PROBABILITY THEORY FOR DATA SCIENCE

Prof. Ishapathik Das

Department of Mathematics and Statistics

Indian Institute of Technology Tirupati

Week - 10

Lecture - 51

Numerical Examples on Joint Probability Density Functions

Let us discuss another numerical example for continuous random variables in the multivariate case. Here, we are given the joint probability density function. Let X, Y, and Z be a trivariate random variable with the joint probability density function given by:

$f(x, y, z) = k * e^{(ax + by + cz)}$,

where a, b, and c are constants, and k is a constant to be determined.

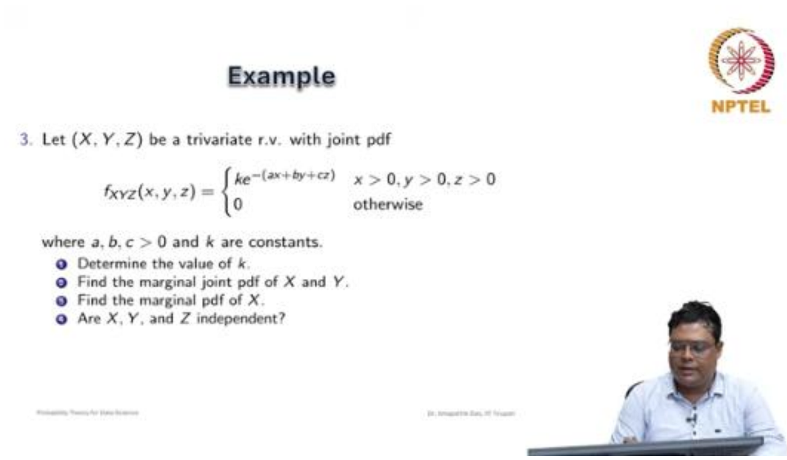
The questions are as follows:

1. Determine the value of k.
2. Find the marginal joint probability density function of X and Y.
3. Find the marginal probability density function of X.
4. Check if X, Y, and Z are independent.

Let us begin by writing down the joint probability density function:

$f(x, y, z) = k * e^{(ax + by + cz)}$.

This is the problem setup. We will first determine the value of k, then proceed to find the marginal joint probability density functions, and finally check the independence of X, Y, and Z. So, we have the joint probability density function:



$f(x, y, z) = k * e^{(-ax + by + cz)}$, where $0 < x, y, z < \infty$, and $0 < a, b, c < \infty$. Also, k is a constant.

a, b, c are given values that can be any value between 0 and ∞. We need to determine k, a constant.

To find the value of k, we must satisfy the conditions for a probability density function:

1. $f(x, y, z) \geq 0$, so $k \geq 0$.
2. $\iiint f(x, y, z)\, dx\, dy\, dz = 1$ over the range $0 < x, y, z < \infty$.

This means:
$\iiint [k * e^{(-ax + by + cz)}]\, dx\, dy\, dz = 1$.

Since $f(x, y, z) = 0$ for $x, y, z < 0$, we integrate from 0 to ∞:
$k * \int_0^\infty \int_0^\infty \int_0^\infty e^{(-ax + by + cz)}\, dx\, dy\, dz = 1$.

First, integrate with respect to x:
$\int_0^\infty e^{(-ax)}\, dx = 1/a$.
This leaves: $k * (1/a) * \int_0^\infty \int_0^\infty e^{(-by + cz)}\, dy\, dz = 1$.

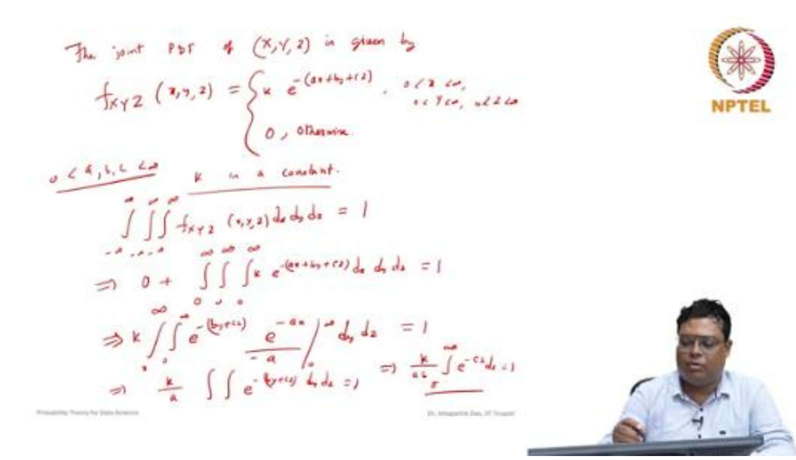
Next, integrate with respect to y:
$\int_0^\infty e^{(-by)}\, dy = 1/b$.
This leaves: $k * (1/a) * (1/b) * \int_0^\infty e^{(-cz)}\, dz = 1$.

Finally, integrate with respect to z:
$\int_0^\infty e^{(-cz)}\,dz = 1/c$.
This results in: $k * (1/a) * (1/b) * (1/c) = 1$.

Solving for k gives: $k = abc$.

The joint probability density function is:
$f(x, y, z) = (1/abc) * e^{(-ax + by + cz)}$.

Now, let's find the marginal joint probability density function of X and Y.



A marginal probability density function is obtained by integrating the joint probability density function over the remaining variable. In this case, we have three variables: X, Y, and Z. To find the marginal joint probability density function of X and Y, we need to integrate the joint probability density function f(x, y, z) with respect to Z.

The marginal joint probability density function of X and Y is given by: $f_{x\gamma}(x, y) = \int f(x, y, z)\,dz$.

We already know the joint probability density function f(x, y, z), which is:
$f(x, y, z) = (1/abc) * e^{(-ax + by + cz)}$.

To find $f_{x\gamma}(x, y)$, we integrate with respect to z:
$f_{x\gamma}(x, y) = \int_0^\infty (1/abc) * e^{(-ax + by + cz)}\,dz$.

Since e^(-ax + by) is independent of z, we can take it outside the integration:
f_{xy}(x, y) = (1/abc) * e^(-ax + by) * $\int_0$^∞ e^(-cz) dz.

Now, integrate e^(-cz) with respect to z from 0 to ∞:
$\int_0$^∞ e^(-cz) dz = 1/c.

So, we have:
f_{xy}(x, y) = (1/abc) * e^(-ax + by) * (1/c).

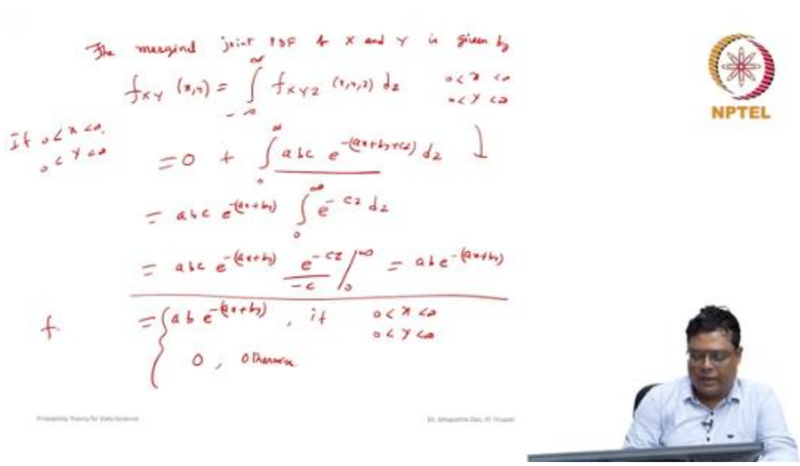
Simplifying, we get:
f_{xy}(x, y) = (ab/c) * e^(-ax + by).

This is the marginal joint probability density function of X and Y. Note that this is true when x > 0 and y > 0. Otherwise, the density function is 0.

So, we can express the marginal joint probability density function of X and Y as:
f_{xy}(x, y) = (ab/c) * e^(-ax + by), for x > 0 and y > 0,
and 0 otherwise.

This is the marginal joint probability density function of X and Y.

Next, let's find the marginal probability density function of X. The marginal probability density function of X is given by:



$f_x(x) = \iint f_{x\gamma}(x, y, z)\, dy\, dz$. We want to keep the variable X, so we integrate over the remaining variables Y and Z. The limits for X are from 0 to ∞. Thus, the marginal probability density function of X is:

$f_x(x) = \int_0^\infty \int_0^\infty f_{x\gamma}(x, y, z)\, dy\, dz$.

We already know that the joint probability density function $f_{x\gamma}(x, y, z)$ is:

$f_{x\gamma}(x, y, z) = (1/abc) * e^{(-ax + by + cz)}$.

Now, we need to integrate with respect to y and z while keeping x constant. This gives us:

$f_x(x) = (1/abc) * e^{(-ax)} \int_0^\infty \int_0^\infty e^{(by + cz)}\, dy\, dz$.

We can factor out $e^{(-ax)}$ because it is independent of y and z. Now, we integrate $e^{(by)}$ with respect to y and $e^{(cz)}$ with respect to z. The integration results in:

$\int_0^\infty e^{(by)}\, dy = 1/b$ and $\int_0^\infty e^{(cz)}\, dz = 1/c$.

Thus, we have:

$f_x(x) = (1/abc) * e^{(-ax)} * (1/b) * (1/c)$.

Simplifying, we get:

$f_x(x) = a * e^{(-ax)}$.

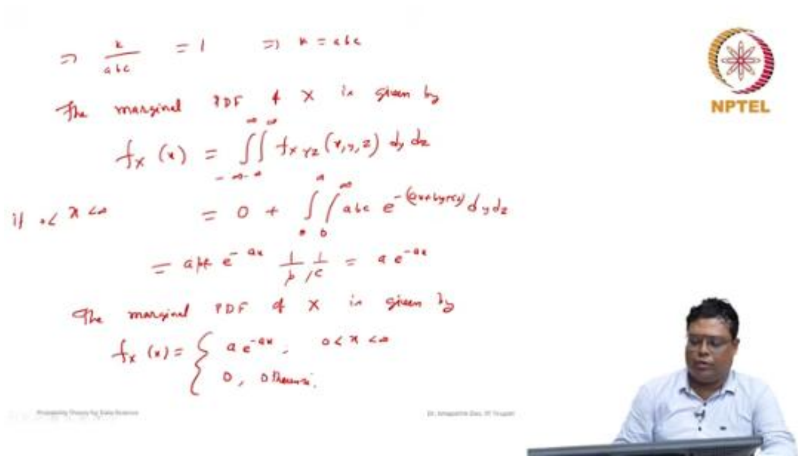
This is the marginal probability density function of X. Note that this is true when $x > 0$, and it is 0 otherwise.

So, the marginal probability density function of X is:

$f_x(x) = a * e^{(-ax)}$, for $x > 0$, and 0 otherwise.

Similarly, we can find the marginal probability density functions of Y and Z by symmetry. The marginal probability density function of Y is:

$f_\gamma(y) = b * e^{(-by)}$, for $y > 0$, and 0 otherwise.



The marginal probability density function of Z is:

$f_{\varphi}(z) = c * e^{\wedge}(-cz)$, for $z > 0$, and 0 otherwise.

Now, we can check whether X, Y, and Z are independent. To determine if X, Y, and Z are independent, we need to check if the joint probability density function can be expressed as the product of the marginal probability density functions.

The joint probability density function is:
$f_{x\gamma\varphi}(x, y, z) = abc * e^{\wedge}(-ax + by + cz)$, for $x > 0$, $y > 0$, $z > 0$, and 0 otherwise.

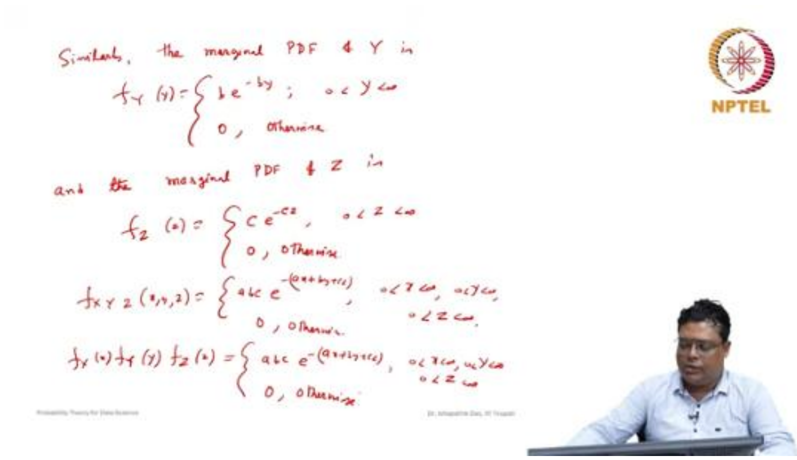
Next, we multiply the marginal probability density functions:
$f_{x}(x) * f_{\gamma}(y) * f_{\varphi}(z) = (a * e^{\wedge}(-ax)) * (b * e^{\wedge}(-by)) * (c * e^{\wedge}(-cz))$.

Simplifying this, we get:
$f_{x}(x) * f_{\gamma}(y) * f_{\varphi}(z) = abc * e^{\wedge}(-ax + by + cz)$.

This is exactly the same as the joint probability density function. Therefore, we can conclude that the joint probability density function can be written as the product of the marginal probability density functions.

Since $f_{x\gamma\varphi}(x, y, z) = f_{x}(x) * f_{\gamma}(y) * f_{\varphi}(z)$ for all $x, y, z \in \mathbb{R}^3$, we can say that X, Y, and Z are independent random variables.

This is the conclusion that X, Y, and Z are independent. Hence, this implies that X, Y, and Z are independent random variables.



This is the last question we covered. I hope you have understood from these numerical examples how multivariate random variables are defined as independent random variables, how to check for independence, and how to determine the properties of the joint probability density function. We used these properties to find the values of k and to compute the sum of the probabilities. In the discrete case, we found the sum of the probabilities, and in the continuous case, we computed the probability $P(Z \geq XY)$ using the joint probability density function.

I hope this explanation is clear. Next, we will discuss some special distribution functions in the multivariate random variable case. We have already covered some discrete random variables, such as Poisson, binomial, and Bernoulli distributions. In the continuous case, we discussed uniform, exponential, gamma, and normal distributions. Here, we will focus on one discrete random variable and one continuous random variable in the multivariate case.