Integral Equations, Calculus of Variations and their Applications
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Lecture 03
Conversion of BVP into integral equations

Hello friends! I welcome you to my lecture on Conversion of Boundary value problem into an integral equation. In my previous lecture we discussed how we can convert any initial value problem into an integral equation and the integral equation which resulted it was a Volterra Integral equation of the second kind.

Then we also recovered back the initial value problem from the integral equation so obtained. Now what we are going to here is that we will be taking a boundary value problem and we shall be converting it into an integral equation and we shall see that we get a Fredholm integral equation of second kind. And then we shall recover that the initial the boundary value problem from the Fredholm integral equations obtained.

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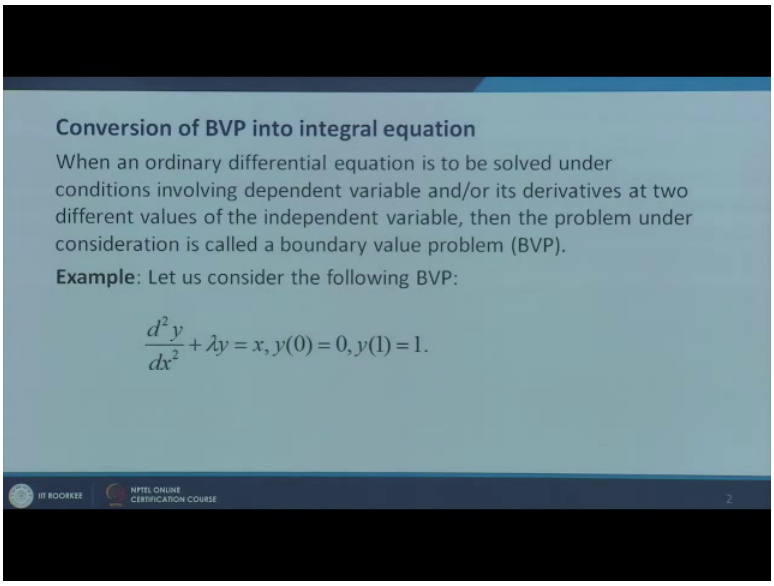

So let us see what do we mean by a boundary value problem first when an ordinary differential equation is to be solved under conditions involving dependant variable or its derivatives at two different values of the independent variable then the problem under consideration is called a boundary value problem.

Let us consider the following boundary value problem you see here we have a second order differential equation d square y over dx square plus lambda y equal to x where the values of the dependent variable y are given at two values of the independent variable that is 0 and 1. We are given y(0) equal to 0 y(1) equal to 1 so we shall convert this boundary value problem into an integral equation and we shall see that we get a Fredholm integral equation of second kind.

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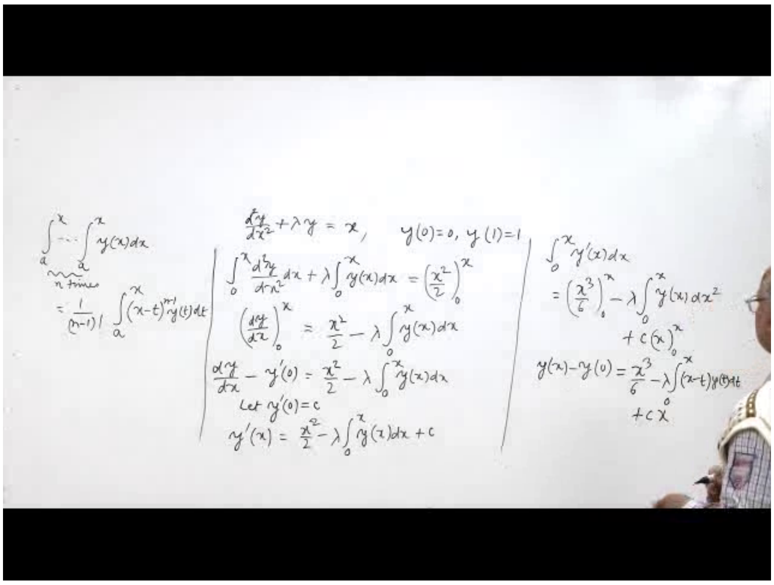

So let us integrate both sides we have d square y over dx square plus lambda into y equal to x y (0) equal to 0 and y(1) equal to 1, ok. Now let us integrate both sides with respect to x from 0 to x. So then what do we have integral 0 to x d square y over dx square lambda is a constant so lambda into integral 0 to x y(x) dx, y is a function of x equal to x square by 2 0 to x.

And when you integrate here what you get is dy by dx which is to be evaluated at the point 0 and x and then we can write it as now x square by 2 at x is x square by 2 and at 0 it is 0 so we get (x square by 2) minus lambda integral 0 to x y(x) dx. Here what do we notice is that dy by dx at x is dy by dx minus dy by dx at 0.

So dy by dx at 0 we can write as y prime 0 and this is x square by minus lambda integral 0 to x y(x) dx. Now we have not given the value y dash 0 so we can assume at to some constant we can take it as C. So let us take y dash (0) equal to C if we take y dash (0) equal to C then we can write it as y dash (x) equal to (x square by 2) minus lambda integral 0 to x y(x) dx plus C, ok.

Again let us integrate with respect to x so again when we integrate we shall have integral 0 to x y dash (x) dx equal to (x square by 2) will come (x cube by 6) and then we shall have minus lambda times 0 to x y(x) dx 2 because we will have double integral here. Ok when we integrate it again we shall have double integral plus C x 0 to x, ok.

Now this will be y(x) minus y (0) on the left side and here we shall have (x cube by 6) and here we have double integral so let us recall the formula which converts the multiple integral the n integrals into a single integral so a to x y(x) dx where this is n times this is equal to 1 over (n minus 1) factorial integral a to x (x minus t) raise to the power (n minus 1) into y t dt.

So let us recall this formula so here n is equal to 2 so we can write 1 over 2 minus 1 factorial which is 1 factorial integral 0 to x and then (x minus t) into y(t) dt and then here we have cx, ok. We are given y (0) equal to 0 so when you put y(0) equal to 0 what you get is y(x) equal to cx plus (x cube by 6) minus lambda 0 to x (x minus t) y(t) dt. Now we have the values of y(x) here and we also have the value of this thing y dash(x) but c is unknown here so what we will do is ok.

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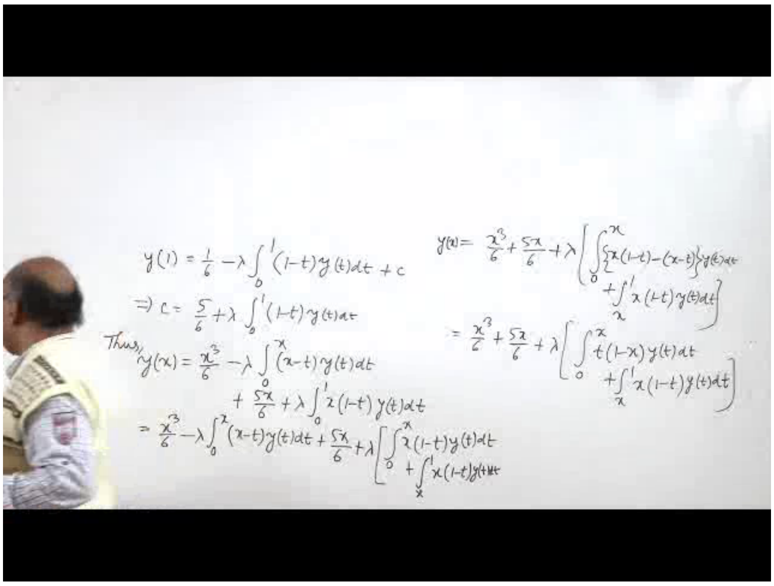

So let us now put x equal to 1 to determine the value of c so y 1 y(0) is 0 so y 1 equal to 1 by 6 minus lambda integral 0 to 1 (1 minus t) into y(t) dt plus c. And y 1 is equal to 1 so we get c equal to 5 by 6 plus lambda integral 0 to 1 (1 minus t) into y(t) dt.

Now the value of c we substitute in this equation, ok. So thus we have y(x) equal to x cube by 6 minus lambda 0 to x (x minus t) into y(t) dt plus c into x. So c into x means x by 6 plus lambda 0 to 1 x into (1 minus t) into y(t) dt. Now we have to write it in terms of what we do is we break this interval from 0 to 1 from 0 to x and then x to 1.

So what we do (x cube by 6) minus lambda integral 0 to x (x minus t) into y(t) dt plus 5x by 6 and then we write lambda times integral o to x x times (1 minus t) into y(t) dt plus x to 1 x times 1 minus t into y(t) dt.

What we will do is we will combine the two integrals where the limits of integration are 0 to x so we shall write it as y(x) equal to (x cube by 6) plus 5x by 6 and then we shall write it as lambda times integral 0 to x x times (1 minus t) minus (x minus t) into y(t) dt plus integral h to 1 h times 1 minus t into y(t) dt.

If you simplify this what we get is (x cube by 6) plus 5x by 6 plus lambda times integral 0 to x, here we have (x minus xt) minus x plus t so we have t times 1 minus x into y(t) dt plus x to 1 x times (1 minus t) into y(t) dt, ok.

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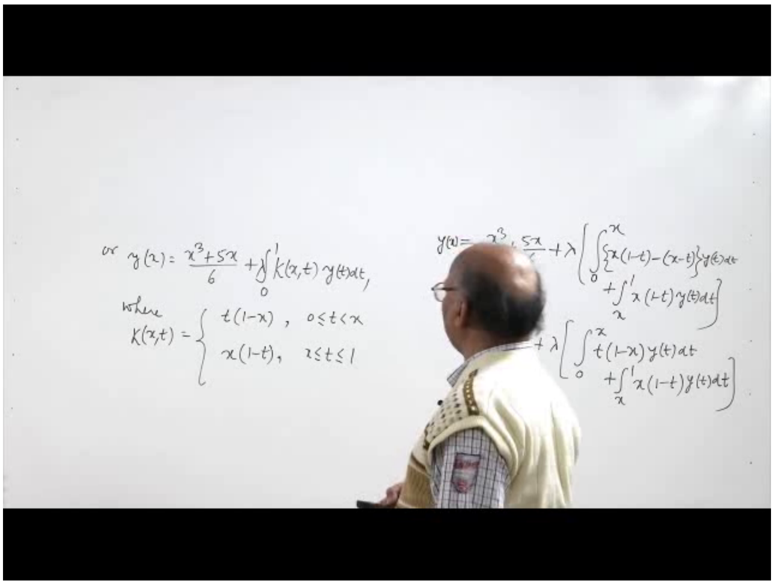

Now what we do is we can write this equation as or y(x) equal to x cube plus 5x by 6 plus integral 0 to 1 k(x, t), k(x, t), the kernel of the integral equation into y(t) dt where k(x, t) is defined as t times (1 minus x) where t lies from 0 to x and x times 1 minus t where x is less than or equal to t less than or equal to 1.

So we see that the given boundary value problem is converted into integral equation y(x) is equal to x cube plus 5x by 6 plus lambda here lambda is missing so lambda times integral 0 to 1 k(x, t) y (t) dt where k(x, t) t times 1 minus x 0 less than or equal to t less than x and x times 1 minus t when x is less than or equal to t less than or equal to 1. So thus we obtain a Fredholm integral equation of the second kind.

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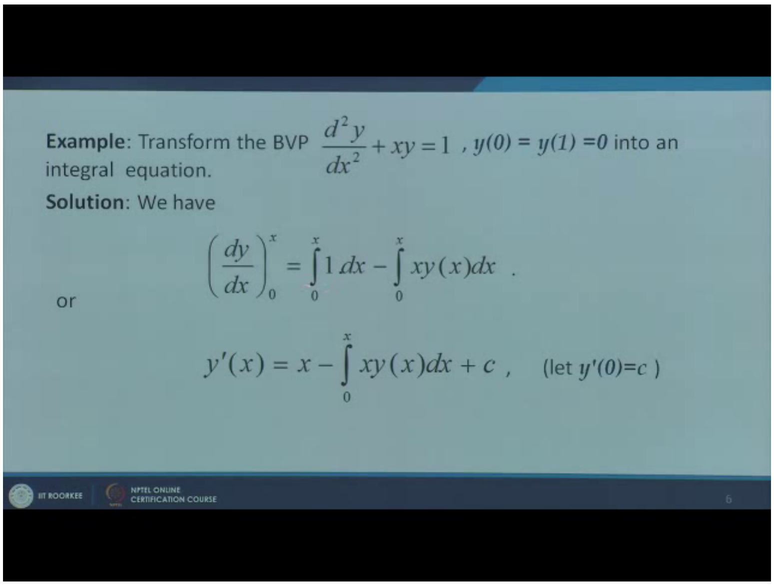

Now let us see how we can recover the boundary value problem from this integral equation. Now let us take one more example of a boundary value problem where we shall see how we can obtain the integral equation we are given a differential equation of second order where the values of the dependent variable y are given at two points 0 and 1.

So again what we will do we will integrate this differential equation with respect to x from 0 to x when you differentiate when you integrate d square y over d square x with respect to x you get dy by dx the limits of integration are 0 x and then integral 0 to x 1 dx for this one on the right side. And then we have brought this term xy on the right side and we have written its integral so minus 0 to x x into y(x) dx, y is a function of x.

Now when we integrate d square y over dx square and then put the limits of integration what we will get dy by dx which we have written as y dash(x) minus dy by dx at x equal to 0 which is y dash (0) as c we have defined, so this c minus y dash (0) we have brought to the right side and written it as c and here we get x, so the limits of integration are 0 and x so we get x here minus 0 to x x into y (x) dx.

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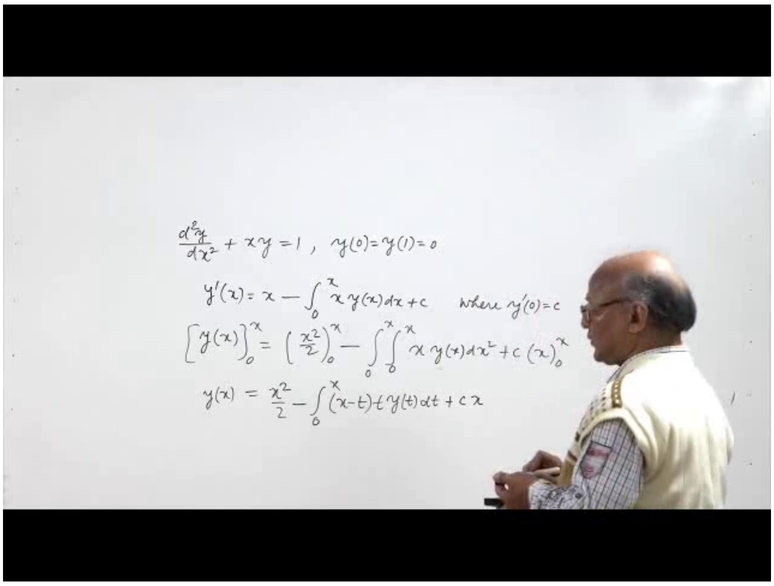

So this is what we get so we have y dash(x) equal to x minus integral 0 to x x into y(x) dx plus c where y dash (0) is equal to c. Now when we again integrate this we get y(x) o to x and here we get x square by 2 0 to x and then we have double integral 0 to x.

So this will become y(x) minus y(0) , y(0) is equal to 0 so y(x) equal to x square by 2 minus this double integral can be converted into a single integral minus 0 to x (x minus t) into t y(t)d t plus c times x.So this is what we get, now let us see value at c by putting x equal to 1

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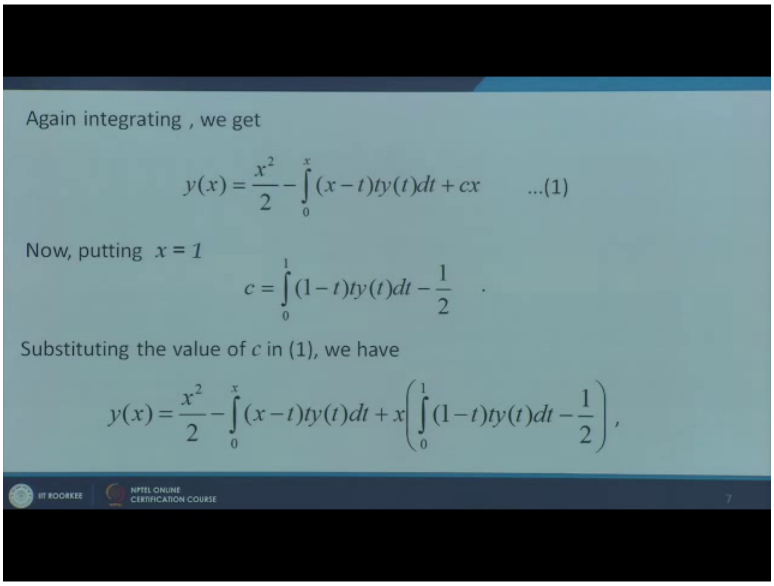

So when you put x equal to 1 here what you get is y (1) is equal to 0. So we get 0 equal to 1 by 2 minus integral 0 to 1 1 minus t into t y(t) dt plus c or we can say c is equal to integral 0 to 1(1 minus t) into t y(t) dt minus half, ok.

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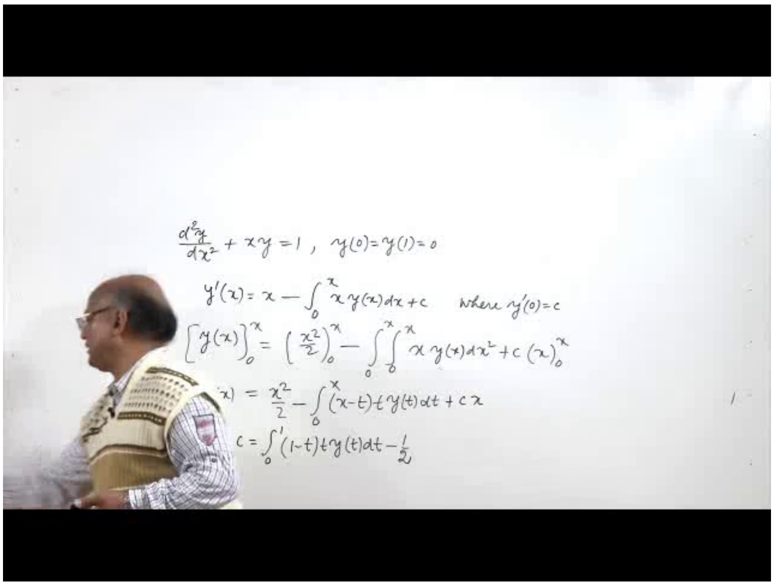

So c is this gives c is equal to integral 0 to 1 (1 minus t) into t y(t) dt minus half, ok.

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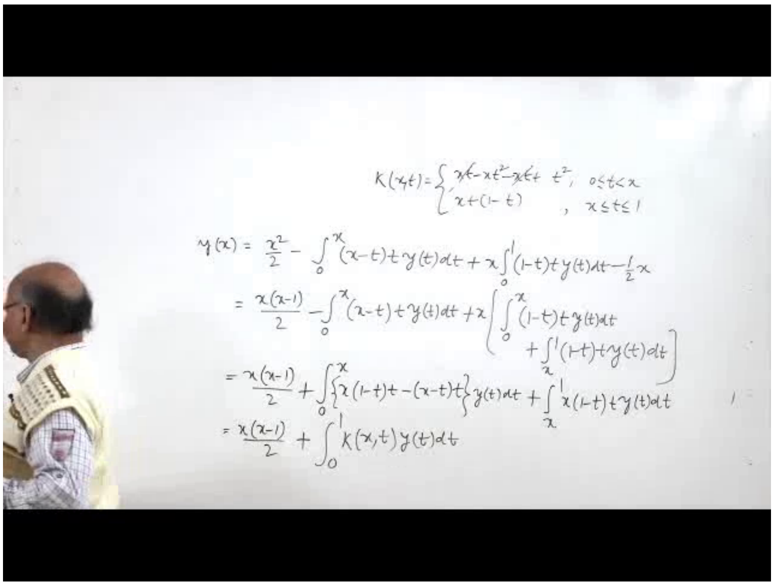

So let us substitute this value of c in this equation what we will have then so substituting the value of c we have y(x) equal to x square by 2 minus integral 0 to x (x minus t) into t y(t) dt plus c x. So x times integral 0 to 1 (1 minus t) into t y(t)dt minus half x.

Or we can say this we can write further as x into x minus 1 by 2 and minus integral 0 to x (x minus t) into t y(t) dt and just we can break into two parts again 0 to x(1 minus t) into t y(t) dt plus x to 1 (1 minus t) into t y(t) dt. And then we can combine the two integrals where the limits of integration are 0 to x and write it as x into (x minus 1) by 2.

So when we combine the two integrals where the limits of integration are 0 to x. We shall have integral 0 to x x times (1 minus t) into t minus (x minus t) into t y(t) dt and then we shall we have integral x to 1 x times (1 minus t) into t y(t) dt, ok. And then this can be further written as integral x times x minus 1 by 2 (integ) plus integral 0 to 1 k(x, t) into y(t) dt where k(x, t) is equal to x into 1 minus t into t minus x minus t into t.

So this is if you simplify it x t minus x t square will have x t minus x t square and here we will have minus xt plus xt square 1 0 is less than or equal to t less than x. And this will be x into t into 1 minus t. When x is less than or equal to t less than or equal to 1.

So this x t and this xt cancel and what we get is x t times 1 minus t. So sorry we have t square xt square minus xt square I think we have not written it correctly just a moment we have t square times 1 minus x so here we have xt minus xt square minus xt plus t square we have.

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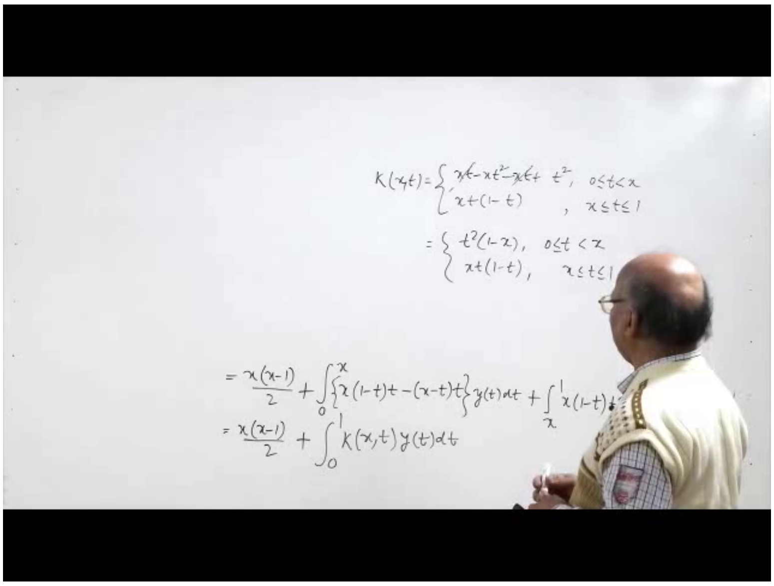

So this is this can be written as so t square times (1 minus x) 0 less than or equal to t less than x and x t times (1 minus t) x less than or equal to t less than or equal to 1.

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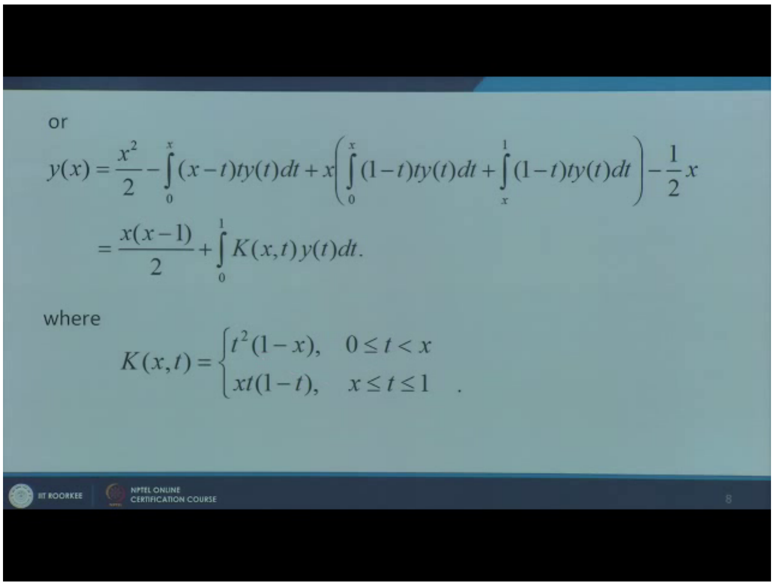

So we see that we again that Fredholm integration equation of the second kind y(x) equal to x into x minus 1 by 2 plus integral 0 to 1 k(x, t) y(t) dt where k(x, t) is t square into 1 minus x 0 less than or equal to t less than x and x t times 1 minus t when x is less than or equal to t less than or equal to 1.

So this is how we convert a boundary value problem into an integral equation and we have seen that when we convert a boundary value problem into an integral equation the resulting integral equation is a Fredholm integration equation of the second kind .

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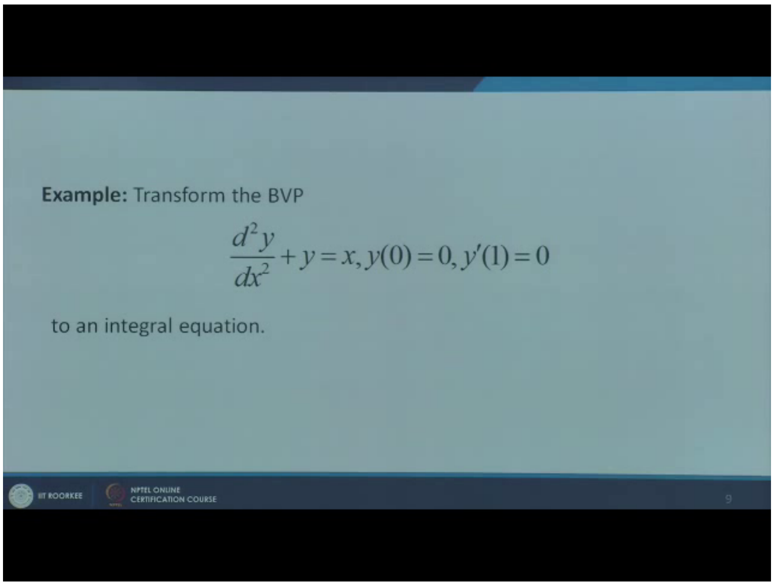

There is one more example d square y over d x square plus y equal to x where we have taken the two conditions on the dependent variable y first condition is y(x) equal to 0 is 0 and the second condition gives the derivative of y at x equal to 1 which is also given as equal to 0. So this is again a boundary value problem and it can be similarly converted into an integral equation which will be a Fredholm integration equation of the second kind.

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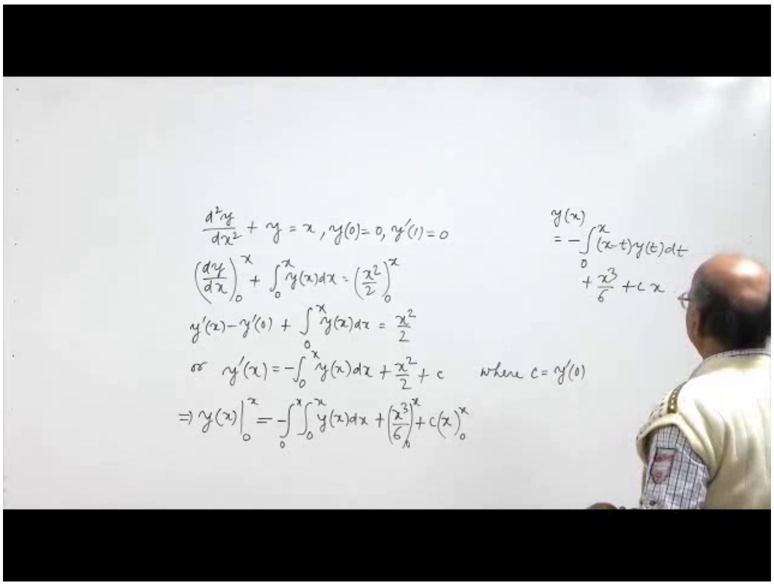

So let us convert this boundary value problem into an integral equation so we have d square y over d x square plus y equal to x where y(0) is 0 and y dash 1 is equal to 0, ok. So we integrated with respect to x from 0 to x, so dy by dx (min) plus integral 0 to x y(x) dx equal to x square by 2 integral 0 to x. So let us integrate the given differential equation with respect to x from 0 to x.

So we would have y dash (x) minus y dash (0) plus integral 0 to x y(x) dx equal to x square by 2 y dash (0) let us assume to be equal to c. So then we will get our y dash(x) equal to integral 0 to x y(x) dx we will have to take it to the right side so we put negative here plus x square by 2 plus c, where c is equal to y dash (0). Y dash (0) we have assumed equal to c so y dash(x) equal to minus integral 0 to x y(x) dx plus x square by 2 plus c.

And let us put h equal to 1 here so putting x equal to 1we have y dash (1), y dash (1) is equal to 0 so 0 equal to minus integral we need not put I think we need not put right now let us integrate it once again let us integrate it once again so what we will get y(x) equal to 0 to x minus double integral y(x) dx plus x cube by 6 plus c x.

We have this so this will give you minus y(0), y(0) is equal to 0 so we will get y(x) equal to y(x) equal to now this double integral can be converted into a single integral minus integral 0 to x, x minus t into y(t) dt plus x cube by 6 plus c x. So we have got the value of y(x) but we still do not know what is the value of c. So the value of c can be obtained from here.

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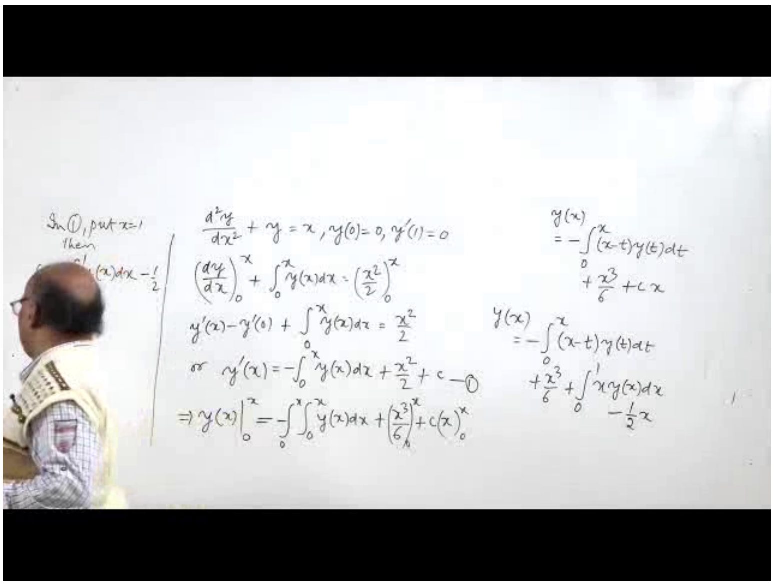

Let us call it equation 1, so in equation 1 let us put x equal to 1, so in 1 put x equal to 1 then we get c here c equal to y dash (1), y dash(1) is 0 so we get integral 0 to 1 y(x)dx minus 1 by 2.

So this value of c we substitute and obtain the integral equation. So put the value of c here so y(x) equal to minus integral 0 to x (x minus t) into y t dt plus x cube by 6 plus we multiply that value of c y(x). So integral 0 to 1 x into y(x) dx minus half x , ok.

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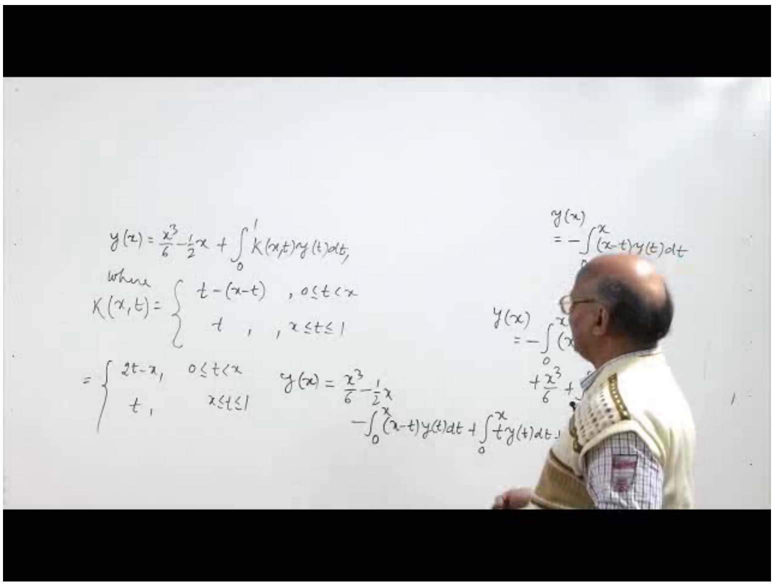

Now what we can combine this, so we can combine this as a y(x) equal to x cube by 6 minus half x and then this can be broken into two parts, now here you can write t y(t) dt because it is a definite integral so we can write 0 to 1 t y(t) dt and write it as minus integral 0 to x (x minus t) into y t dt and that can be broken into two parts 0 to x t y(t) dt plus x to 1 t y(t) dt.

And then this can be written as y(x) equal to x cube by 6 minus half x plus integral 0 to 1 k(x, t) y(t) dt where k(x, t) is what we have 2 minus x minus t where 0 is less than or equal to t less than x. And we have t when x is less than or equal to t less than or equal to 1. This is nothing but 2t minus x 0 less than or equal to t less than x and t when x is less than or equal to t less than or equal to 1.

So we can obtain the Fredholm integral equation of second kind y(x) equal to x cube by 6 minus half x plus integral 0 to 1 k(x, t) y(t) dt where k(x, t) is given by this expression. So this how we convert this boundary value problem into a Fredholm integral equation of second kind. This is what I have to say in this lecture, thank you very much for your attention.