PROBABILITY THEORY FOR DATA SCIENCE

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Week - 11

Lecture - 56

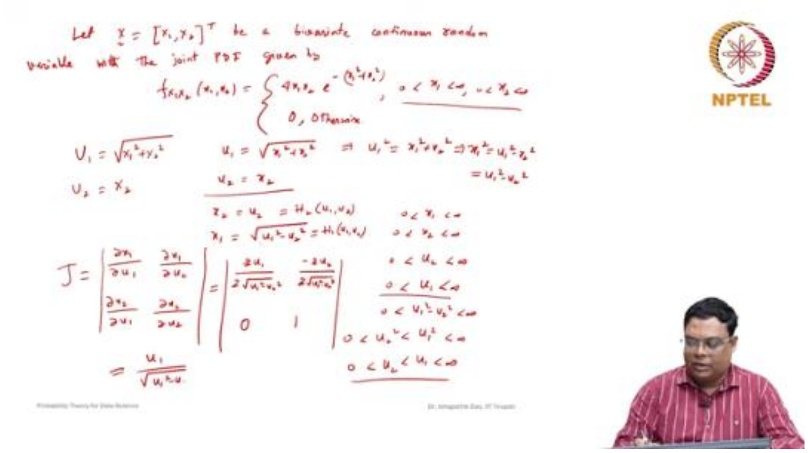
Examples of Transformation of Bivariate Random Variables

Let X be a bivariate continuous random variable with a joint probability density function where both variables, x_1 and x_2, are greater than 0, and the function is 0 otherwise. We define a transformation by introducing two new variables. The first new variable, u_1, is the distance from the origin in a two-dimensional space, calculated as $\sqrt{(x_1^2 + x_2^2)}$. The second new variable, u_2, is simply equal to x_2. From this transformation, we know that u_1 represents the distance between the point (x_1, x_2) and the origin, and u_2 is directly equal to x_2.

This allows us to express x_1 in terms of u_1 and u_2, where $x_1 = \sqrt{(u_1^2 - u_2^2)}$. The original density function is non-zero only when both x_1 and x_2 are greater than 0. This means that for the transformed variables, u_2 must also be greater than 0 and less than ∞. Additionally, since u_1 is the square root of the sum of the squares of x_1 and x_2, u_1 must also be greater than 0 and less than ∞. Moreover, since u_1 is the result of adding x_1^2 and x_2^2, we can conclude that u_1 must always be greater than u_2.

To calculate the Jacobian of this transformation, we need to look at how the original variables x_1 and x_2 change with respect to the new variables u_1 and u_2. This involves calculating the partial derivatives of x_1 and x_2 with respect to u_1 and u_2. The derivative of x_1 with respect to u_1 is proportional to $u_1 / \sqrt{(u_1^2 - u_2^2)}$. The derivative of x_1 with respect to u_2 is proportional to $-u_2 / \sqrt{(u_1^2 - u_2^2)}$. The derivative of x_2 with respect to u_1 is 0, while the derivative of x_2 with respect to u_2 is 1.

The Jacobian is the determinant of a matrix of these partial derivatives. After performing the necessary calculations, we find that the Jacobian is equal to $u_1 / \sqrt{(u_1^2 - u_2^2)}$. Since u_1 is always positive in the range of the transformed variables, the Jacobian is always positive. Therefore, the Jacobian is $u_1 / \sqrt{(u_1^2 - u_2^2)}$. We can find the joint probability density function using the theorem.



The joint probability density function of Y is given by the vector U, which consists of U_1 and U_2. This is expressed as |J| * f, where J is the Jacobian and f represents the joint probability density function of the original variables. This is f(x). We can write it as $f(x_1, x_2)$, or for simplified notation, we use f(x). This is the inverse of y.

The Jacobian, which we know, is equal to the ratio of u_1 to the square root of the difference between the square of u_1 and the square of u_2. Therefore, the Jacobian is $u_1 / \sqrt{(u_1^2 - u_2^2)}$. Next, we need to determine the values of x_1 and x_2. When transforming in this region, we have $4 * x_1 * x_2$. To clarify, x_1 is the square root of the difference between u_1^2 and u_2^2.

Therefore, $x_1 = \sqrt{(u_1^2 - u_2^2)}$, and $x_2 = u_2$. Then, we apply the inverse transformation for Y. The expression becomes $4 * x_1 * x_2$, which is the product of $\sqrt{(u_1^2 - u_2^2)}$ and u_2. Then, we have the exponential term, $e^{(-x_1^2 - x_2^2)}$. Since the sum of x_1^2 and x_2^2 is equivalent to u_1^2, this simplifies to $e^{(-u_1^2)}$.

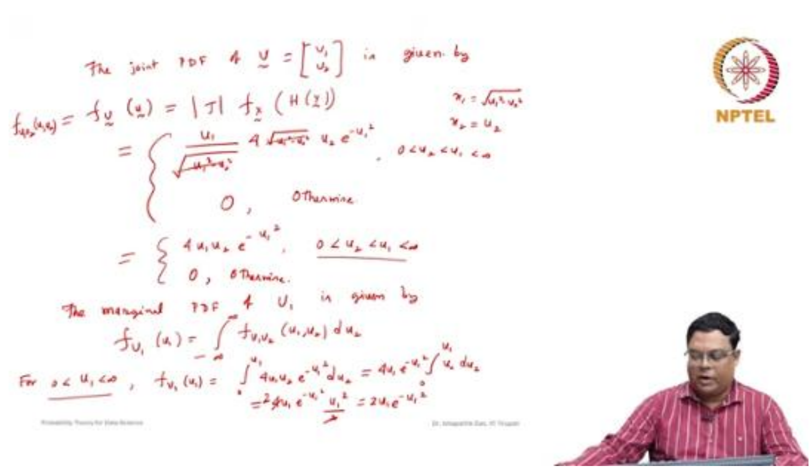
This value holds true when $0 < u_2 < u_1 < \infty$. For any other values, the result will be 0. Now, if we simplify the expression, we can cancel out terms. We will then obtain $4 * u_1 * u_2 * e^{(-u_1^2)}$, valid when $0 < u_2 < u_1 < \infty$. For any other values, the result will be 0.

It is important to note that u_1 and u_2 are dependent on each other in this range. Therefore, they are dependent random variables. You can verify this by examining their marginal distributions. We will now find the marginal probability density function of U_1, as the question asks for it. Since we have already found the joint probability density function, the marginal probability density function of U_1 is determined by integrating the joint density function with respect to U_2. The marginal probability density function of U_1, denoted as $f\_U_1(u_1)$, is equal to the integral of $u_1 * u_2$ with respect to u_2, over the range from 0 to u_1.

This function is non-zero when the condition is satisfied. Now, we need to determine the range for U_2. For any given value of U_1, U_2 ranges from 0 to U_1. Additionally, U_1 can take any value between 0 and ∞, as U_1 cannot be negative. The probability density function of U_1 is given by the integral over the range of U_2, which is from 0 to U_1. For any other case, the value will be 0. Therefore, the function becomes $4 * u_1 * u_2 * e^{(-u_1^2)}$, integrated with respect to u_2.

The expression $4 * u_1 * e^{(-u_1^2)}$ will be taken outside the integral. Then, the integral becomes from 0 to u_1, with respect to u_2, resulting in $u_2\, du_2$. The integral of u_2 with respect to u_2 is $(u_2^2) / 2$. Using the limits, this gives $(u_1^2) / 2$. Therefore, the expression becomes $4 * u_1 * e^{(-u_1^2)} * (u_1^2 / 2)$.

The 2s cancel out, resulting in $2 * u_1 * e^{(-u_1^2)}$. This holds for values of u_1 between 0 and ∞. For any other value, the result is 0. Finally, we can write the marginal probability density function. This is the marginal density function of U_1.



The marginal probability density function of U_1 is given by $2 * U_1 * e^{(-U_1^2)}$. This is valid for U_1 values ranging from 0 to ∞. The expression is derived from the fact that $U_1 = \sqrt{(x_1^2 + x_2^2)}$. The joint probability density function is expressed as $2 * U_1^3 * e^{(-U_1^2)}$. This is valid when $U_1 > 0$ and $U_1 < \infty$. Outside this range, the probability density is 0. For the function to be a valid probability density, the total probability must equal 1. This is checked by integrating the function from 0 to ∞. If the result of the integration is 1, then the function is valid. If not, the function would need to be adjusted to ensure the total probability equals 1. This is the density function.

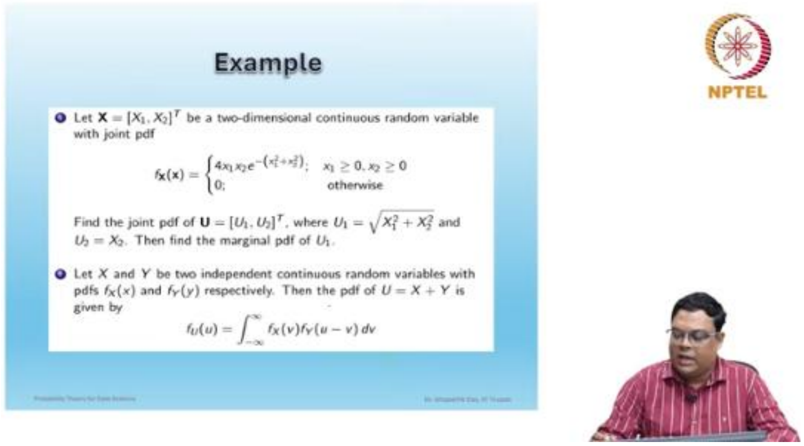
I just wanted to verify whether the calculations are correct. To do this, we need to check if the total probability integrates to 1. Specifically, the integral of the probability density function from 0 to ∞ should be equal to 1. The function we are working with is non-zero between 0 and ∞.

The integral involves the expression where U_1 is raised to the power of 3 and multiplied by e^(-U_1^2). To solve this integral, we can apply a transformation using the gamma function, which is a well-known mathematical function used to evaluate integrals of this type. We will apply a transformation where we set U_1^2 = Z. In this case, we have the relationship 2 * U_1 * dU_1 = dZ. The limits of Z will range from 0 to ∞. Now, we can express the probability density function in terms of Z.

The function becomes Z * e^(-Z), and we substitute 2 * U_1 * dU_1 with dZ. The equation simplifies to Z * e^(-Z) * dZ. This integral has the form of an expression from 0 to ∞, where Z is raised to the power of 1, which is derived from subtracting 1 from 2. This expression is then multiplied by the exponential function e^(-Z). This type of integral is a standard form and is known to be equal to the gamma function evaluated at 2.

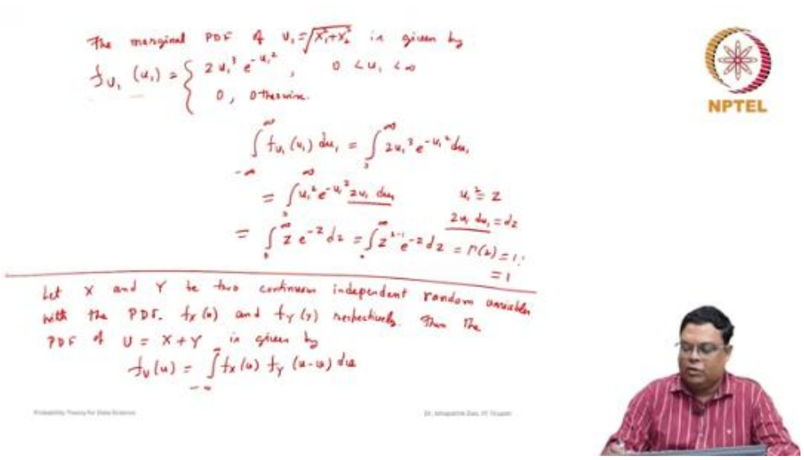
The gamma function for the value 2 is equivalent to 1 factorial, which is simply 1. As a result, the value of this integral is 1, which confirms that the function is a valid probability density function. So, I was trying to verify the calculations and I realized there was a mistake in the computation. After correcting that, I arrived at the final value. This shows that by using the transformation concept in the theorem, we were able to determine the marginal probability density function of U_1.

Additionally, we also found the joint probability density function of U_1 and U_2. Next, let's discuss another example. Here, we are given that X and Y are two independent continuous random variables with their respective probability density functions, denoted as $f_X(x)$ and $f_Y(y)$. Since X and Y are independent, we can determine their joint probability density function. The probability density function of U, where U = X + Y, can be found using the transformation method.



The formula for the probability density function of U is given. Let's write down this formula and then work through how it is derived. Let X and Y be two independent continuous random variables, with their respective probability density functions, denoted as $f_x(x)$ and $f_Y(y)$. The probability density function of U, where U = X + Y, is given by the following expression.

You can observe that the formula is written as the integral of the product of $f_x(b)$ and $f_Y(u - b)$, integrated over b. Mathematically, this operation is called the convolution of two functions. In this case, we are convolving the two functions, f_x and f_Y, to find the probability density function for the sum of two independent random variables. Let us now examine how this works in detail. Since X and Y are independent random variables, the joint probability density function of the vector (X, Y) is given by $f_{xY}(x, y)$.



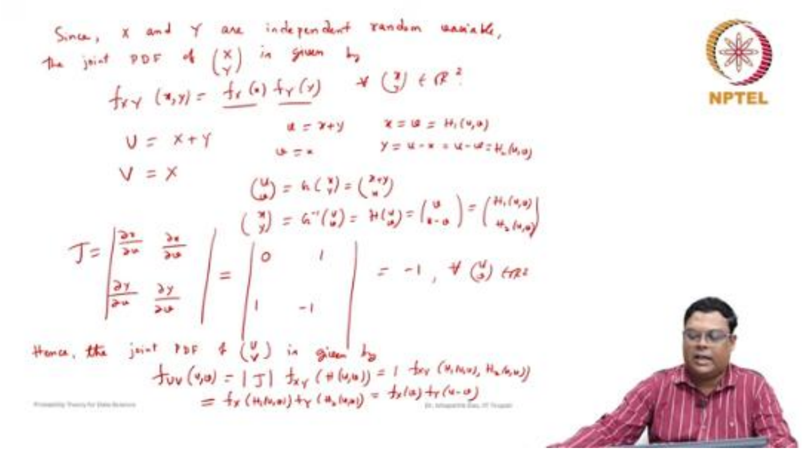
Since X and Y are independent, this is equal to the product of their marginal probability density functions, meaning $f_{XY}(x, y) = f_X(x) * f_Y(y)$, where $(x, y) \in \mathbb{R}^2$. Now, we will consider the transformation where U = X + Y. Since we do not have a pre-existing formula for this transformation, Now, we will consider the transformation where U = X + Y. Since we do not have a pre-existing formula for this transformation, For this transformation, we are considering a change from $\mathbb{R}^2$ to $\mathbb{R}$, but we need a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ in order to use the previous theorem.

To achieve this, we introduce another function. Let's define a new variable, V, which is simply equal to X. So, we now have two new variables: U = X + Y, and V = X. From these equations, we can solve for the inverse relationships: X = V, and Y = U - V. Thus, the transformation is represented by the pair of variables U and V, where U = X + Y and V = X.

The inverse transformation is then given by X = V and Y = U - V. Therefore, the transformation from (X, Y) to (U, V) can be expressed as the function g(X, Y) = (X + Y, X), and the inverse transformation, $g^{-1}(U, V) = (V, U - V)$, gives us the corresponding values for X and Y. Since the inverse transformation exists, we now need to check the Jacobian. The Jacobian is defined as the determinant of the matrix formed by the partial derivatives of the transformed variables with respect to the original variables, and we denote this determinant by J. For the transformation from U and V to X and Y, we calculate the following partial derivatives: the partial derivative of X with respect to U, the partial derivative of X with respect to V, the partial derivative of Y with respect to U, and the partial derivative of Y with respect to V.

This gives us a matrix of partial derivatives, and the determinant of this matrix is the Jacobian. The values are as follows: Since X = V, the partial derivative of X with respect to U is 0, and the partial derivative of X with respect to V is 1. For Y, which is equal to U - V, the partial derivative of Y with respect to U is 1, and the partial derivative of Y with respect to V is -1. Therefore, the determinant of the Jacobian matrix is -1 for all values of U and V in $\mathbb{R}^2$. Thus, the determinant is -1.

The joint probability density function of the vector (U, V) is given by the absolute value of the Jacobian determinant multiplied by the joint probability density function of X and Y, evaluated at the inverse transformation. Since the Jacobian determinant is 1, the joint probability density function simplifies to the product of the individual marginal probability density functions of X and Y. Because X and Y are independent, the joint probability density function is the product of the densities of X and Y, evaluated at the transformed variables. Specifically, for $h_1(U, V) = V$ and $h_2(U, V) = U - V$, the joint probability density function becomes the product of $f_x(V)$ and $f_Y(U - V)$. This is the joint probability density function for the transformed variables (U, V).



Hence, the joint probability density function of u and v is given by f(u, v). This is equal to what we found earlier, which is f(x, v) multiplied by f(y, u - v). Now, when we know the joint probability density function, we can find the marginal probability density function of u, which is equal to x + y. The marginal probability density function of u is given by the integral of the joint probability density function over the variable v. Specifically, it is defined as the integral from -∞ to ∞ of f(u, v) with respect to v.

This is equal to the integral from -∞ to ∞ of f(x, v) and f(y, u - v) with respect to v. This is the theorem we used. You can observe that when x and y are two independent continuous random variables with probability density functions f(x) and f(y) respectively, the probability density function of u, which is the sum of x and y, is given by the formula: f(u) = the integral from -∞ to ∞ of f(x, v) and f(y, u - v) with respect to v. I hope this explanation clarifies the concept.

If it's still unclear, you can review it again. Additionally, we will discuss numerical examples based on this theorem, where we consider specific distributions for X and Y and determine the resulting distribution for U.