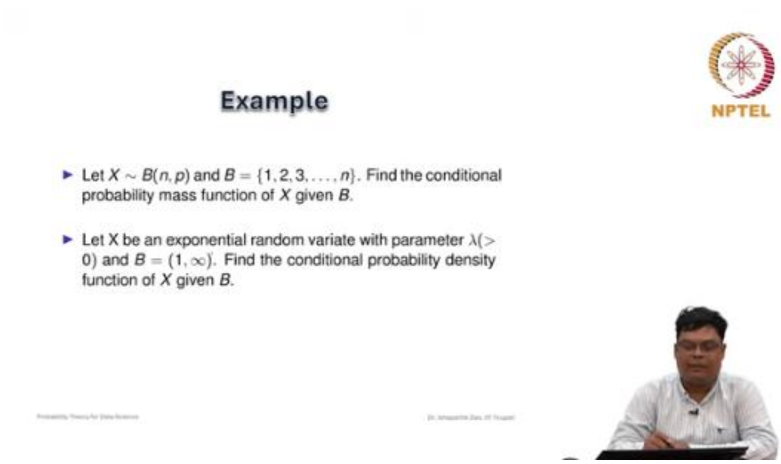


So, we can see that this distribution function is 0 when x ≤ 1 or x > -∞. Now, if you take the derivative with respect to x, this is nothing but λ * e^(-λx). So, this is finally the conditional probability density function of X given B. So, this is one example of how to find the conditional probability density function given event B. Now, next, we will discuss the conditional probability mass function.



For this example, whenever X is a binomial distribution with parameters n and p, and B = {1, 2, ..., n}, we will find the conditional probability mass function of X given B. Next, we will discuss this. Now, we will discuss the conditional probability mass function with an example. The conditional probability mass function for a given event B is as follows: Let X be a random variable with a binomial distribution with parameters n and p. Let B be the event that X takes values from 1 to n.

We need to find the conditional probability mass function of X given B. So, we have already discussed the conditional probability density function. To summarize, when we compute the conditional probability density function, we first need to find the cumulative distribution function (CDF). Once we have the CDF, we take its derivative to obtain the probability density function (PDF). Now, for the discrete case, the cumulative distribution function may not be differentiable for all values of x.

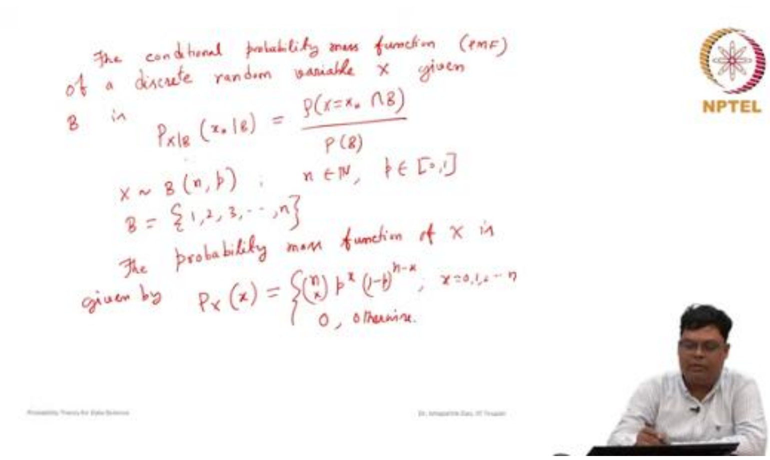


In that case, we discuss the probability mass function (PMF). Here, we will directly find the conditional probability mass function. So, recall that the conditional probability mass function for a given random variable is given by the conditional probability mass function. The conditional probability mass function (PMF) is the conditional probability mass function of a discrete random variable. Basically, for a discrete random variable X given B, where B = {1, 2, ..., n}, so here, B is given as {1, 2, ..., n}.

Let us discuss, first, for any event B. The conditional probability of X given B is the probability that X = xk and event B occurs, divided by the probability of event B. Now, for this example, X is binomial with parameters n and p, and B = {1, 2, 3, ..., n}. So, X has a binomial distribution with parameters n and p, where n is a natural number and p is a probability between 0 and 1. We discussed the binomial distribution earlier. Now, how can we find the probability mass function when B is given as the set {1, 2, ..., n}? First, let's find the probability mass function of a binomial random variable. This is the probability mass function of a binomial distribution. The probability mass function of X, which is a binomial random variable, is as follows. So, we write the probability mass function of X as follows.

The value of X can range from 0 to n because it follows a binomial distribution. This means that the probability of X taking a particular value is determined by the binomial distribution with parameters n and p. So, for X = x, when x ∈ {0, 1, 2, ..., n}, this is equal to: P(X = x) = C(n, x) * p^x * (1 - p)^(n - x), for x ∈ {0, 1, 2, ..., n}, and 0 otherwise.

So, we now know the probability mass function. Now, we use this formula to find, by this definition, the conditional probability mass function of X given B.



So now, how can we find the conditional probability mass function of X given B? So now, what is xk? xk can take values from 0 to n. So, we write it as P(X = x | B). This is nothing but P(X = x ∩ B) / P(B).

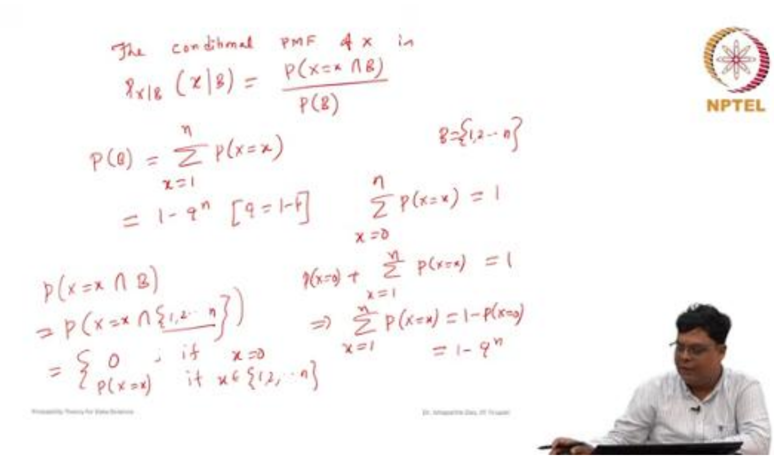
First, we will find P(B). So, P(B) is nothing but the summation of probabilities. Since B is given as {1, 2, ..., n}, P(B) is the summation, because it is a discrete random variable. It can take values from 1 to n, so the summation is for x = 1 to n, and it represents P(X = x). So, this is P(B) because it is the union of the probabilities: P(X = 1) + P(X = 2) + ... + P(X = n).

Since it is a binomial random variable, we know that the summation of the probabilities, from X = 0 to n, has already been shown. For binomial distribution, the PMF of a binomial random variable should be equal to 1. This means that P(X = 1 to n), when summed, is equal to 1. So, this is nothing but P(X = 0) + Σ P(X = x) for x = 1 to n. Since the total probability for X from 0 to n is 1, this is P(X = 0) + Σ P(X = x) for x = 1 to n. This is equal to 1, which implies that Σ P(X = x) for x = 1 to n = 1 - P(X = 0). So, what is P(X = 0)? P(X = 0) is just P(X = 0), which is (1 - p)^n. So, let q = 1 - p. This is equal to 1 - (1 - p)^n.

Therefore, this is simply q^n. So, this is equal to 1 - q^n. Now, the probability of X = x, as we found, is x = 1 to n, P(X = x). This is nothing but 1 - q^n, where q = 1 - p. Now, we found this value as 1 - q^n.

We've found $P(B)$. Now, we need to find $P(X = x \cap B)$. So, this is for $x = 1$ to n. Essentially, it's $P(X = x \cap \{1, 2, ..., n\})$. Note that x can be any value from 0 to n, because X follows a binomial distribution.

So, this intersection will be a null set if $x = 0$, because 0 does not belong to the set $\{1, 2, ..., n\}$. Therefore, $P(X = x \cap B) = 0$ if $x = 0$. Otherwise, for any other value of x, such as $x = 1$ or $x = 2$, the intersection will be a subset of $\{1, 2, ..., n\}$, because this set is a superset of that. So, if x belongs to the set, then for any other value of x from 1 to n, this is simply $P(X = x)$. Because this is nothing but a subset of this random variable, the probability of $X = x$ is something we already know.



So, I will write here. Let us move to the next page. Hence, the probability that $X = x$ given B can be written as 0 when $X = 0$. For any other values, it is also 0. However, because X has a non-zero probability at X, this intersection is 0 when $X \in \{1, 2, ..., n\}$. So, this probability is nothing but $P(X = x)$.

So, we can write it as 0 when $X = 0$. $P(X = x)$ is already known to be the PMF of the binomial random variable, which is written as (n choose x) * p^x * $(1 - p)^{(n - x)}$. So, this is (n choose x) * p^x * $(1 - p)^{(n - x)}$. This holds when $x \in \{1, 2, ..., n\}$; for any other value, it is actually 0. So, basically, we can combine these things.

This is equal to (n choose x) * p^x * (1 - p)^(n - x) if x ∈ {1, 2, ..., n}; otherwise, for any other values of x, this is equal to 0. So now, finally, what is the probability mass function? Hence, the conditional probability mass function of X given B is P(X = x | B). So, P(X = x | B) = P(X = x ∩ B) / P(B).

This is non-zero when x ∈ {1, 2, ..., n}. So, this is (n choose x) * p^x * (1 - p)^(n - x) / P(B), which we found to be 1 - q^n. So, P(B) = 1 - q^n, where q = 1 - p. This is 1 - q^n whenever x ∈ {1, 2, ..., n}; it is equal to 0 otherwise. So, these are the two examples we discussed here: one using the conditional probability mass function to find the conditional probability mass function, and the other using the conditional probability density function. With this, we conclude the discussion of univariate random variables. Next, we will discuss more later as needed. Some of the basic concepts and examples have already been covered, including important distributions and their numerical examples. Now, we will move on to multivariate random variables, starting with bivariate random variables.