Integral Equations, Calculus of Variations and their Applications
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Lecture 10
Solutions of Integral Equations by Successive Approximations: Resolvent Kernel

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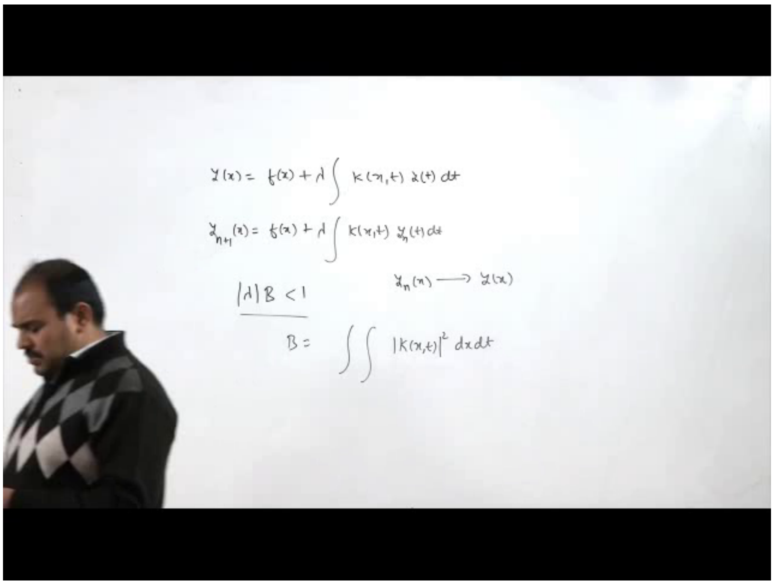

Ok! So welcome to welcome my friends and here in previous lecture if (you) we have already discussed the successive approximation method to solve Fredholm integral equation of second kind and we have seen this thing that we have y(x) is equal to see f(x) plus lambda times k(x, t) (f of) y(t) dt not f(t)it is y(t) dt d(t) here and we have seen that by approximation here we have created a approximation scheme like this y n plus 1 (x) equal to f(x) plus lambda times k(x,t) and y n (t) dt and we have shown that under certain condition here we assume that this kernel k(x,t) and this function f(x) both are square integral with function.

And in previous class we have seen the convergence criteria here we have seen that f modulus of lambda b is less than 1 then this series is going to be this y n(x) will converge to y(x) absolutely and uniformly so this convergence is uniform here. So here b is basically what b is basically modulus of k(x, t) whole square dx dt. So here this is the b rotation and if modulus of lambda b is less than 1 this convergence is uniform.

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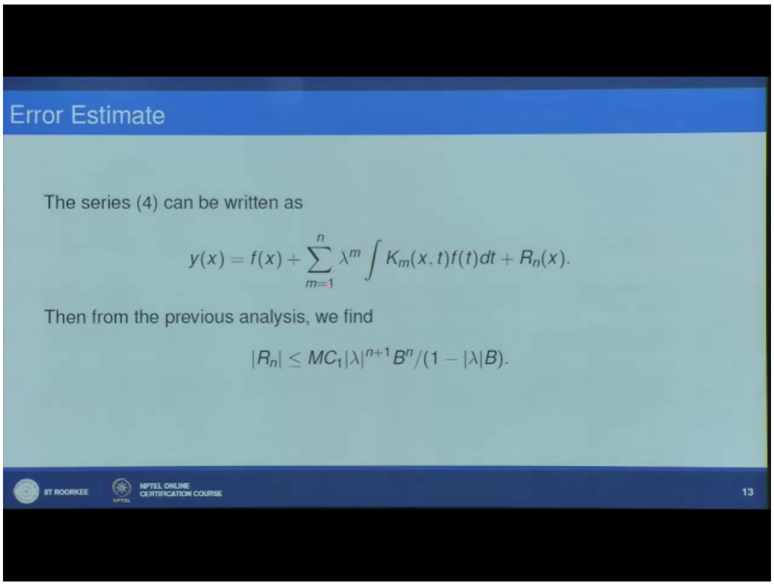

Now we want to show that if we approximate the solution of this by (n plus 1)th approximation here then of course there should be a truncation error we try to see that what should be the truncation here. So here we say that if this series can be written as like this. So here y(x) equal to f(x) m equal to 1 to n lambda m k m(x, t) dt. So this is your general term here and on an x I am writing here after (n plus 1) th term.

So here we can say that this term if we truncate a word y(x) by this n th term we can say that there is a truncation error and that truncation error is bounded by this quantity, modulus of R n m c 1 modulus of lambda n plus 1 b to power n divided by this.

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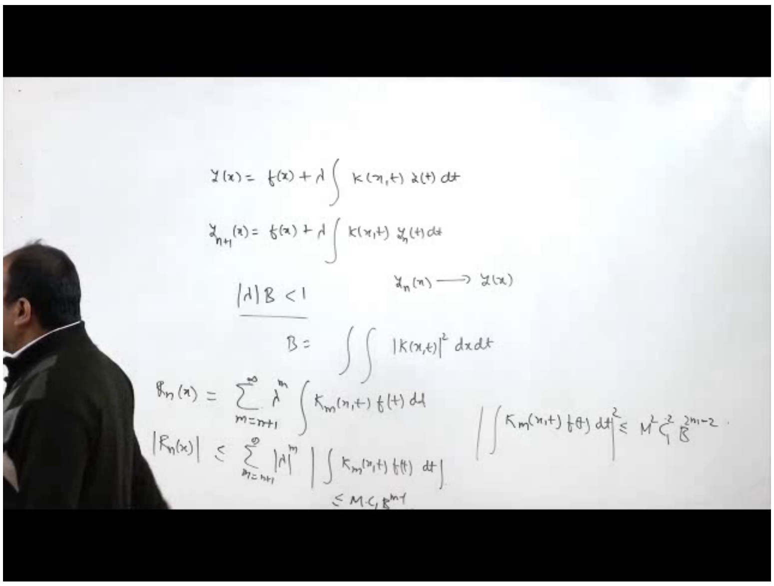

So this we can see it like this that your r n is going to be what r n is basically our summation your m equal to 1 to n lambda to power m and it is inside your k m (x, t) f(t) dt and oh sorry here limit will start from n plus 1. So it is n plus 1 to infinity.

So the modulus of R n (x)you can find out by so this is going to be less than or equal to now here it is summation m equal to n plus 1 to infinity modulus of lambda to power m here and modulus of this k m(x , t) f(t) dt dt here this thing, right?

So this we can calculate this quantity we can calculate using previous and this is we can see that this is this will reduce to so here let me write it what is modulus value k m(x,t) f(t) dt square of this , so square of this is given as capital M square and this is what c 1 square and it is going to be b to power 2 m minus 2.

So you can say that this is going to be bounded by lambda to power m this is you can say that that is bounded by say m c 1 and p to power m minus 1, right? And yeah, so we can say that there is something missing here lambda to power m is also there, is that ok?

Then you can write it here this is what this is the geometric series here whose first term is started at m plus 1. So we can simply write the summation here as first term divided by 1 minus common ration.

Common ration is basically modulus of lambda and b, right? So you can , you can simplify and you can see that the modulus of R n is given by bounded by m c 1 modulus of lambda to power n plus 1 b n that is your (n plus 1) th term divided by 1 minus common ratio here that is modulus of lambda b, ok.

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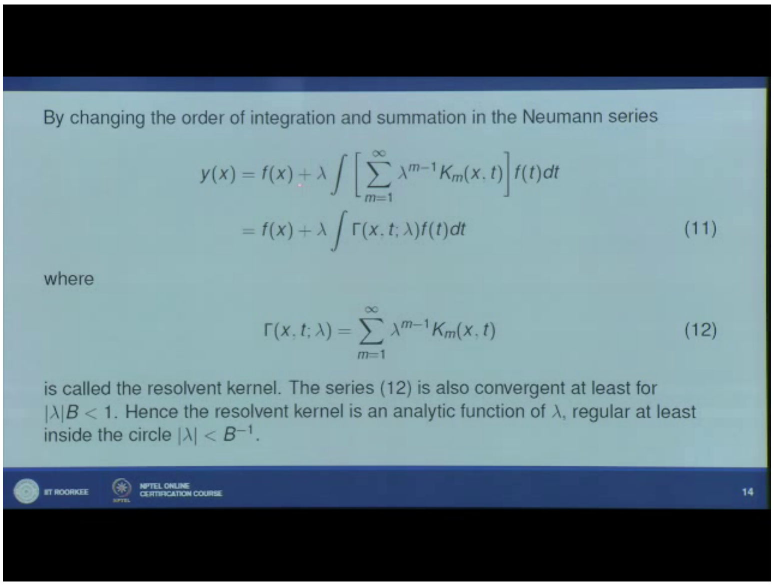

Now here we since convergence is uniform here so I can write that this integral and the summation can be interchangeable. So if you remember our solution is given by this Neumann Series y(x) equal to f(x) plus lambda here we are interchanging the summation and integral sign we are taking summation sign inside and we can write it like this. So here this notation we are calling this as gamma (x,t) lambda f(t) dt where gamma (x, t) lambda is given by defined by this 0 1 to infinity lambda to power m minus 1 k m(x,t) here.

I am taking lambda here then that is why this lambda to power m is (trunca) appearing here. So we are saying that we are calling this gamma (x, t) lambda as Resolvent Kernel. Again we we can prove that this gamma(x, t) lambda is given by this power series in terms of this lambda here and we can say that this says now I am taking function of a lambda here . And this convergence we have already seen that it is convergent in the for modulus of lambda b is less than 1.

So this is also an analytic function of lambda whose radius of convergence is a given by this formula modulus of lambda is less than b inverse, b is defined here, right?

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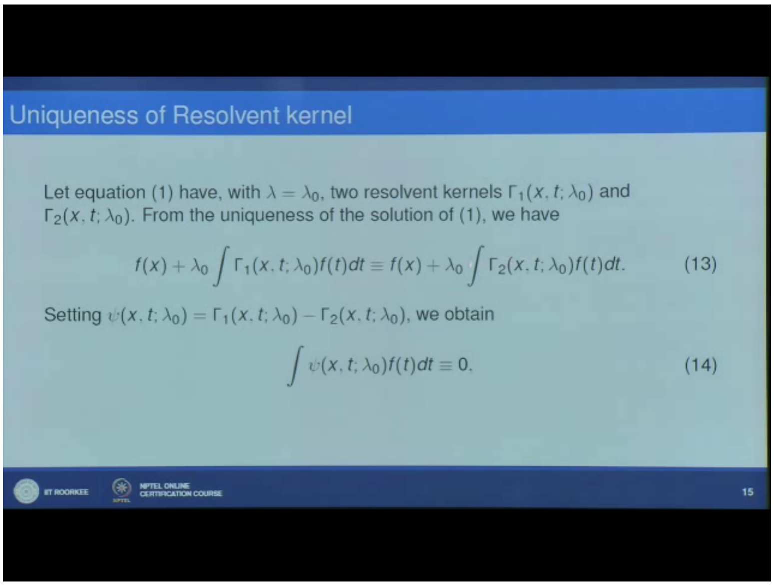

So here we can say that if solution is unique we can say that not only solution is unique even this resolvent kernel is also unique. So for that let us say that we have two resolvent kernels corresponding to the same solution. And call it gamma 1 and gamma 2 and we try to show that this gamma 1 is equal to gamma 2 for that we just equate y(x) here which is obtained by gamma 1 and y(x) which is obtained by this gamma 2.

And since these are equal we can say that these are equal. So f(x) f(x) will be cancelled out and you can write it that lambda not we are assuming that it is some non zero constant and we are writing this as denoting psi(x, t) lambda as difference of these two and we can write it that psi(x, t) lambda 0 f(t) dt. Here I am using a particular value lambda 0 for this lambda. And this is true for all function f(t) here.

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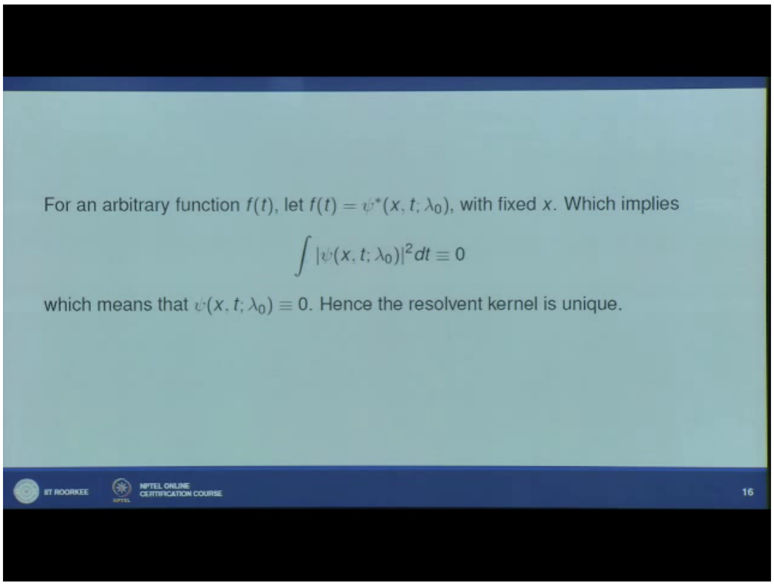

So here we can choose this f(t) as complex conjugate of this psi(x, t) lambda. So since this is true for every f(t) we can say that even it should be true for its f(t) which is given as complex conjugate of this side stamp. If we use it then we have this last equation given as 14 is reduced this integral of psi(x, t) lambda 0 square dt equal to 0.

So here this is not equality this is equivalent is not correct word it is equality. So here we can say that this is possible only when, when this integrant is unequal equal to 0. So if it is a integrant is unequal to 0 means your this gamma 1 is unequal to gamma 2. So it means that under the condition that modulus of lambda b is less than 1 your solution is unique your resolvent kernel is unique where resolvent kernel is defined by equation number 12, is it ok?

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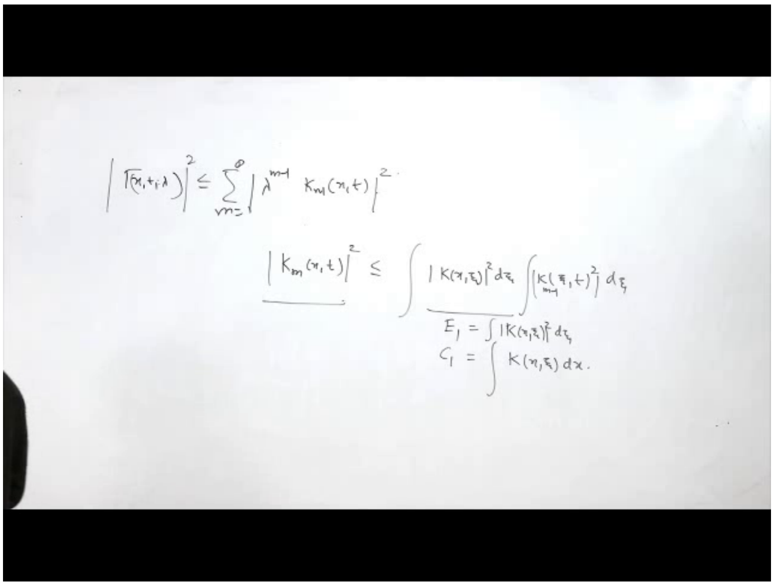

Now they are certain mode properties corresponding to your resolvent kernel and the solution we just writing here and we say that for every l to kernel k(x, t) they correspond a unique resolvent kernel gamma (x, t) lambda this uniqueness we have already proved. And which is gamma (x, t) lambda is defined by this m equal to 1 into infinity lambda to power m minus 1 k m(x, t).

And we can prove that this is absolutely and uniformly convergent fpr all values of x and t in the circular modulus of lambda is less than b inverse and further more we have just proved that if f(x) is also an l 2 function then the unique l 2 solution of the certain integral equation 1 valid in the circle modulus of lambda less than b inverse that we have proved in the previous lecture, ok.

So this proves that this gamma(x, t) lambda is convergent in this reason I can give you small hint that is all I can simply write it here as this. So here if you look at gamma(x, t) lambda it is denoted as summation m equal to 1 to infinity lambda to power m minus 1 k m(x, t), right? So here we take modulus here and take the square here so this less than or equal to modulus here and square here.

And then you can find out k m(x, t) and square of this again using the Caushy's unequality so which is it is like k(x i) square dx i and here we have k m minus 1 x i t square d x i and using the bond of this. So that you can do and you can say that now please remember here it is different from c 1. This is not a c 1 this is infinite quantity but it is different, c 1 is defined by what ? c 1 is defined as k (x, x i) t x, so this is your c 1 so we can call this as any quantity let us say this is e 1 , e 1 I am denoting as modulus of k (x x i) square d x i as some e 1.

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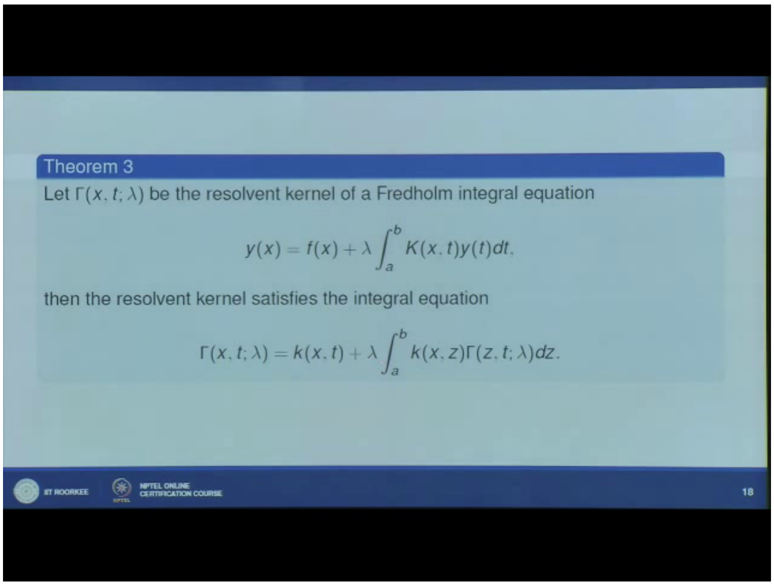

So using these notations you can prove that this series is also absolutely and uniformly convergent for the same bound that is modulus of lambda less than b inverse, ok. I am not giving any proof of this, ok. Now there is one more theorem which is important to know that if gamma (x, t) is the resolvent kernel of this Fredholm integral equation then this gamma(x, t) lambda is also satisfying the (kern) this integral equation given in terms of gamma (x, t) lambda.

And define as gamma (x, t) lambda is k(x, t) plus lambda a to b k(x , z) gamma x t lambda b z. There is one and this is done as Fredholm Fredholm identities and there is one more Fredholm identitites given in terms of gamma(x, t) lambda. So that we write when it is required, ok. So this is the integral equation satisfied by the resolvent kernel.

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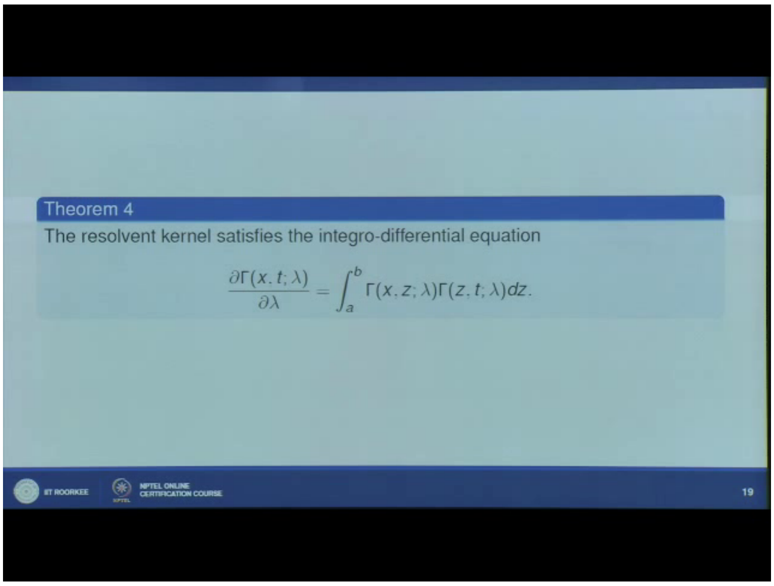

Now there is one more theorem which I am writing just for sake of completeness that resolvent kernel satisfy the integro differential equation given by this, ok. Why we are integro differential equation there is a differentiation of gamma(x, t) lambda with respect to lambda is given in terms of integral equation. So that is why it is integro differential equation. So it is just for completeness.

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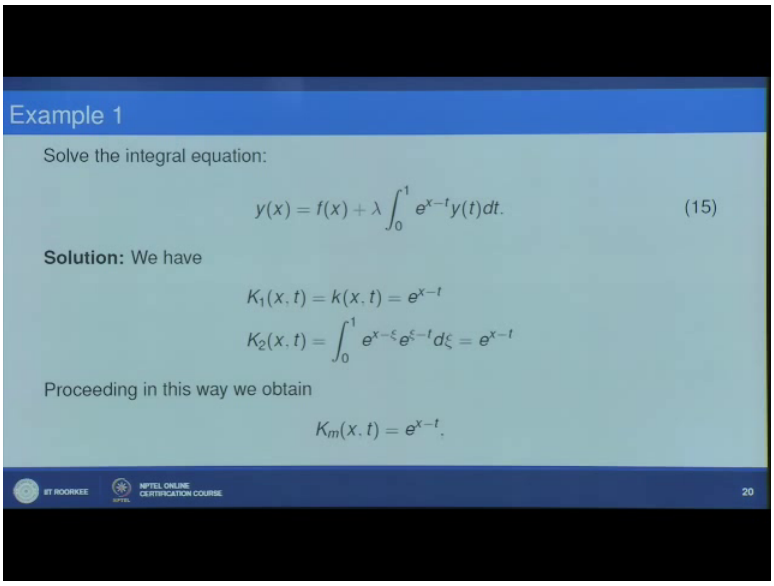

Now we try to solve some problem with the help of the theory developed earlier by the way here I just want to say one word for the theory part that before doing any problem we may start attacking in the problem itself but before attacking to any problem we must know that they exist a solution and not only a solution we should not know that the iterative procedure we are considering that should converge somewhere not only converge it will converge to a unique limit.

That is why we have discussed all the theory part. Now let us see towards our theory and try to discuss these examples. So here we have y(x) equal to f(x) plus lambda 0 to 1 e to power x minus t y(t) dt. So here your kernel is given in terms of e to power x minus t. So k 1is defined as this you can define k 2(x, t) as like this. So I hope you will remember the formula here.

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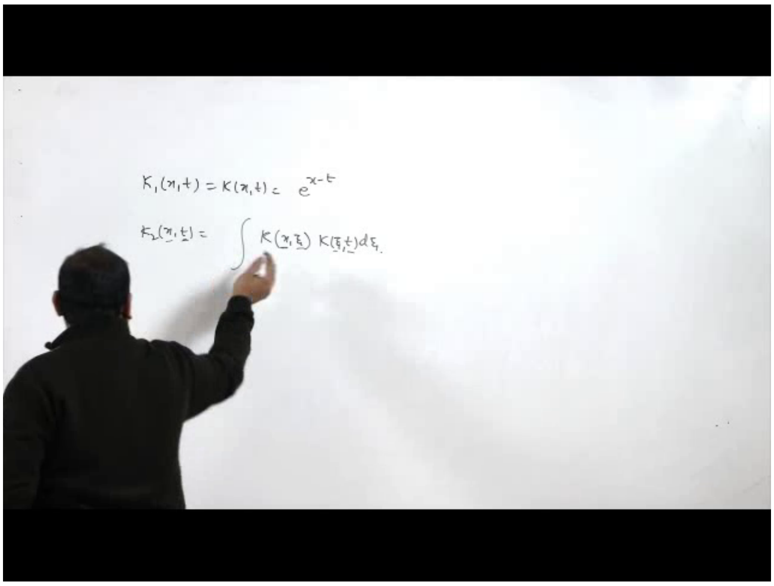

So what is your so k 1(x, t) is same as k(x, t), so k(x, t) is given as e to the power (x minus t) and k 2(x, t) is basically what k 2(x, t) is basically integration of k(x, t) k(x i) k(x i, t) v (x i). Please remember here this starting point x is a starting point here end point here is a end point here. And this n is the integral variable. So you can use this formula and you can find out k 2 (x, t) and if you simplify it is given by e to power x minus t.

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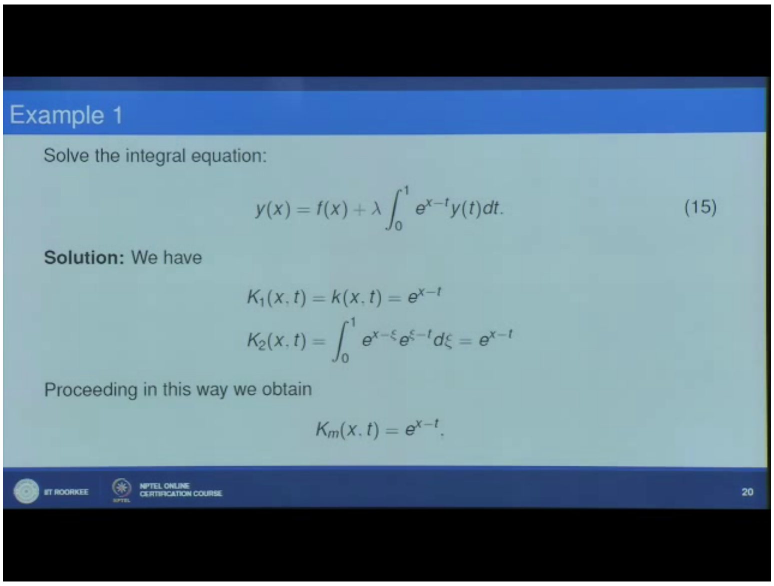

Here a is 0 and b is 1 and if we proceed in a similar way we can say that k m(x, t) is given by e to power x minus t. So once we have (())(14:44) iterations for this kernel then we can write we can find out resolvent kernel.

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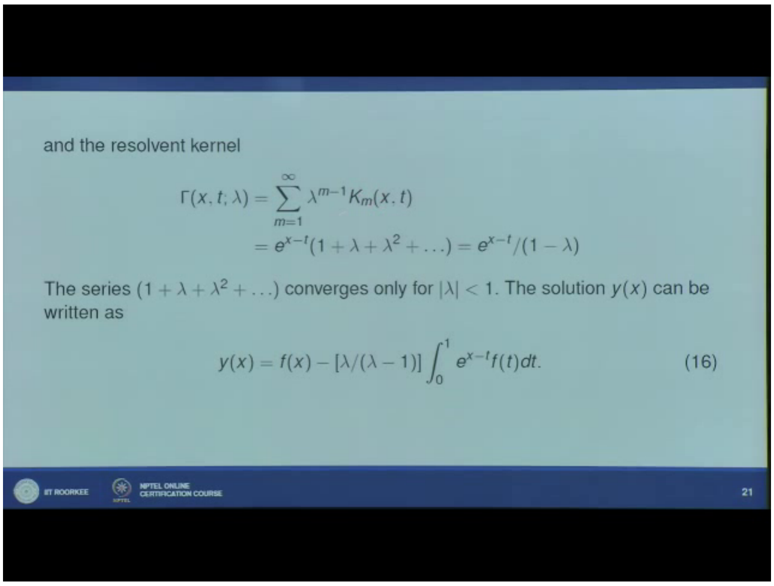

Resolvent kernel is defined as lambda to power m minus 1k m(x, t) we can we have already find out k m(x, t) as e to the power x minus t.

And if you look at this is geometric series with common issue lambda so here this will converge provided that modulus of lambda is less than 1 and we can say that gamma (x, t) lambda is given by e to power x minus t divided by 1 minus lambda and solution is given by y(x) f(x) minus lambda divided by lambda minus 1 0 to 1 e to power x minus t f(t) dt.

So once you know your n th iteration for kernel you can define resolving kernel and you can write down your solution in terms of resolvent kernel and in this case it is coming out to be given by equation number 16, is that ok.

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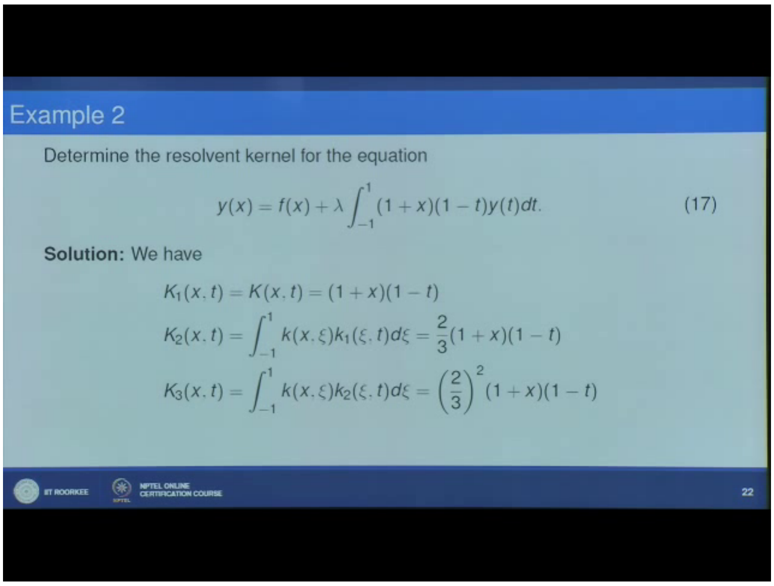

Now let us move to one more non trivial I can say this is quite trivial because here k 1(x, t) is coming out to be e to the power x minus t and all these k 1 to k n is coming out to be e to power x minus t.

Let us take some more difficult problem here and we say that here as he does the second problem here your k(x, t) is given by 1 plus x into 1 minus t again it is not very difficult but any way it is not too easy also. So here a is equal to minus 1 and b is equal to 1. And again we want to write this solution here. So here k 1(x, t) is same as k(x, t) so it is (1 plus x) (1 minus t) 1 minus t here.

And you can find out k 2(x, t) is what a to b k(x i) k 1(x i) t b (x i). So when you solve this it is simple integration and you can say that it is 2 by 3 (1 plus x) (1 minus t). And if you keep on doing this you can see that this is a kind of pattern we are getting that k 3 is basically 2 by 3 square (1 plus x) (1 minus t). So if you look at here 3 corresponding to the second power of this quantity 2 by 3.

So if you define you will get k m(x, t) is defined as (2 by 3) m minus 1 (1 plus x) (1 minus t) and if you find out the resolvent kernel, resolvent kernel is given by this and you can put the value of k m (x, t) here and you can say that it is simple (1 plus x) (1 minus t) and this summation. Now this is again a geometric series in terms of 2 lambda by 3. So here we can say that this will converge and summation is given by 1 upon 1 minus common ratio that is 3 by 3 minus 2 lambda.

And this will converge only when this quantity is less than 1or I can say that modulus of lambda 3 by 2 in this particular case your gamma (x, t) lambda is given by this quantity. So gamma (x, t) lambda is given by this provided modulus of lambda less than 3 by 2. So once you know your gamma (x, t) lambda you can find out your solution. And ok, so you can write all this solution here, ok.

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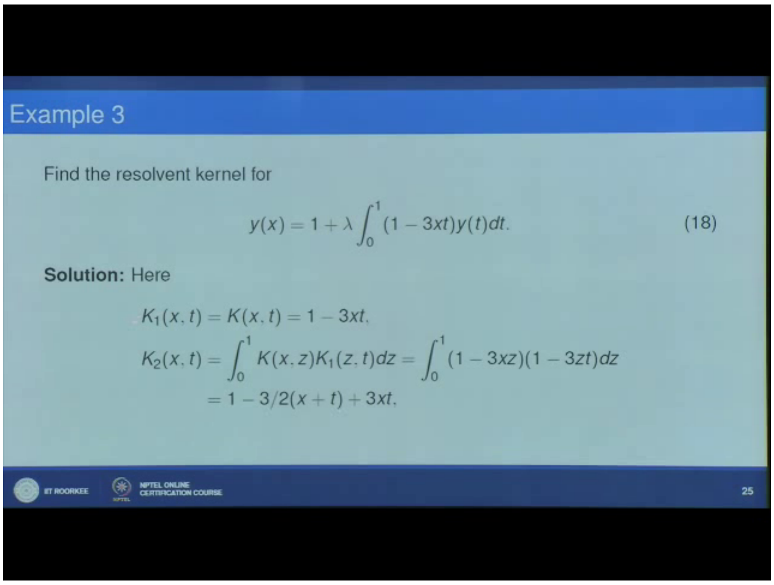

Now let us discuss example number 3 here y(x) equal to 1 plus lambda 0 to 1 minus 3x t y(t) dt. If you remember we have discussed this kind of problem for f(x) for general f(x) there we have assumed 1 i is equal to f(x) if you look at this is the example with the separable kernel here k(x, t) is given by 1 minus 3x t. And we have already discussed the solution procedure in the lecture of a separable kernel. We have already solved for.

Now here we are solving for a particular case when f(x) is equal to 1 here, so again as we want to find out say resolvent kernel of this so for that k 1(x, t) is same as k(x, t) it is given by 1minus 3x t. Similarly you can find out k 2(x, t) and a it is given by this formula k(x,z) k 1(x, t) dz you don't worry about these k(x z) k(x i) whatever, you simply remember only this thing that the first variable here will be here, and the last the second variable is will be here.Here it is your integral variable ok. So using this formula you can write down k2(x, t) like this.

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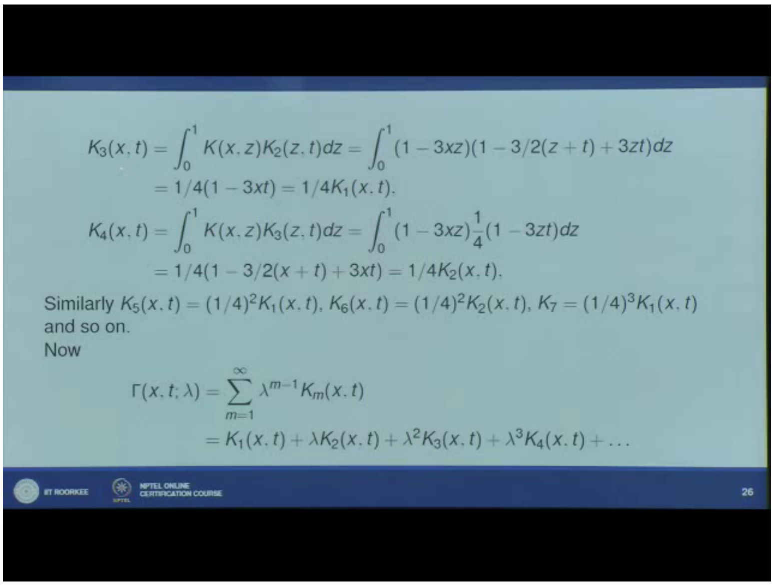

Similarly you can calculate k 3(x, t) k 4(x, t) and so on. And when you calculate k 3(x, t) it is coming out to be 1 by 4 k 1(x, t). Similarly k 4(x, t) is given in terms of k 2(x, t) so it is 1 by 4 k 2(x, t). If you keep on doing this then you can see that all odd number of iterations k 3 , k 5 , k 7 all are written in terms of k 1(x, t).

Simialry your even say k 2, k 4, k 6 all are written in terms of even that is k 2(x, t). So I can write this write resolvent kernel as gamma(x,t) lambda as lambda power m minus 1 k m(x,t) as this series k 1(x, t) plus lambda k 2(x, t) and so on and there we can put the value of k 3 , k 4 and s o on in terms of lambda 1 and lambda 2 . So I can write it like this.

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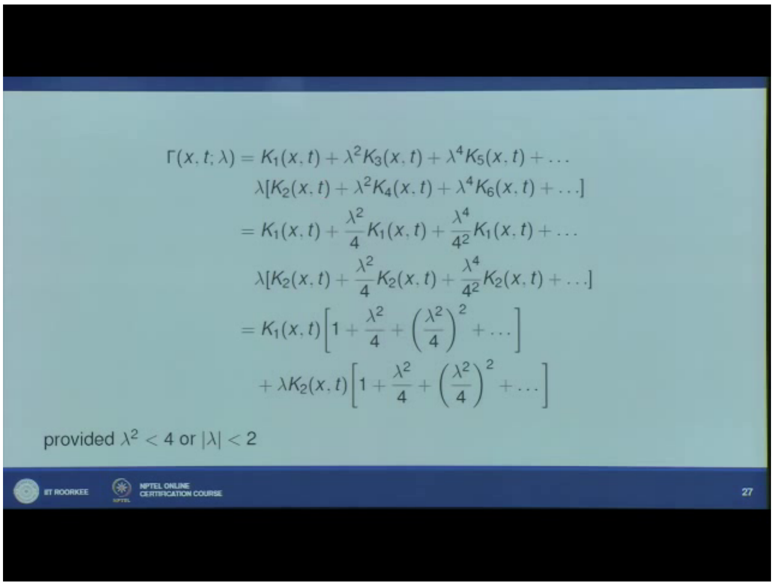

So k 1(x, t) as it is lambda square (x, t) and this is k 2(x, t) and then we are writing all odd terms together and all even terms together. So here when you use the value of k 3(x, t) it is coming out to be 1 by 4 k 1(x, t) , ok. Similarly k 5(x, t) is you just look at your calculation k 5(x, t) is 1 by 4 square k 1(x, t) similarly k 7 is 1 by 4 q k 1(x, t). So using this you can find out the summation and it is coming out to be k 1(x, t) times this and k lambda k 2(x, t) times the same series if you look at there is same series here.

And if you look at this is a geometric series with the common ratio lambda square by 4, so this will converge provided that modulus lambda square is less than 4 or you can say that modulus of lambda is less than 2 so you keeping this thing in mind that modulus of lambda by modulus of lambda is less than 2.

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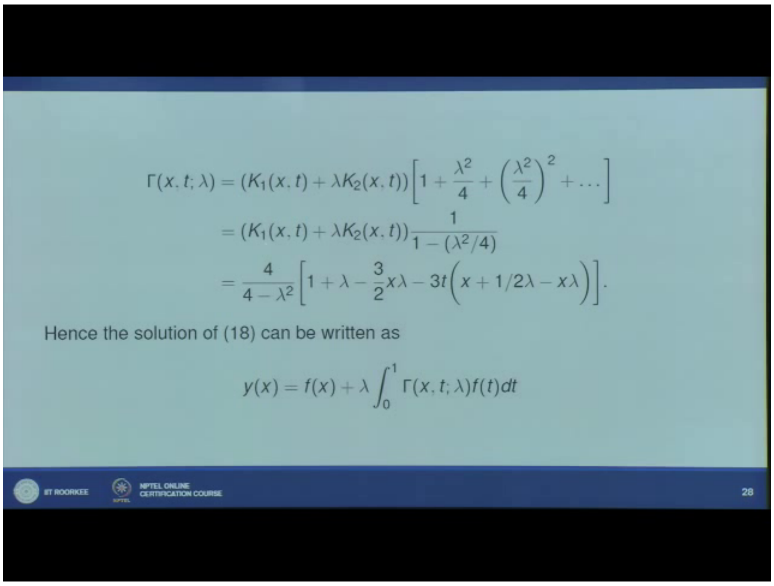

You can simplify and this written as k 1(x, t) plus lambda k 2(x, t) into 1 upon 1 minus lambda square by 4.And if you simplify using k 1(x, t) and k 2(x, t) it can be given by this. So here once you have gamma (x, t) lambda you can write down the solution as y(x) equal to f(x) plus lambda 0 to 1 gamma (x,t) lambda is we can put it here provided that modulus of lambda is less than 4.

So your solution if you simplify f(x) is given as 1 here so putting the value 1 here you can see that y(x) is equal to this quantity 4 plus 2 lambda 2 minus 3 x divided by 4 minus lambda square provided this condition modulus of lambda less than 2.And we can say that this will converege only when we are modulus of lambda less than 2.

So it means that we donot have any criteria to say that if what happen if modulus of lambda equal to 2 or modulus of lambda greater than 2, but if we remember the same problem we have discussed in the case of separable kernel there we have we are able to find out the solution of this in the case when modulus of lambda is equal to 2 or you can say lambda equal to 2 lambda equal to minus 2 and even when modulus of lambda is not equal to 2 modulus of lambda is not equal to 2.

So it means that here this is a quite method of successive approximation is quite useful but not in all case. Here I can say that this is in this particular example your method of separable kernel is quite useful. That gives you solution for all values of modulus of lambda here in fact you can check that for modulus of lambda less than 2 the method of separable kernel and method of successive approximation both will match.

But this will not give any result for modulus of lambda greater than 2 or modulus of lambda equal to 2, so in that case this since it is a problem of separable kernel so we suggest that whenever we have problem of separable kernel we always use the method of separable kernel and we may say why it is happening like this why we are not able to get that the entire reason for modulus of lambda.

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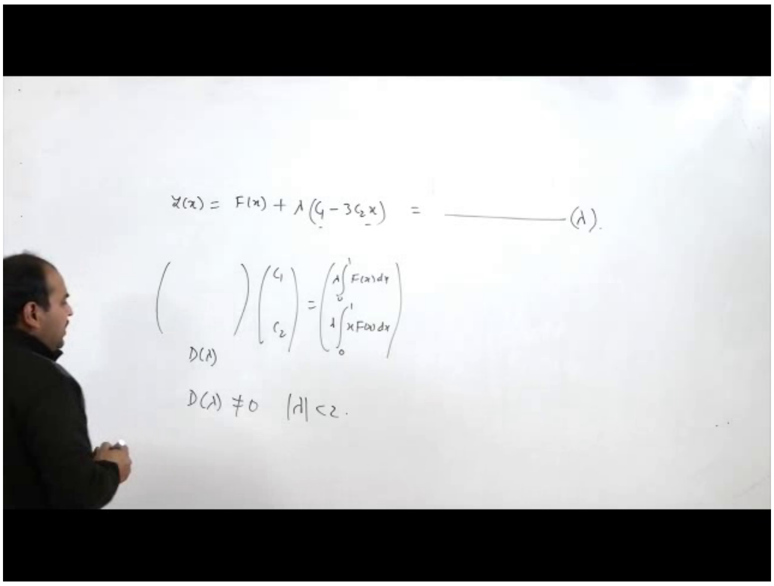

So if you remember in a method of separable kernel we are getting solution like this , we are getting solution like y(x) equal to sum of f(x) here it is 1 plus some lambda times c 1 minus 3 c 2 x kind of thing if you remember we have we can do it like this if you apply a method of successive separable kernel you can get it solution like this.

And when you find out your c 1 and c 2 then you have say d lambda and here we have c 1 and c 2 and it is some integration f(x) d(x) plus integration x f(x) dx , right? 0 to 1 here limit is 0 to 1 here and you can find out your c 1 and c 2. So if you remember I can say that here your lambda is there right so here we can say that it is given in terms of c 1 c 2 you can find out in terms of your numerator and denominator numerator is also given in terms of lambda denominator also given in terms of lambda.

So you can say that it is kind of some function during lambda divided by some function given in terms of lambda and if you divide it then you will have a function in terms of you can write it in terms of say lambda some polymal functions in terms of lambda. So in successive approximation we are writing our solution in terms of your series in terms of lambda.

So if you say that here if we do this then this ration will be valid when modulus of lambda is say here we have some radius of convergence here we have some radius of convergence so this whole expression is valid when we have modulus of lambda less than or equal to say any radius of convergence here, so if you remember this will converge when modulus this d lambda is non zero when modulus of lambda is less than 2.

So that is why this method of separable kernel and method of approximation successive approximation will match when modulus of lambda is less than 2, is that ok. So but we suggest that whenever we have a problem with the separable kernel we start solving with the help of method with the given in the lecture solving solution of a integral equation with the help of separable kernel, is it ok.

So thanks for listening us and we will meet again in next lecture , Thank you!