PROBABILITY THEORY FOR DATA SCIENCE

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Week - 09

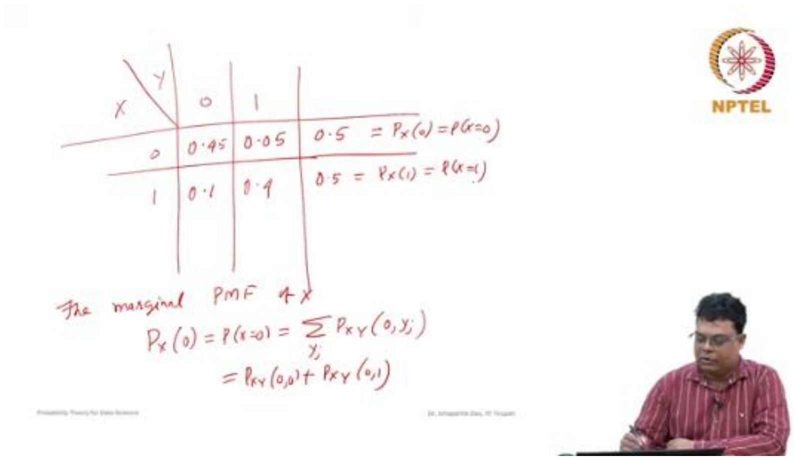
Lecture - 43

Numerical Examples on Moments for Bivariate Random Variables

The next question is to find the marginal probability mass function of X and Y. The marginal probability mass function of X is already given, with P(X = 0) = 0.5. However, we can also verify this using the joint probability mass function to ensure our computation is correct. To find the marginal probability mass function of X, consider the two possible values of X: 0 and 1. For X = 0, we take the sum of the probabilities across all values of Y.

This involves summing the joint probabilities where X = 0 and Y takes its possible values, which are 0 and 1. This gives us P(X = 0). Similarly, for X = 1, we take the sum of the joint probabilities where X = 1, summing over all values of Y. The result confirms that P(X = 0) = 0.5 and P(X = 1) = 0.5. This approach ensures consistency with the given marginal probabilities. Essentially, the row sums of the joint probability mass function correspond to the marginal probabilities of X.

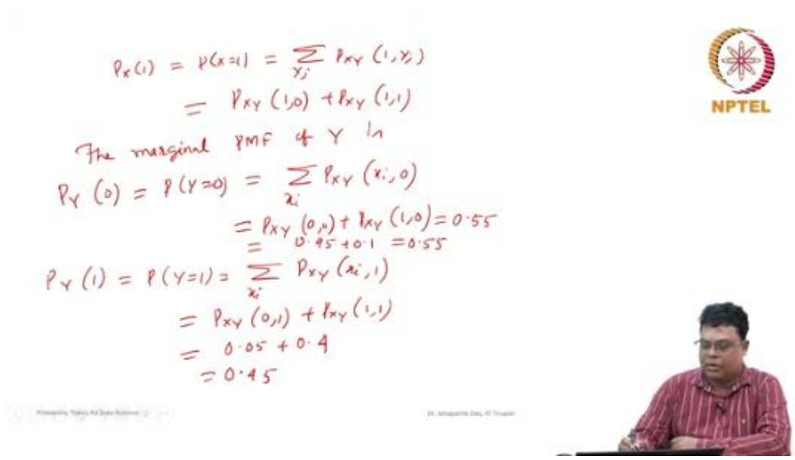
This is P(X = 1). To calculate this, you sum up all the values of Y in the joint probability mass function while keeping X fixed. Y can take two values, 0 and 1. So, you add P(X = 1, Y = 0) and P(X = 1, Y = 1). This corresponds to summing the values in the second row of the table.



The result is 0.1 + 0.4 = 0.5. Thus, the probability mass function for X is as follows: P(X = 1) = 0.5, and P(X = 0) = 0.5. These values were already provided in the problem, and the calculations from the joint probability mass function confirm that our results are correct. This means the computed values align with the given information. Now, let's determine the marginal probability mass function of Y.

To find this, we calculate the probabilities for each possible value of Y. For Y = 0, we sum the probabilities for all X values while keeping Y fixed. This involves adding P(X = 0, Y = 0) and P(X = 1, Y = 0). The result is 0.45 + 0.1 = 0.55. Therefore, P(Y = 0) = 0.55. Similarly, to find P(Y = 1), we sum the probabilities for X = 0 and Y = 1, and X = 1 and Y = 1. The result is 0.05 + 0.4 = 0.45. Thus, P(Y = 1) = 0.45. These computations ensure that the marginal probability mass function of Y is clear and accurate. We have verified this by summing over the respective rows and columns of the joint probability mass function.

Note that the sum of probabilities in a probability mass function must always equal 1. For example, 0.45 + 0.55 = 1.



Similarly, the sum of 0.5 + 0.5 = 1. This confirms that the marginal probability mass functions are correct as stated in the question.

What we have to check is if X and Y are independent. Then, if X and Y are independent, P(X = xi, Y = yj) can be represented as the product of their marginals, P(X = xi) * P(Y = yj) for all xi and yj.

Next, we consider whether X and Y are independent. To check independence, we need to verify if the joint probability for any pair of values can be represented as the product of the corresponding marginal probabilities. If this condition is satisfied for all possible values, then X and Y are independent.

Let's take a specific example where X = 0 and Y = 0. The joint probability for X = 0 and Y = 0 is 0.45. According to the condition for independence, this should equal the product of P(X = 0) and P(Y = 0). Here, P(X = 0) = 0.5, and P(Y = 0) = 0.55. However, when we multiply these two probabilities, the result is 0.5 * 0.55 = 0.275, which does not equal 0.45. This shows that the joint probability is not equal to the product of the marginals, meaning X and Y are not independent random variables.

Now, we will find the mean and variance of X. To do this, we use the marginal probability mass function for X. Similarly, the marginal probability mass function for Y can be used for calculations involving Y. This is a good practice and aligns with our earlier discussions about finding mean and variance.

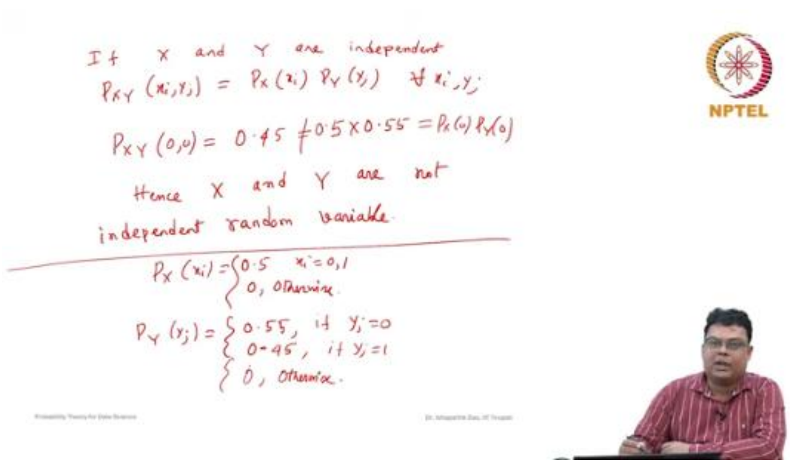
The marginal probability mass function for X shows that P(X = 0) = 0.5, and P(X = 1) = 0.5. For Y, P(Y = 0) = 0.55, and P(Y = 1) = 0.45. All other probabilities are zero for both

X and Y. From these marginal probabilities, we see that both X and Y follow Bernoulli distributions. For X, the probability of success is 0.5, while for Y, the probability of success is 0.45.

If you recall the properties of a Bernoulli random variable, these probabilities can be directly used to calculate the mean and variance. Alternatively, you can rely on the marginal probability mass functions to compute these values.

Now, let's determine the mean. The expected value of X is calculated as:

$E(X) = \Sigma [X * P(X)] = 0 * P(X = 0) + 1 * P(X = 1) = 0 * 0.5 + 1 * 0.5 = 0.5$



Since X can take the values 0 and 1, the calculation involves multiplying 0 by $P(X = 0)$, and adding 1 multiplied by $P(X = 1)$. The result is simply 0.5.

To calculate the variance of X, we first need to find the expected value of X squared. This is defined as the sum of the squares of each value of X, multiplied by their respective probabilities. For X, this involves squaring 0 and multiplying by $P(X = 0)$, and squaring 1 and multiplying by $P(X = 1)$.

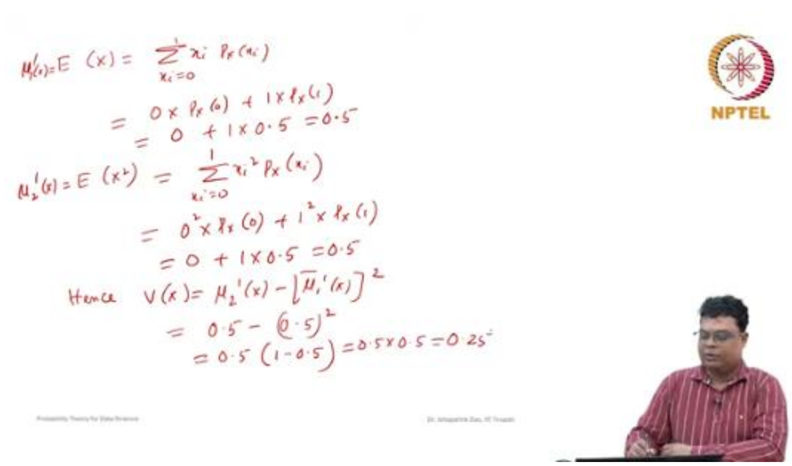
$E(X^2) = 0^2 * P(X = 0) + 1^2 * P(X = 1) = 0 * 0.5 + 1 * 0.5 = 0.5$

With these values, we can now compute the variance of X. The variance is obtained by subtracting the square of the mean (E(X)) from the expected value of X squared. Using this formula:

Var(X) = E(X²) - [E(X)]² = 0.5 - (0.5)² = 0.5 - 0.25 = 0.25

This is the variance of X.

Similarly, we can calculate the mean and variance of Y using the same approach. Now, let’s calculate the mean of Y. The expected value of Y, denoted as E(Y), is the sum of each possible value of Y multiplied by its corresponding probability. Y can take the values 0 and 1. For Y = 0, we multiply 0 by P(Y = 0).



For Y = 1, we multiply 1 by P(Y = 1). The result is 0 + 0.45, as P(Y = 1) = 0.45. Thus, the mean of Y is:

E(Y) = 0 * P(Y = 0) + 1 * P(Y = 1) = 0 + 0.45 = 0.45.

Next, we calculate the expected value of Y squared, denoted as E(Y²). This is the sum of the squares of each possible value of Y, multiplied by their respective probabilities:

E(Y²) = 0² * P(Y = 0) + 1² * P(Y = 1) = 0 * 0.55 + 1 * 0.45 = 0 + 0.45 = 0.45.

To find the variance of Y, we use the formula:

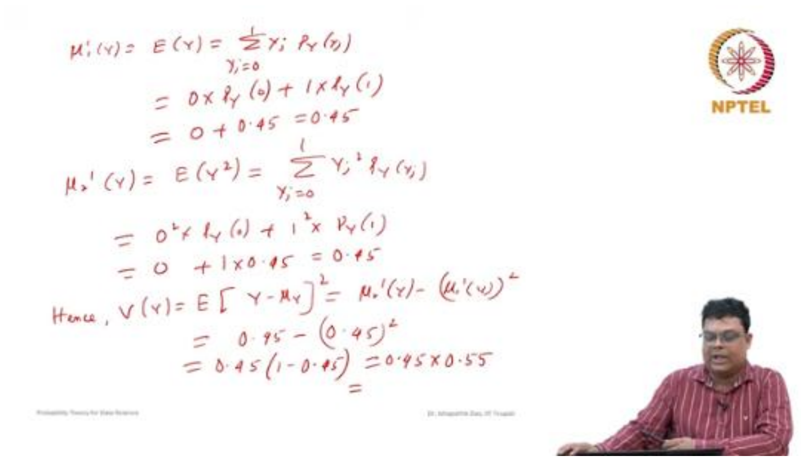
Var(Y) = E(Y²) - [E(Y)]².

Substituting the values, the variance becomes:

Var(Y) = 0.45 - (0.45)² = 0.45 - 0.2025 = 0.2475.

Thus, the variance of Y is 0.2475.

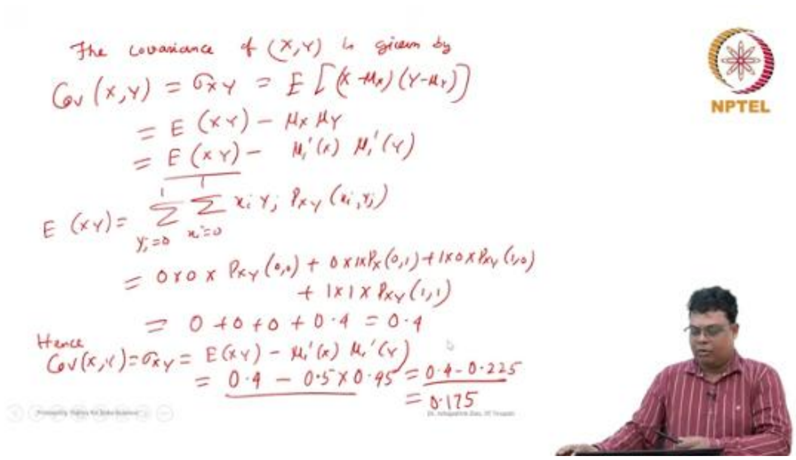
This is how you find the mean and variance of Y. I hope you understand how to find the mean and variance of X and Y.



The covariance of X and Y is denoted as $\sigma(XY)$ and is calculated using the formula: $\sigma(XY) = E(XY) - (\mu_X * \mu_Y)$, where $\mu_X = 0.5$ (mean of X) and $\mu_Y = 0.45$ (mean of Y). To calculate the expected value of the product of X and Y, $E(XY)$, we sum over all combinations of X and Y. When $X = 0$ and $Y = 0$, we multiply $0 * 0 * P(X = 0, Y = 0)$. When $X = 0$ and $Y = 1$, we multiply $0 * 1 * P(X = 0, Y = 1)$. When $X = 1$ and $Y = 0$, we multiply $1 * 0 * P(X = 1, Y = 0)$. When $X = 1$ and $Y = 1$, we multiply $1 * 1 * P(X = 1, Y = 1)$.

From the earlier calculation, $P(X = 1, Y = 1) = 0.4$, so the expected value of XY becomes: $E(XY) = 0 + 0 + 0 + (1 * 1 * 0.4) = 0.4$. Now, we substitute the values into the covariance formula: $\sigma(XY) = E(XY) - (\mu_X * \mu_Y) = 0.4 - (0.5 * 0.45) = 0.4 - 0.225 = 0.175$. Thus, the covariance of X and Y is 0.175.

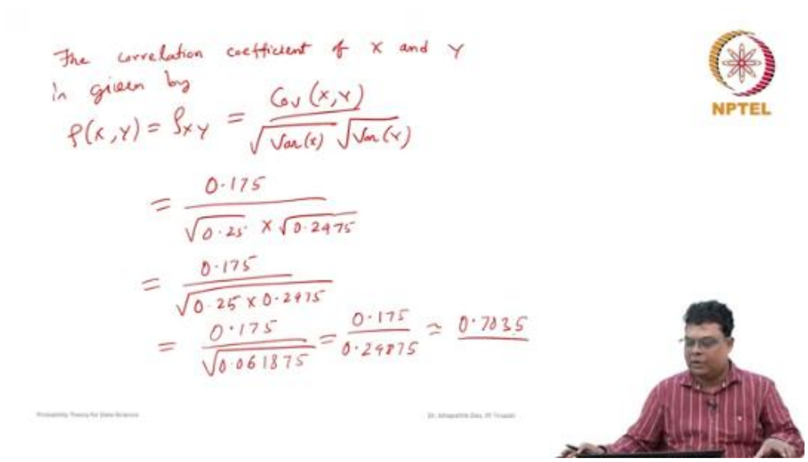
Next, we compute the correlation coefficient of X and Y, denoted as $\rho(XY)$, using the formula: $\rho(XY) = \sigma(XY) / (\sqrt{Var(X)} * \sqrt{Var(Y)})$. We know that $\sigma(XY) = 0.175$ (covariance of X and Y) and $Var(X) = 0.25$ (variance of X). We still need to compute $Var(Y)$ before calculating the correlation coefficient.



The variance of Y is calculated by multiplying 0.45 and 0.55, which gives 0.2475. Now, to find the correlation coefficient, we divide the covariance of X and Y (0.175) by the square root of the product of the variances of X and Y. This simplifies to $0.175 \div \sqrt{(0.25 \times 0.2475)}$.

After calculating, you will get approximately 0.703 when rounded to four decimal places. If the other computations are correct, the final value will be approximately 0.703.

If the other computations are correct, this value should come out like this. Please go through the process again. The remaining part is to compute the values and check the result. I have used a simple calculator to obtain this result, but please verify if there are any mistakes and correct them if necessary. This is the method for finding the correlation coefficient.



If the calculation is correct, the correlation coefficient is positive. A positive correlation coefficient means that X and Y are positively correlated. If the correlation coefficient is negative, then X and Y are negatively correlated. If the correlation coefficient is close to 0, it indicates that X and Y are uncorrelated. This concludes the concept of covariance, the correlation coefficient, and their interpretation.

I hope you followed along and understood it clearly.