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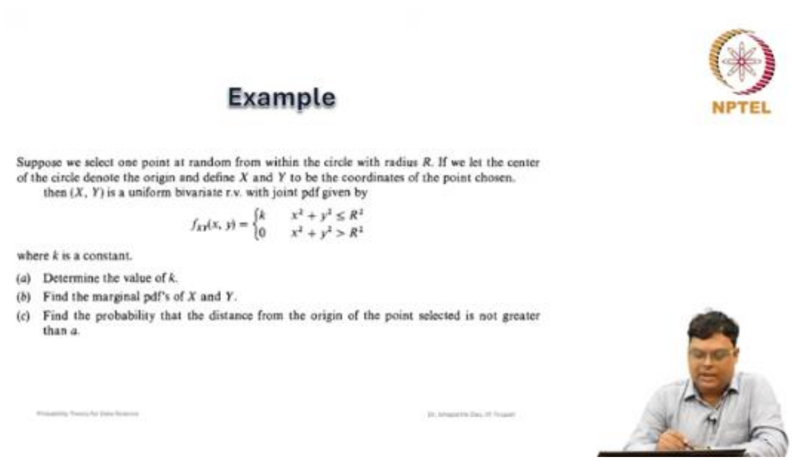
**Week - 08**

**Lecture - 38**

# Numerical Examples on Probability Density Function

Let us discuss another example of a joint probability density function. Suppose we select one point at random from within a circle with radius R. If we let the center of the circle denote the origin, define X and Y as the coordinates of the point chosen. Then, (X, Y) is a uniform bivariate random variable with a joint probability density function given by this. Essentially, suppose we consider a circle centered at the origin.



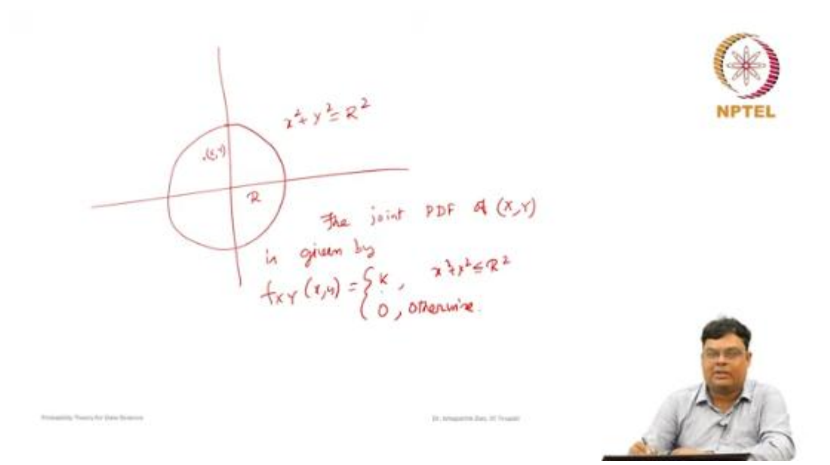

We select one point at random from within the circle with radius R, where the center of the circle denotes the origin. So, this is the origin, the center, and the circle may look like this. The equation of the circle, we know, is $x^2 + y^2 = R^2$ because we are considering the radius as R. So, this is given. We are randomly selecting a point inside the circle, and since it's a uniform distribution, the joint probability density function applies.

So, let x and y be any point here. X and Y are the coordinates of the point chosen. Then, (X, Y) is a uniform bivariate random variable with a joint probability density function given by this. So, the joint probability density function of (X, Y) is given by $f_{X\gamma}(x, y)$. This is equal to some constant k because it is a uniform distribution.

So, inside this region, it is k. We will denote this by $x^2 + y^2 \leq R^2$, then it is k; otherwise, it is 0. So, this k is a constant we have to find. First, we need to determine the value of k. Then, we will find the marginal probability density functions of X and Y.

After that, we will find the probability that the distance from the origin of the point selected is not greater than a. So, let us find each step one by one. So, first, we need to find the value of k. The probability density function is non-zero inside the region of the circle; otherwise, it is 0. Since it is a probability density function, we will check the properties of the probability density function.

The properties state that k must be $\geq 0$. When the joint probability density function $f_{X\gamma}$ is always $\geq 0$ for all x and y, k must also be $\geq 0$ to satisfy this condition. The second condition is that when you take the integral over the entire region, from $-\infty$ to $+\infty$, it must equal 1. This will hold true when the function is non-zero inside the circle. So, basically, this is the integral of k with respect to dx and dy over the region where $x^2 + y^2 \leq R^2$.

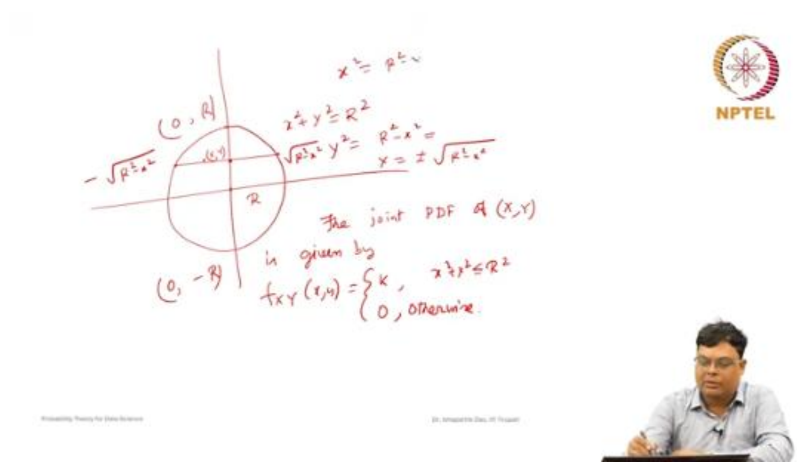



This equals 1. Since k is a constant and the function is even, we can proceed with the integration. To find this, we already know that it's simply $k * \iint (x^2 + y^2 \leq R^2)\, dx\, dy$. So, this integration is nothing but the area under the circle, which implies the area under the equation $x^2 + y^2 \leq R^2$. We know that this area is $\pi R^2$.

So, we can directly write $\pi R^2$, and this equals 1. This implies that $k = 1 / \pi R^2$. Or, you can just find the area by doing the integration here. So, whenever you take this, the limit of y will be from -R to +R. It is an even function because it is a constant.

So, you can also change the limits to 0 to R and multiply by 2. Whenever, sorry, if you want to take the limit of y first, it will be from -R to +R. So, this is (0, R) and this is (0, -R) for any y value. So, then the x value will go because it will be nothing but $y^2 = R^2 - x^2$. So, it will be from $y = \pm\sqrt{(R^2 - x^2)}$.

For a particular y value, there will be two values. So, this is $-\sqrt{(R^2 - x^2)}$, and this is $\sqrt{(R^2 - x^2)}$. So then, your integration will be nothing but, if you want to do the integration dx dy, it will be from $-\sqrt{(R^2 - y^2)}$ to $\sqrt{(R^2 - y^2)}$. So, we are finding the limit of x. So, we did the opposite for the y limit.




Now, for the x limit, we will have $x^2$ for given y values, which will be $R^2 - y^2$. So, actually, this will be $y^2$, and this will be $x^2$. Thus, x will be $\pm\sqrt{(R^2 - y^2)}$ for the given y value. So, this will be the integral from $-\sqrt{(R^2 - y^2)}$ to $\sqrt{(R^2 - y^2)}$, and then this will be the integral from -R to +R, dx dy. So, this will be $2 * \sqrt{(R^2 - y^2)}$, dy, from -R to R.

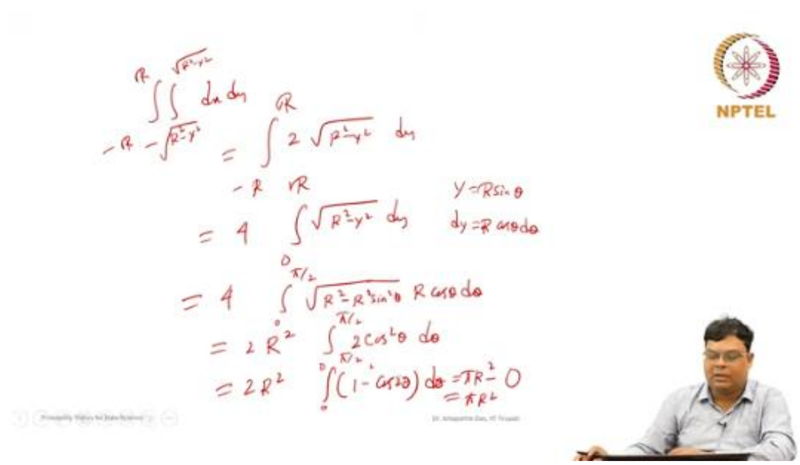
So, this can be represented as an even function. If you replace y with -y, it will become $4 * \int(0 \text{ to } R) \sqrt{(R^2 - y^2)}$, dy. Now, we can substitute $y = R * \sin(\theta)$. By making this substitution, we can perform the integration, and finally, we will find the result. So, what we will get is that $dy = R * \cos(\theta)\, d\theta$.

The limits of θ will be from 0 to π/2. When θ = 0, y = 0, and when θ = π/2, y = R. So, this becomes the integral from 0 to π/2 of $\sqrt{(R^2 - R^2 * \sin^2(\theta))} * R * \sin(\theta) * \cos(\theta)\, d\theta$. So, then what will we finally get? So, this is R * 4 * R. So, $R^2 * 2R$, it will be coming.

So, 1 - $\sin^2(\theta)$ will be $\cos^2(\theta)$, and the square root of this is cos(θ). Then, we have the integral from 0 to π/2. So, sin(θ)—okay, I made a mistake. So, here actually, it will be nothing but R * cos(θ), dθ. So, it will be R * R * cos(θ), dθ.

Then, this is the square root of 1 - $\sin^2(\theta)$, which is $\cos^2(\theta)$, again the square root of cos(θ). So, this will be $4R^2 * \cos^2(\theta)$, $\cos^2(\theta)$, dθ. Now, if you simplify it, it becomes $2R^2 * 2 * \cos^2(\theta)$. $\cos^2(2\theta)$ can be represented as 1 - cos(2θ). Then, the integration goes from 0 to π/2.

Sorry, this is just the integration; you can do it, dθ. So, this is $\pi R^2$, and minus this is sin(2θ) / 2. If you take the limit, this becomes 0. So, the result is $\pi R^2$. Why did we do this integration?




Because we need it for the next problem. We have to find the marginal probability density function of X and Y. So, what will the marginal probability density function be? How will we find it? Note that the area of the circle is $\pi R^2$, which we have already found.

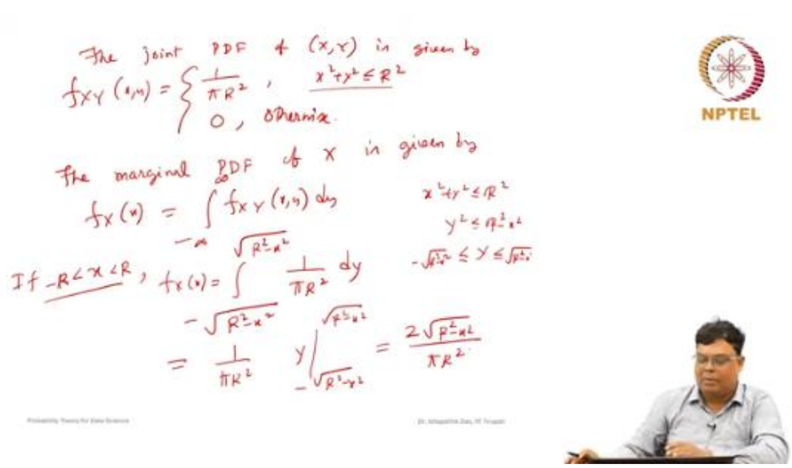
So, k becomes $1 / (\pi R^2)$. The joint probability density function of X and Y is given by a constant, $1 / (\pi R^2)$, when $x^2 + y^2 \leq R^2$. Otherwise, it is 0. Now, we need to find the marginal probability density function of X and Y. The marginal refers to finding the distribution of a subset of a random vector, which means considering one of the components.

In the case of multivariate random variables, there are multiple subsets, but here we are only discussing X and Y. The marginal probability density function of X is given by f(x). By definition, this involves integrating over the other variable, y, from $-\infty$ to $+\infty$. Now, for a fixed x, the previous integration was required to determine the limits where the density function is non-zero. We know that the density function is non-zero whenever $x^2 + y^2 \leq R^2$.

For a particular value of x, the possible values of y range from a minimum of -R to a maximum of R. If x is between -R and +R, the density function will be non-zero. Otherwise, it will be zero. For a particular value of x within the range of -R to +R, the limit for y will be such that $y^2 \leq R^2 - x^2$. Therefore, y will lie between $-\sqrt{(R^2 - x^2)}$ and $\sqrt{(R^2 - x^2)}$.

This is why the limit for the marginal probability density function of X will be defined in this way. For this region, the value will be non-zero, and outside of this region, the value will be zero. The limits for y will range from $-\sqrt{(R^2 - x^2)}$ to $\sqrt{(R^2 - x^2)}$. The value will be $1 / (\pi R^2)$ by definition. Then, you integrate with respect to y.

This is the definition of the marginal probability density function. This is the correct method. If you make any mistakes, you won't be able to find the exact probability density function. So, you need to find the limit properly. That's why, even though we already know that the area of the circle is $1 / (\pi * R^2)$, we went through this process to understand how we do the marginal calculation.



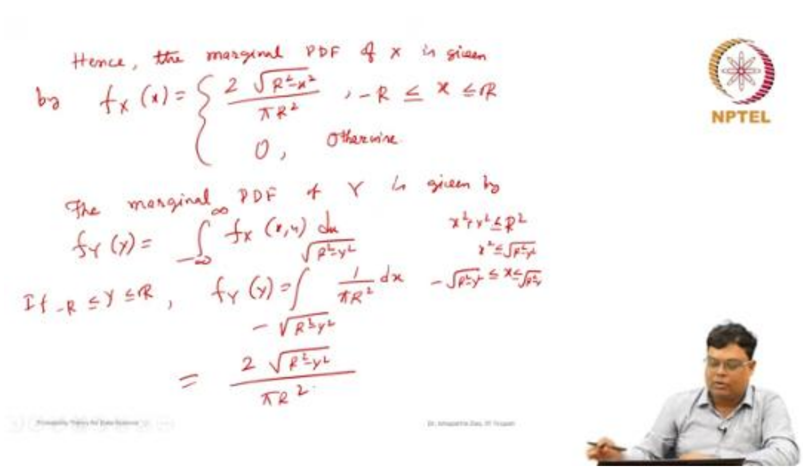

Now, this is a constant. When you integrate with respect to y and take the limit, it becomes $1 / (\pi R^2)$. The integration involves $\sqrt{(R^2 - x^2)}$. So, if you apply the limit, you'll get $2 * \sqrt{(R^2 - x^2)} / (\pi R^2)$. So, we will write it properly; otherwise, it will be zero. Hence, the marginal probability density function of X is given by f(x), which is $2 * \sqrt{(R^2 - x^2)} / (\pi R^2)$, whenever x is between -R and +R. Since it is a continuous density, you can use "equal to" or "strictly less than"; it doesn't matter, as the probability at a single point will be 0, and 0 otherwise. Hence, the marginal probability density function of X is given by f(x), which is $2 * \sqrt{(R^2 - x^2)} / (\pi R^2)$, where x is $\geq$ -R and $\leq$ R, and 0 otherwise.

Similarly, we can find the marginal probability density function of Y. Let's proceed with the computation.

The marginal probability density function of Y is given by the integral from $-\infty$ to $+\infty$, where we perform the integration with respect to x. Now, whenever Y is between -R and +R, the probability density function is non-zero. The condition for this is that $x^2 + y^2 \leq R^2$. Fixing Y within the range of -R to +R, we know that $y^2 \leq R^2 - x^2$. For a fixed value of $y^2$, we need to determine the limits of x because the integration is performed with respect to x.

The value of $x^2$ must be less than or equal to $R^2 - y^2$. Therefore, the range of x is from $-\sqrt{(R^2 - y^2)}$ to $\sqrt{(R^2 - y^2)}$. This means the integration limits for x are from $-\sqrt{(R^2 - y^2)}$ to $\sqrt{(R^2 - y^2)}$. The probability density function is constant, given as $1 / (\pi R^2)$, and the integration is carried out with respect to x. So, when you perform the integration similarly, you will obtain the result as $2 * \sqrt{(R^2 - y^2)} / (\pi R^2)$.

So, finally, we write the marginal probability density function of Y. It is $2 * \sqrt{(R^2 - y^2)} / (\pi R^2)$, whenever y is $\geq$ -R and $\leq$ +R. Outside of this range, it is equal to 0. Whether it is a density or not, because it has to be a probability density, you have to do the integration from -R to R. Then you can show that it is equal to 1 because we computed the area under the circle.



We already did a similar type of integration here. So, you can see that we already computed this, like $2\sqrt{R^2 - y^2}$. We computed it from $-R$ to $R$. This comes out to $(\pi R^2) / (\pi R^2) = 1$. So, that is why we did this computation, and this is the marginal probability density function.

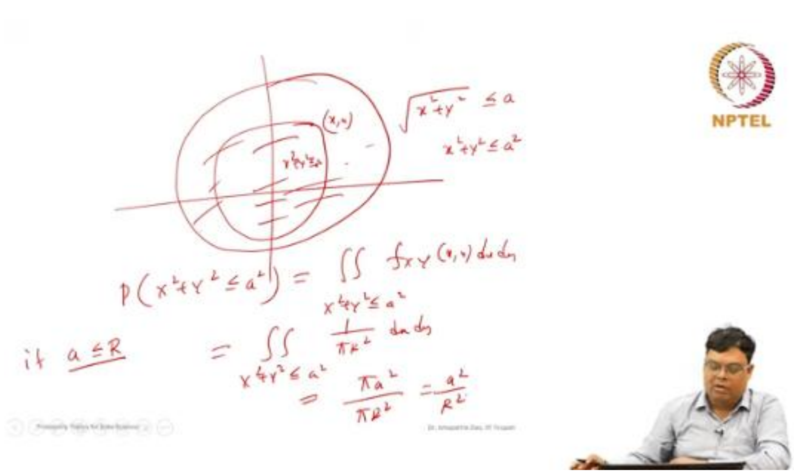
So, let's see what the other problems are. The first one is finding the marginal probability density function of X and Y. The next question asks to find the probability that the distance from the origin of the point selected is not greater than a. So, it's asking about the distance from the origin. I just, okay, so we already have this image, so this is...

Now, for any radius, suppose the distance from the origin is given by a. The question asks for the probability that the distance from the origin is not more than a. So, how do we find the distance from the origin for any point in the plane? For any point, say $(x, y)$, the distance from the origin can be represented by $\sqrt{x^2 + y^2}$. So, this is the distance, and since we're taking the square root, it will be a positive value.

So, this should be $\leq a$. It's asking for the probability that this should be $\leq a^2$. So, we have to find the probability that $x^2 + y^2 \leq a^2$. Now, this is essentially the integration of $x^2 + y^2 \leq a^2$, multiplied by $1 / (\pi R^2)$, with respect to dx and dy. So, this is, by definition, the density function $f(x, y)\, dx\, dy$. Now, this is non-zero if $a \leq R$.

This is essentially $x^2 + y^2 \leq a^2$. This value will be $1 / (\pi R^2)$ if $x^2 + y^2 \leq R^2$; otherwise, it is 0. Now, for a, let's find this value because it is a constant. We know that the area under $x^2 + y^2 \leq a^2$ is the area of the circle with radius a. This means that if $R^2$ is like this and the circle is like this, then this represents $x^2 + y^2$.

The smaller region is $x^2 + y^2 \leq a^2$, while the larger region is $x^2 + y^2 \leq R^2$. So, this is nothing but the area under the circle, which will be $(\pi a^2) / (\pi R^2)$, or $(a^2) / (R^2)$. Now, if $a \leq R$, then the probability will be $(a^2) / (R^2)$. But if $a > R$, then the probability that $x^2 + y^2 \leq a^2$ will be 1. Now, in that case, by definition, this is nothing but $x^2 + y^2$.



In this region, we need to integrate the joint density function. So, when $x^2 + y^2 \leq a^2$ and $a > R$, the density will be 0. Essentially, this integration can be divided like this:

1. $x^2 + y^2 \leq R^2$, $f(x, y)$.

   This integration is essentially the density function, $dx * dy$ within the region.

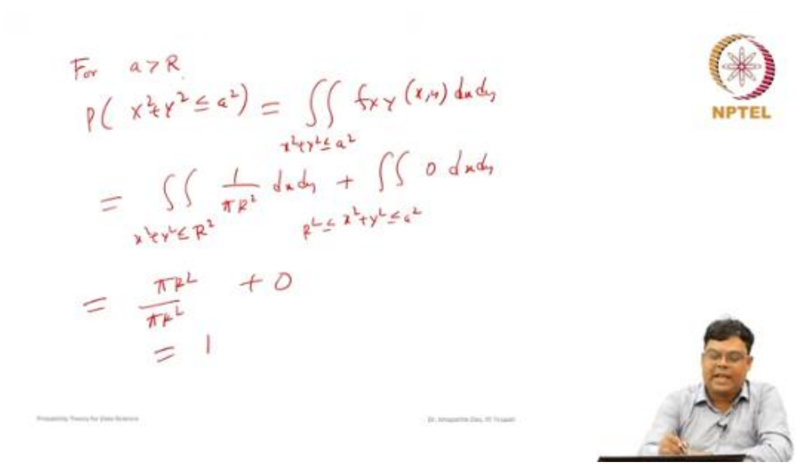
2. $R^2 < x^2 + y^2 \leq a^2$.

   Since $a > R$, the density in this region is 0.

Therefore, this is equivalent to $(\pi * R^2) / (\pi * R^2) = 1$. The remaining part is 0, so the result is simply 1.

So, for $a > R$, the probability will be 1, and for $a \leq R$, the probability will be $a^2 / R^2$.

We have to specify the value because it's in the problem, but they didn't ask for or specify it.




So, here, we need to find the probability that the distance from the origin of the point selected is not greater than a. It's very important to know where a is situated. Suppose, if we try again to draw a slightly better circle, this is $x^2 + y^2 = R^2$. If $a < R$, then the probability that the distance from the origin, $x^2$, will be no more than a, is the ratio of this area, $\pi * a^2$, to the whole area. So, $(\pi * a^2) / (\pi * R^2)$, which simplifies to $a^2 / R^2$.

But if a > R, then the density will be 0 inside this region. So, this integration will be (π * R^2) / (π * R^2), plus the region where the density is 0. This will be 1. That's why we wrote earlier that when a > R, the probability is 1. If a ≤ R, then the probability is a^2 / R^2.

This is the solution to the problem. Now, we will discuss some other topics. We have covered two numerical examples for joint probability density functions and two numerical examples for joint probability mass functions. I hope you have understood the concept of joint distributions and the joint cumulative distribution function of a bivariate random variable (X, Y). If the variables are discrete, we discussed joint probability mass functions and their related properties, along with some numerical examples.

Then, if X and Y are both continuous, we covered joint probability density functions and their related properties, along with two numerical examples. I hope you understand the concepts, and you can try solving more problems related to these theories. Now, we will discuss conditional probability mass functions. We have already discussed conditional probability for a given event. Now, we will discuss a more general case.

Suppose there are two random variables, X and Y, both discrete. If the observation of one variable is given to us, we will explore how the probability mass function of the other variable can be modified. This is known as the conditional probability mass function in the case where both random variables are discrete. Here, you can see that the conditional probability mass function, when X and Y are discrete random variables with a joint probability mass function for specific values of X and Y, is defined as the probability of Y given that X = a specific value.