Integral equations, Calculus of Variations and their applications.
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Lecture-12.
Fredholm Integral Equations with Symmetric kernels: Hilbert Schmidt theory.

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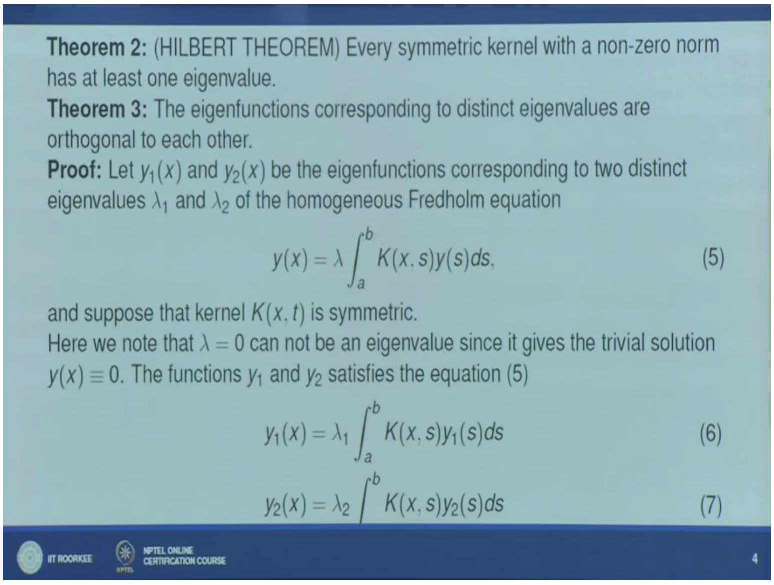

Hello friends, welcome to the lecture of integral equation, fredholm integral equation with the symmetric kernel. So if you look at, if you remember, we have discussed certain theorem corresponding to fredholm integral equation which symmetric kernel. So 1st theorem we have discussed is that if a kernel is symmetric, then all its iterated kernels are also symmetric. And 2nd result which is a kind of hunting license to start with, that is every symmetric kernel with a nonzero norm has at least one eigenvalue. So this is the beginning point by which we want to start our theory.

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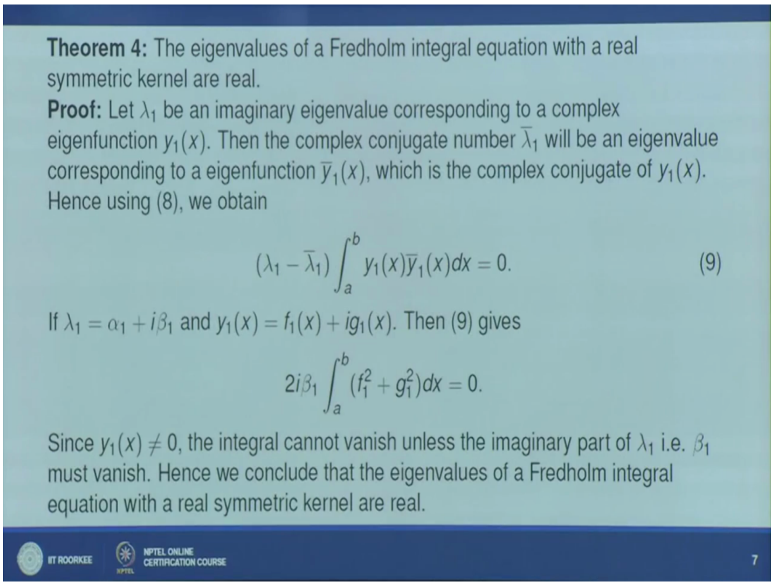

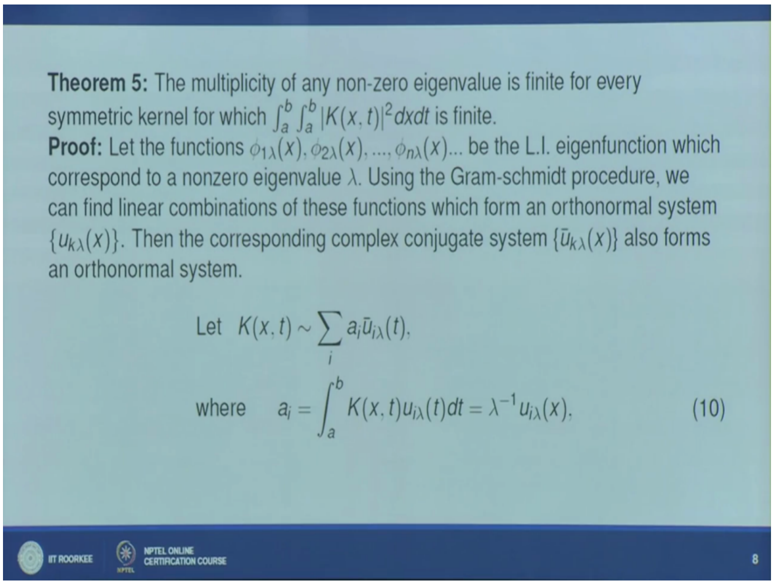

So 3rd is that if we have more than one more than one Eigen functions corresponding to distinct eigenvalues, then they are orthogonal to each other. And theorem 4 says that the eigenvalues of a fredholm integral equation with the real symmetric kernel are real, okay. And next result is which we have discussed states that the multiplicity of any nonzero eigenvalue is finite. So multiplicity is the number of linearly independent Eigen functions corresponding to a given eigenvalue is always finite, provided we have a symmetric kernel and this quantity is finite.

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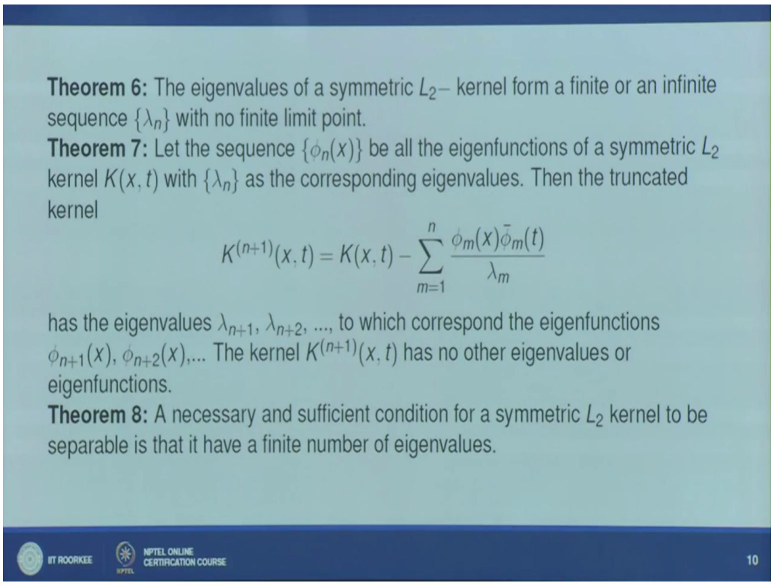

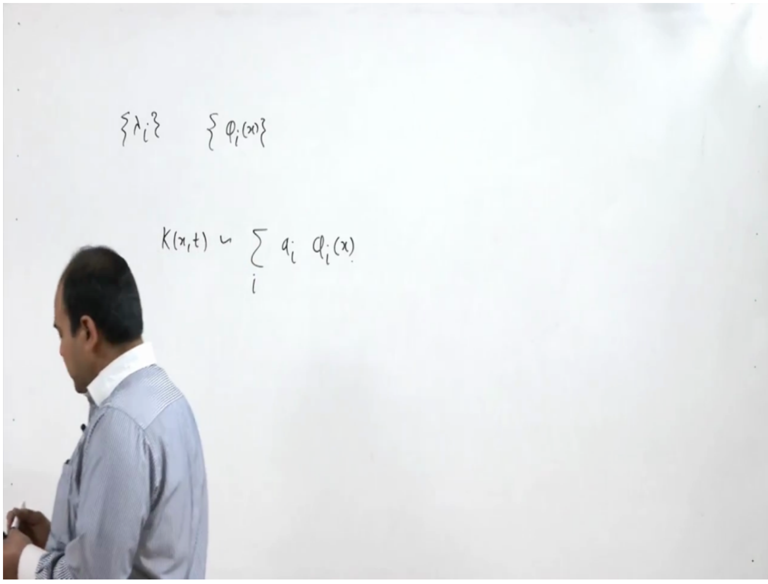

So this we have discussed in previous lecture, now let us utilise the proof of this theorem 5 to prove one more result which says that the eigenvalues of a symmetric L2 kernel from a finite or infinite sequence with no finite limit point. It means that when we have a sequence of Eigen functions, if it is finite, then no problem, but if it is infinite, then we do not have finite limit point, so they will converge to infinity. So for that we just look at that, suppose we have a sequence, say lambda I, so we have a sequence say lambda I and the corresponding sequence Eigen functions we are denoting as Phi of x.

So without loss of generality we can assume that all the Phi I x are all orthonormalized by a gram Schmidt process. Now if you remember, in previous proof, we have simply approximated your k of xt as a summation your ai Phi I x here. So I is equal to whatever Ix we have. So here we are taking the summation of over ai. So this is the beginning point here we are assuming. So here I am assuming that U I bar lambda I lambda t is your Eigen functions corresponding to lambda. So if you drop this notation lambda, then you can say that it is nothing but this.

So here we are assuming that k xt is approximated by ai U I bar t. So here U I bar t is the corresponding, so Eigen functions corresponding to your eigenvalues lambda I, is that okay. So in the same way you can define your ai, ai is basically a to b k xt, U I t dt. So I am dropping this lambda because we are considering all the Eigen, eigenvalues. So if you remember that U I is the Eigen functions corresponding to say lambda I eigenvalues. So here I am assuming that if we have repeated say eigenvalues, we count them as lambda 1, lambda 2 and so on.

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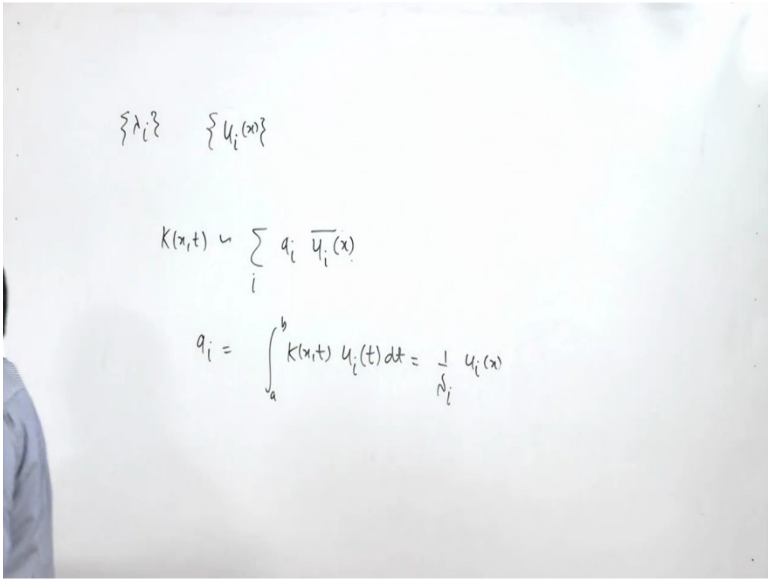

So it may happen that lambda 1, lambda 2 may be equal and same as lambda, but we are counting all the eigenvalues, okay. So we are saying that this UI is the Eigen functions corresponding to lambda I. So here ai is nothing but lambda I inverse U I x. So here I am writing here, let me use this notation, since I am using this, let us assume this, this Phi I is replaced by your UI. So this is U I, x is the Eigen functions corresponding to this lambda I, so I can write it like this. So here we have say U I bar, because if this is Eigen functions, then these are also Eigen functions, okay.

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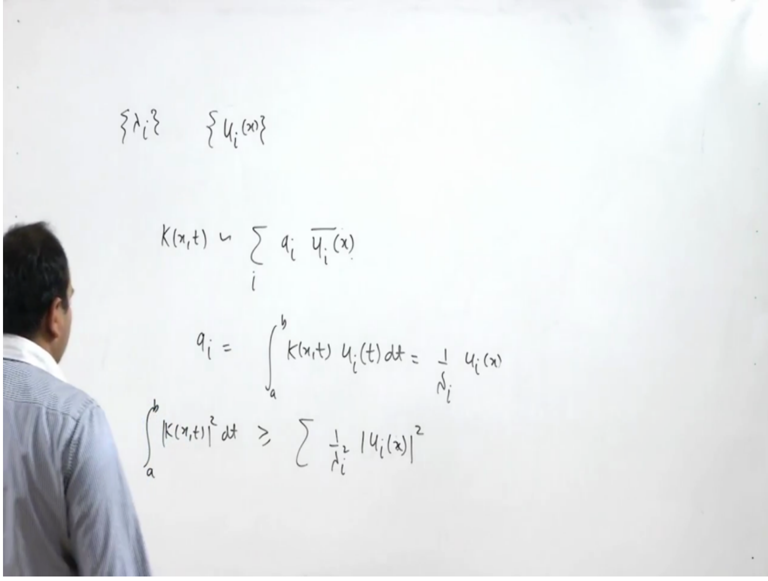

So here I can approximate k xt by this. So here ai is basically what, ai is a to b, your k of xt and U I t dt. Now this is, since by the property of Eigen functions, this is 1 upon lambda I, your U I x, is that okay. So here ai you can get it like this, 1 upon lambda I UI x. Then again we can use Bessel's inequality and you can have this property, this equation 11. So along equation number 11, you have this a to b k of xt square dt is greater than or equal to summation 1 upon lambda I square and it is what, modulus of U I t square, this is X I think, so this is x here.

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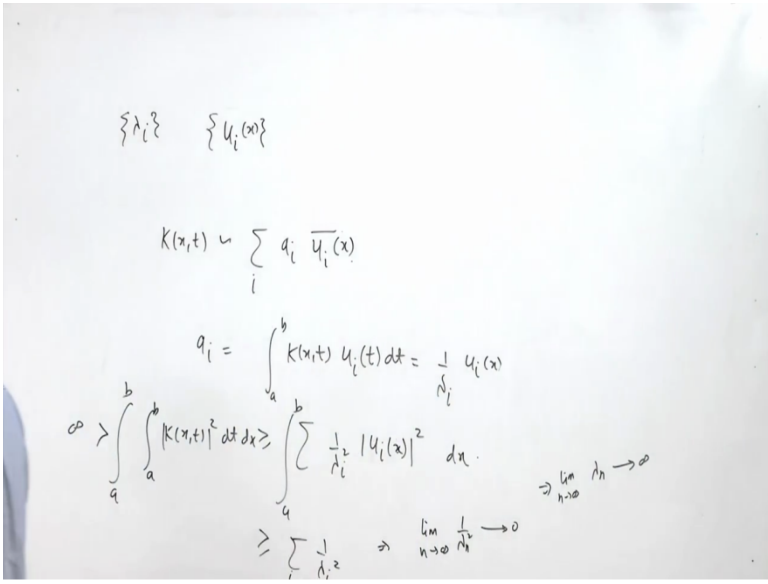

Okay, so now, again now let us, again integrate with respect to x. So a to b dx here, then here we have a to b t of x, right. And we simply assuming, okay. This I can consider, since this U I x and all normalised, then I can write this as, this is greater than equal to summation 1 upon lambda I square. Is that okay. So let me use the, okay. So here we have this thing. Is that okay. Now, if this quantity is finite, okay so here is this quantity is finite, then we can say that, okay, let me write it here. Then this series, summation 1 upon lambda I square is sum is going to be finite.

Now sum is going to be finite means you are infinite series of 1 by lambda I square is convergent series. And if you remember, there is a small result that if a series converges, then its nth term is tending to 0 as n tending to 0. So it means that, this implies that limit n tending to infinity, your 1 upon lambda n square is basically tending to 0. Or equivalently we can say that, this implies that limit n tending to infinity, your lambda n is going to be infinity. So there are only 2 possibilities that this sum is finite, if this sum is finite, no problem and if this sum is not finite, then we can use this property of convergent series which says that your nth term is tending to 0, it means that your lambda n is tending to infinity.

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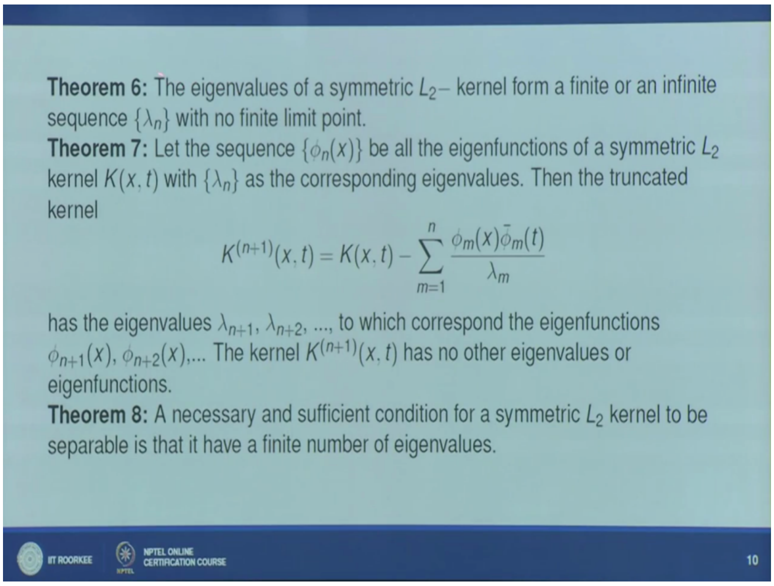

So it means that either your, so this proves this thing that the eigenvalues of a symmetric L2 kernel form a finite, if it is finite set, fine, or if it is an infinite sequence with no finite limit point. Now this is a rough sketch of the proof, the exact proof (())(8:24) proof of this book is, this proof is given in the book by RP kawal, linear integral equation. So now let us move little bit further, which says that theorem 7, it says what, let the sequence Phi nx be all the Eigen functions of the symmetric L2 kernel k xt with lambda n as the corresponding Eigen values.

So here we have, we are able to calculate all the Eigen values and Eigen functions. Then with the help of this we try to define new quantity, which is known as truncated kernel. Which is what, if it is 1st kernel, so it is k1 xt is your k xt and your k2 xt is basically kxt - this quantity Phi 1x phi m bar t lambda m. So it is kind of, we are approximating your kernel k xt by the this thing.

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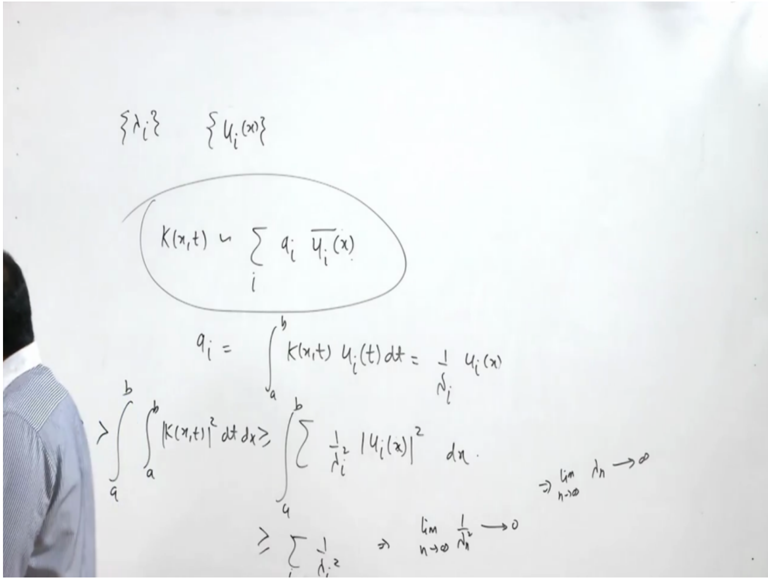

So here basically we are doing like this. Is that okay. So here we try to find out, say as accurate as possible. So here we are truncating I equal to1, 1st approximate is 2, 3, and so on. So here we are defining that nth +, n +1th place, it is given by this. So k xt - m equal to 1 to n, phi mx into some constant, that constant I am writing as phi m bar t upon lambda m. So it is kind of approximation of k xt with the help of m, 1st m Eigen functions. So we call it n +1 th truncation of this k xt. So this is known as truncated kernel.

Now we can prove that this truncated kernel has the eigenvalues lambda n +1, lambda n +2 and so on, which is corresponding to Phi n +1 x, Phi n + 2x Eigen functions. So and this kernel will not have any other eigenvalues or any other Eigen functions. So this is corresponding, so it means that if we already know that eigenvalue is an Eigen function, then we can always find out eigenvalues an Eigen functions corresponding to this truncated kernel. Now what is the use of this, we are trying to, with the help of this we are trying to approximate your symmetric kernel with the help of this kind of separable kernel kind of thing.

So here if you remember there is a result in separable kernel, that if we have a separable kernel, then eigenvalues are finite eigenvalues. Now here if I look at that theorem 8, which says that, a necessary and sufficient condition for asymmetric L2 kernel to be separable is that it have a finite number of eigenvalues. So it means that, this is an if and only if result, that if we have finite number of eigenvalues, then it means that at some point this process will stop. So it means that suppose we have say, we have only n eigenvalues, then we can have say only m Eigen functions.

So it means that at kn +1th stage, this is nothing but, we cannot get kn +1 xt, so it is simply 0. So in that case your k xt is written as m equal to 1 to n phi mx, phi m bar t lambda m. Or we can say that this is nothing but given in terms of separable form. So if we have only finitely many Eigen values, we can say that in that case your k xt is given by this separable form. Or if I say that we do not have, if it is separable, then we already know that we have only finite number of eigenvalues. So if it is finite number of eigenvalues, then it is separable, if it is separable, then we have only finitely many eigenvalues, that we have already done.

So here with the help of theorem 7 we can say that if we have finite number of eigenvalues, then k xt can be written as this kind of form, which is nothing but a separable form for this k xt. So that it means that a symmetric L2 kernel is separable if and only if we have finite number of eigenvalues, okay. So now let us move to the next result which is a very very important result of Hilbert Schmidt, known as Hilbert Schmidt theorem. It says that, again we are taking this without proof but statement is very very important.

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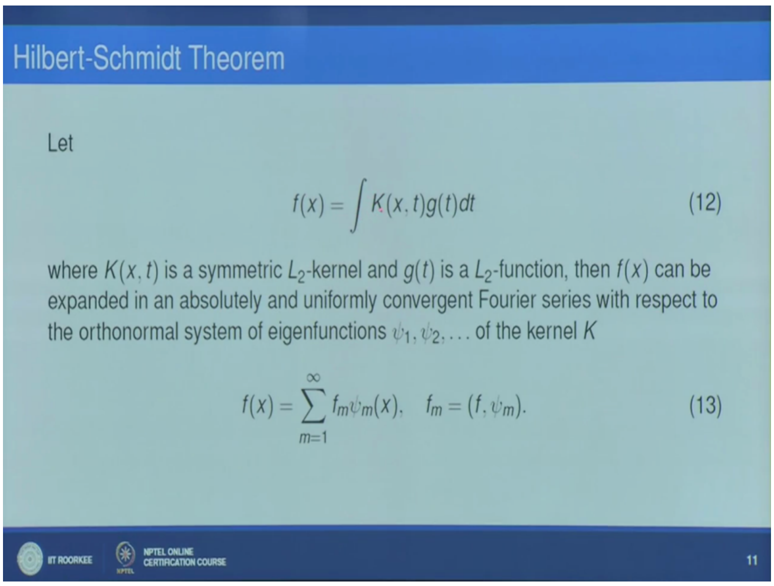

Let us say that that if we have a function f of x, which is given by k xt gt dt, we can say it is generated by a kernel k and a function g. Here we are assuming that this k kernel k is symmetric L2 kernel and g also we are assuming that it is L2 function. Then Hilbert Schmidt theorem says that, then this function f of x which is generated by this k and g can be expanded in an absolutely and uniformly convergent fourier series with respect to orthonormal system of Eigen functions psi 1, psi of the kernel k. That means that, what it says, let me explain in a little bit detailed manner.

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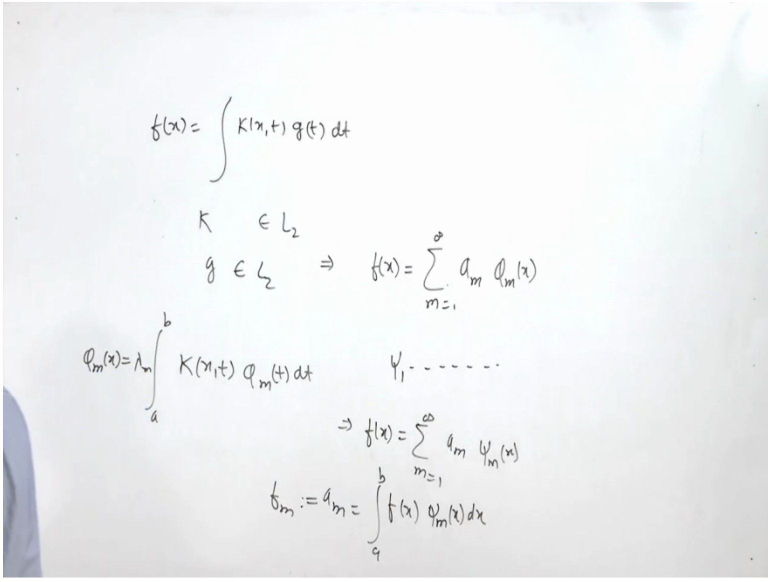

It means that if we have f of x is equal to your k of xt gt dt, so here we are assuming that k xt belongs to L2 and similarly your gt, k and g, both belongs to L2. Then you are Hilbert Schmidt theorem says that fx can be written as summation of some am phi mx and summation over this m, m is say 1 to infinity. Now what is this Phi m, so here phi m is the Eigen function corresponding to this kernel k xt. Means what, that it satisfies this property, that phi mx is equal to lambda a to b and k xt phi mt dt.

So it means that that this phi mx is basically Eigen function corresponding to some eigenvalues for let us say that lambda n and it satisfies this property. And in addition we are just assuming that this can be orthonormalized to say psi 1, 2, and so on, psi 1, psi 2, and so on. So here we can write this as f of x equal to summation m equal to1 to infinity am psi mx. Now what is the difference between this and this, difference between this and this is that here it is Eigen functions, now by Gram Schmidt orthonormalization, we can convert into new systems having equal number of elements here but now with the property that they are orthonormalized, is that okay.

So it means that fx can be written as am psi mx and here am you can write it as fx a to b and psi mx d of x. And just for simplicity we are writing this as f of, we are denoting this as f of x. So if you look at, look at equation number 13, it says that if we have this function which is given by this, then this fx can be represented as this infinite series which is absolutely and uniformly convergent. So here f of m is given by this f psi m which is denoted by this, a to b f of x psi mx d of x. So this is the inner product we are defining as a to b f of x psi m x d of x, is that okay.

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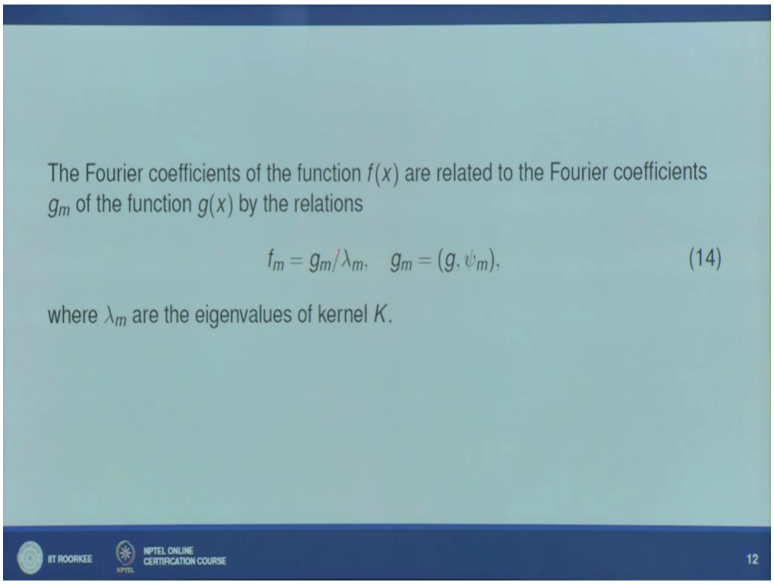

So these coefficients are known as fourier coefficients here and then this fourier coefficient fm is related to fourier coefficient corresponding to g. So here I can say that this fm is given by g of m divided by lambda m, where gm is the fourier coefficient of g and lambda m are the eigenvalues corresponding to say psi m and it is eigenvalues of the kernel k. This is not very difficult here, so what we can do here is , to find out this relation 14, fm equal to gm divided by lambda, what we can do here, we have this. Then we simply multiply by say phi mx and then use a property of orthogonality, let me write it here, we have this, okay.

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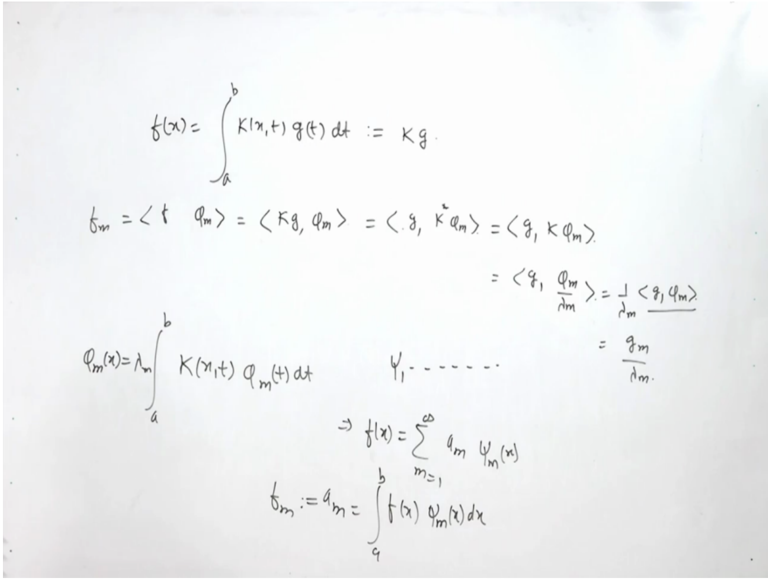

So here let us say we have this, we have say f of x phi mx here, so here let me write it f phi m, right. So this I can write it as f Phi, f is what, let me denote this as kg, so this is denoted as, defined as k operating on g, so this is kind of an operator on this. So k on g, kg is defined as this, okay, limit is say a to b, so kg Phi of m. So this you can prove that this is same as g k star and phi m. Now since k star is same as k, because we are assuming that it is a symmetric kernel, then it is nothing but gk of phi m.

Now we already know that k phi m means this, k xt phi mt dt. So this is going to be phi mx divided by lambda. So this is what, this is nothing but g, your phi m divided by lambda, is that okay. So this is you can write it, you can take out 1 upon lambda m out, this is g phi m and this is nothing but gm, we are defining it like gm, gm by lambda m. So here is fm, this is nothing but your f of m. So fm which is a fourier coefficient of S with respect to phi m, so fm is given by gm divided by lambda m.

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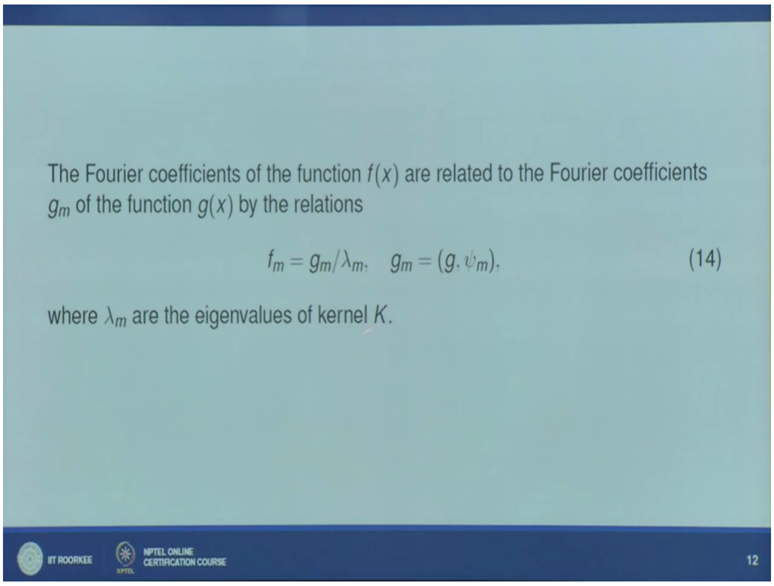

So here you can look at that, equation number 14 is valid in this way, that fm is given by gm divided by lambda m, is that okay. So using this now let us proceed to solve your fredholm integral equation of, 1st we will try to solve for 2nd type and then we, if we have, we will discuss for 1st integral also. So let us try to solve, solution of a symmetric integral equation of non-homogeneous fredholm integral equation. So please remember if it is homogeneous, then we already have solved, that is nothing but your eigenvalue Eigen function problem.

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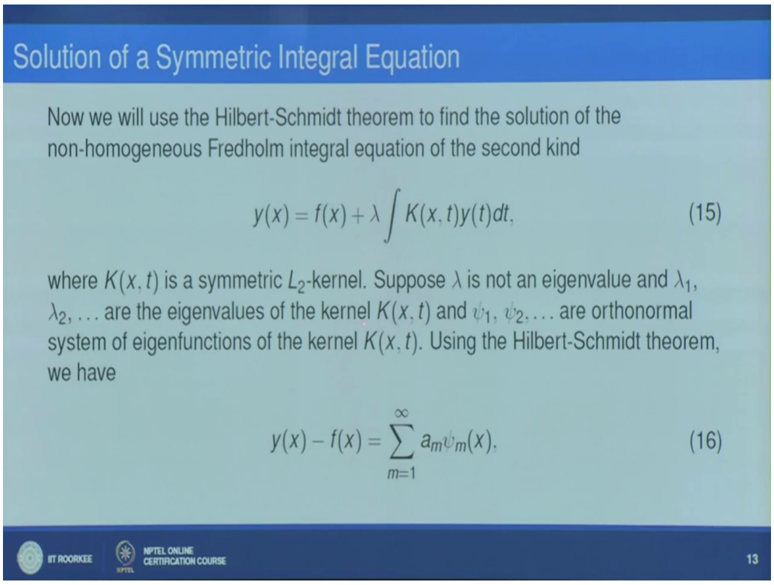

So now let us proceed for solving the non-homogeneous problem, that is y of x equal to f of x + lambda a to b k xt yt dt. Here I am assuming that k is L2 kernel, so that we can utilise the theory which we have discussed earlier. Okay. Now, we assume that this lambda is not an eigenvalue, we will consider the case when lambda is an eigenvalue, but for the starting point let us assume that lambda is not an eigenvalue. And we are able to solve the homogeneous problem, means we are able to find out all the eigenvalues corresponding to this symmetric kernel k.

So here we are assuming that lambda 1, lambda 2, all the eigenvalues of the kernel k xt and this psi 1, psi 2 are orthonormal system of Eigen function of the kernel k xt. So that we already have enquired. So theory says that you can always do it, okay. So now using the Hilbert Schmidt theorem, this by x - fx is written as lambda k xt yt dt. So now you can use Hilbert Schmidt theorem and say that y x - fx can be expressed as this infinite series in terms of Eigen functions corresponding to this k xt which is uniformly, uniformly and absolutely convergent here.

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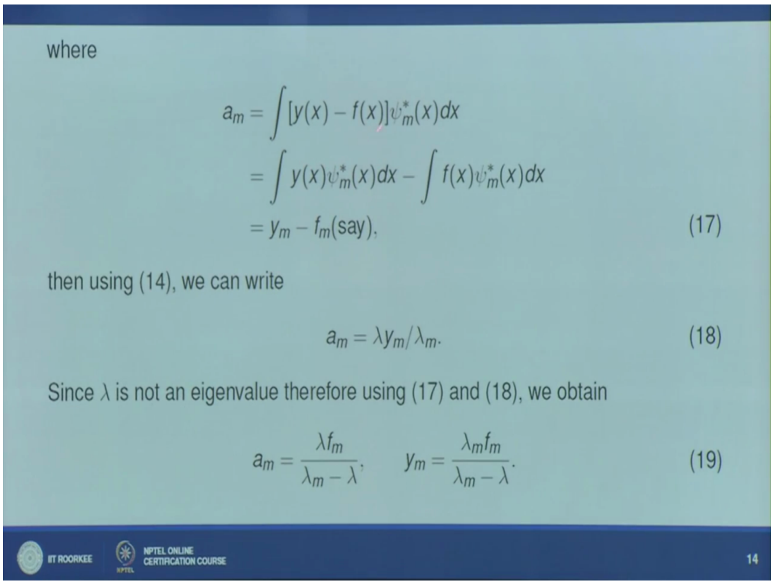

So this is by your Hilbert Schmidt result. And here we try to find out now this fourier coefficient that is am. So that, we know that am is basically what, am is nothing but y x - fx psi star mx dx.

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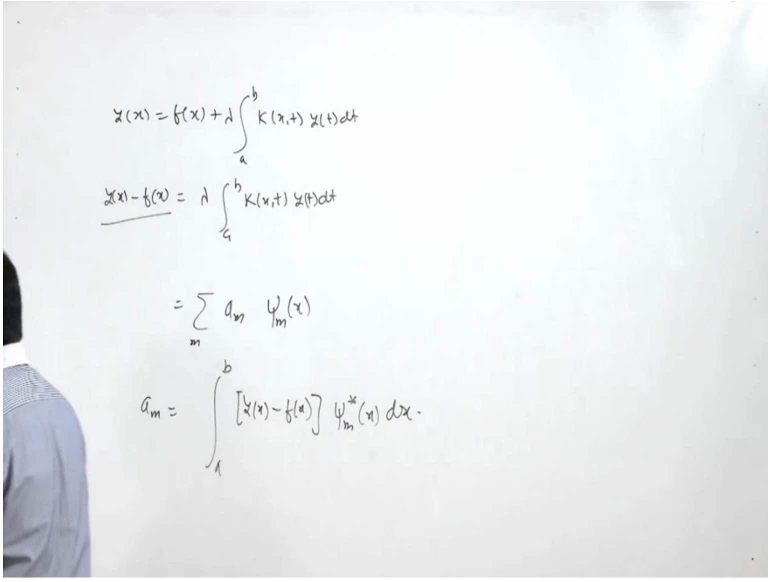

So here your inner product is defined like this, let me use, we already know this that ym, yx, yx is equal to f of x + lambda a to b k xt yt dt, now you can take this out, so y x - f of x this side and it is lambda a to b k of xt yt dt. And then using Hilbert Schmidt theorem, you can always write it like this as am and here we are assuming that psi m, psi mx dx and how to find out this am, am is nothing but fourier coefficient corresponding to this yx - f of x. So that is yx - f of x and then U psi m star x dx a to b.

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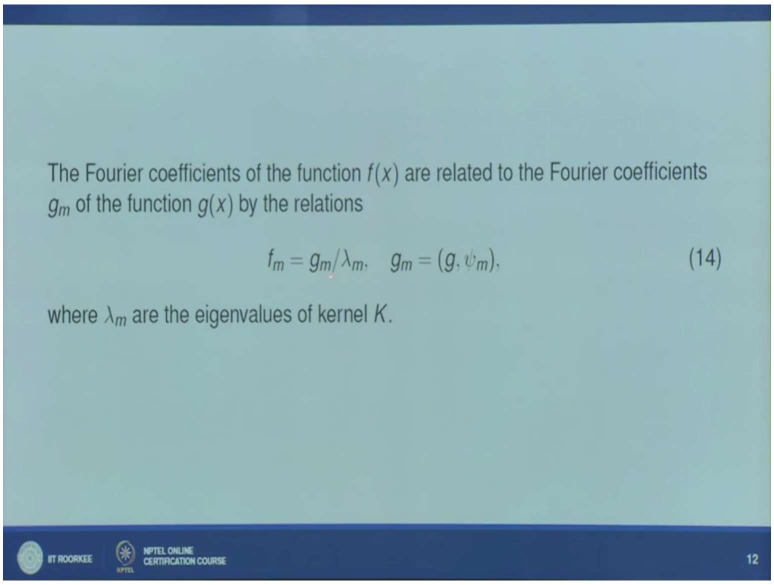

So if you calculate this, this is what, this you can write it, you just take it, you separate these 2 integral. So yx psi star mx dx - fx, psi star mx dx, so where psi m star is complex conjugate of psi mx. But if you look at your previous thing, that your fourier coefficient corresponding to your function f of x can be written as fourier coefficient of the unknown function, given function gx as this fm equal to gm by lambda m. So here, in analogous manner you can say that am which is a fourier coefficient of y x - fx can be written as, sorry can be written as fourier corresponding, fourier coefficient corresponding to this yt.

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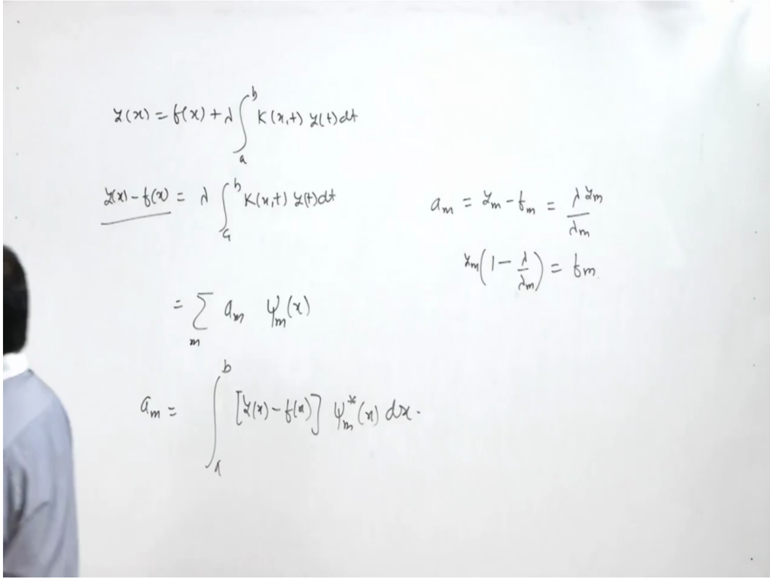

So I can write it here am as lambda ym divided by lambda, so you can equate these 2 things. So when you equate these 2 things, you can get your am and y m. So am is the fourier coefficients here and it is written as lambda fm divided by lambda m - lambda, this is very easy, I can say that here we have am as ym - fm and this is coming out to be lambda y m upon lambda m. So if you will compare you will get y as 1 - lambda upon lambda m is equal to f of m. So you can get, once your ym is calculated, then you can calculate your am also.

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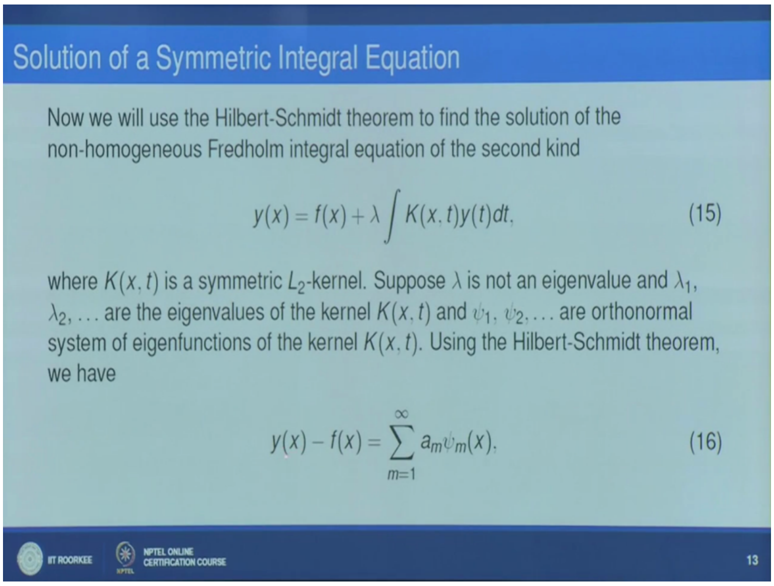

So here you can get am as lambda fm upon lambda m - lambda, y equal to this. Okay, fine, so this is something we want to find out, ym, okay. So now I can write it, yx as what, so y, since, look at here, equation number 16, so yx - fx is equal to, n equal to, this summation m equal to 1 to infinity am. Am you have already obtained and psi m is already known to you, so you can get yx in terms of fx + this infinite series. So you can write it here, yx equal to fx + lambda, I am writing the value of am. So value of am is lambda fm upon lambda m - lambda.

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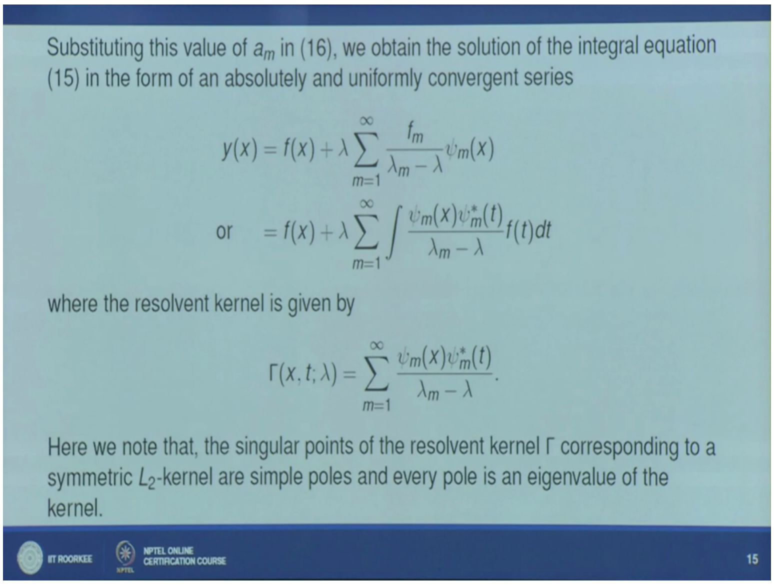

So I can write it here, lambda fm, lambda fm upon lambda m - lambda psi mx dx. Now here you can utilise the value of fm, fm is what, fm is the fourier coefficient of f, which is nothing but this a to b, I am not writing the limit because your interval maybe anything. So here fm is basically what, ft psi star mt dt. So using the expression for f of m and I am using t as the integrable variable because we are already having x. So here fm t I am writing as a to b psi m star t ft dt.

So when you write it here, and we already know that this series is absolutely and uniformly convergent, so we can always interchange the integral sign and summation sign. So we can write it like this, f of x + lambda, n equal to1 to infinity, this thing. Now if we denote this m equal to1 to infinity fm upon lambda m - lambda or you can say that if we denote this gamma xt lambda as this, m equal to1 to n, psi m is x psi m star t lambda m - lambda and say that it is resolvent kernel, then your solution, it can be written as this.

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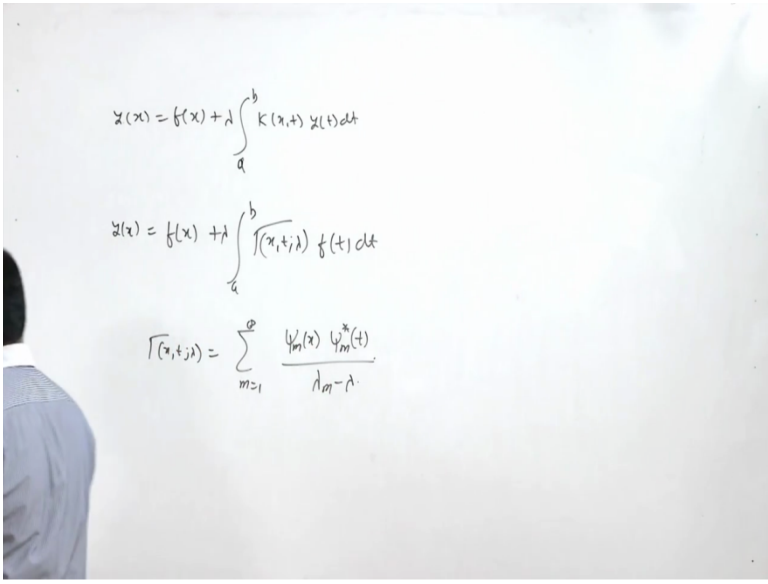

So here your solution is given as y of x is equal to f of x + your lambda times a to b gamma xt lambda your f of t dt where gamma xt lambda is given as this infinite series m equal to1 to infinity psi, sorry, psi mx, psi m star t divided by lambda m - lambda, okay. And this series is absolutely and uniformly convergent. Okay. So here solution is given by this. Now if you look at here, your lambda, choice of lambda is very very important because if lambda is one of the Eigen value, then there is gamma xt lambda will not exist.

So here I am assuming that the singular point of this resolvent kernel is the values of lambda which is equal to lambda m or you can say that the singular point of the resolvent kernel gamma corresponding to a symmetric L2 kernel are simple poles because at the simple pole every pole is an Eigen values of the kernel. Or you can say that for lambda equal to eigenvalues is your the singular point of this kernel gamma, resolvent kernel gamma xt lambda. So using this, now let us try to apply the result for, actually solving the Fredholm integral equation of the 2nd kind, okay. So that we are going to do it in the next lecture, thank you very much.