Integral equations, calculus of variations and their applications

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Lecture 32

Cauchy type integral equations- 2

Hello friends, welcome to my second lecture on Cauchy type integral equations. We will begin with Cauchy type integral equation on the real line, the equations that are of the first kind, in the first time we shall consider the similar integral equation integral 0 to 1 this is star means we are considering the principal value of the integral. So integral 0 to 1 gt dt upon t minus s.

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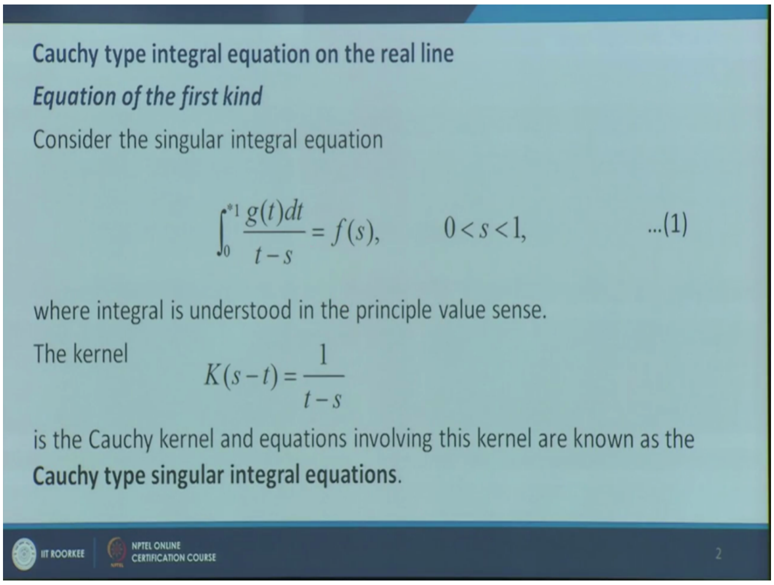

You can see that because of t minus s in the denominator and s lying in the open interval 0 to 1 the integral becomes unbounded over the interval of integration 0 1. So integral 0 to 1 gt dt upon t minus s equal to fs, where the integral is understood in the principal value sense, so this is Cauchy principal value, star denotes the Cauchy principal value. Now the kernel here is is Ks minus t upon t minus s. This kernel is said to be Cauchy kernel and the equations involving such type of kernel are called as Cauchy type singular integral equations.

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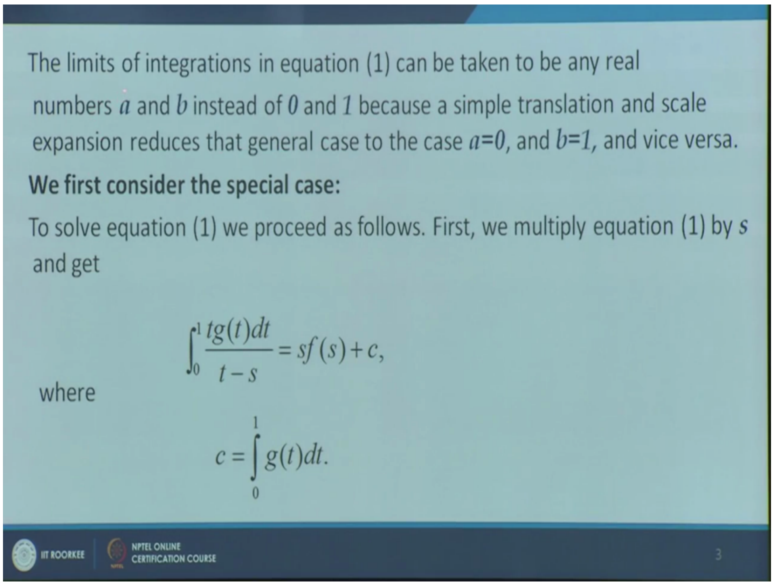

Now here we have taken the limits of integration to be 0, 1 but they could be any real numbers a and b by suitable substitution we can make the limits of integration to 0, 1. So the limits of integration in equation one can be taken to be any real numbers a and b instead of 0 and 1 because by a simple translation and scale expansion we can reduce the integral over a to b to the case integral over 0, 1.

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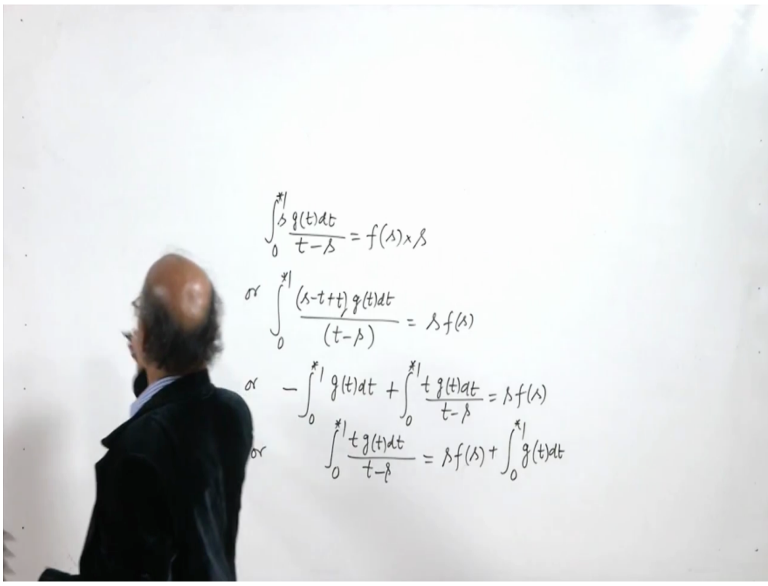

Now let us first consider the special case where the limits of integration are 0 and 1. So to solve the equation 1 we proceed as follows the integral 0 to 1 gt dt upon t minus s, gt dt upon t minus s equal to fs. We can write it as, let us multiply both sides of this equation by s.

So when you multiply by s, since the integration is with respect to t so I can take s inside and this will become then fs into s. So now I can write it as integral 0 to 1 instead of s I can write s minus t plus t times gt dt divided by t minus s equal to s fs. or I can say this is minus integral 0 to 1 gt dt s minus t will cancel with t minus n and we will have a minus1 here plus integral 0 to 1 t times gt dt divided by t minus s which is equal to sfs or we can say integral 0 to 1 t times gt dt divided by t minus s equal to sfs plus integral 0 to 1 gt dt.

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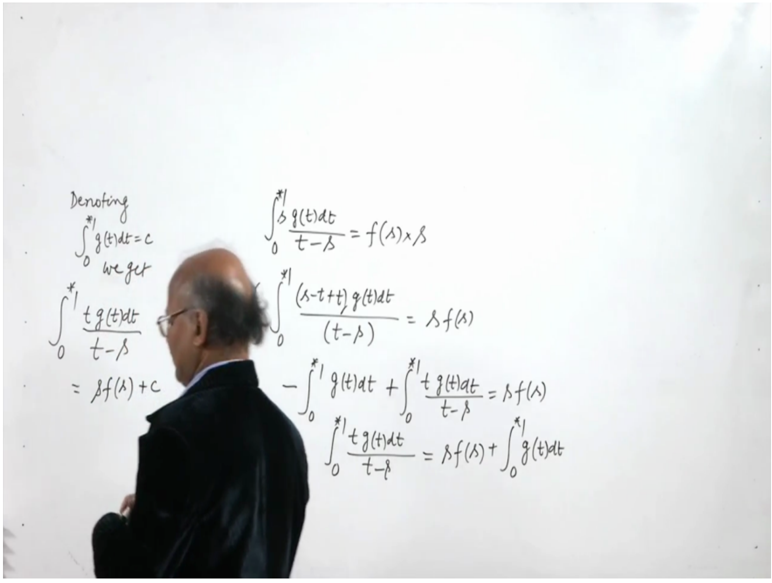

Let us denote because integral 0 to 1 gt dt will be a constant, so let us denote it by c so denoting integral 0 to 1 gt dt by c we have we get integral 0 to 1 t gt dt upon t minus s equal to sfs plus c. Now for convenience we are not writing here star but it is always assume that we are considering the principal value of the Cauchy integral. So for convenience we can simply write integral 0 to 1 without star but it will always mean that we are considering the Cauchy principal value.

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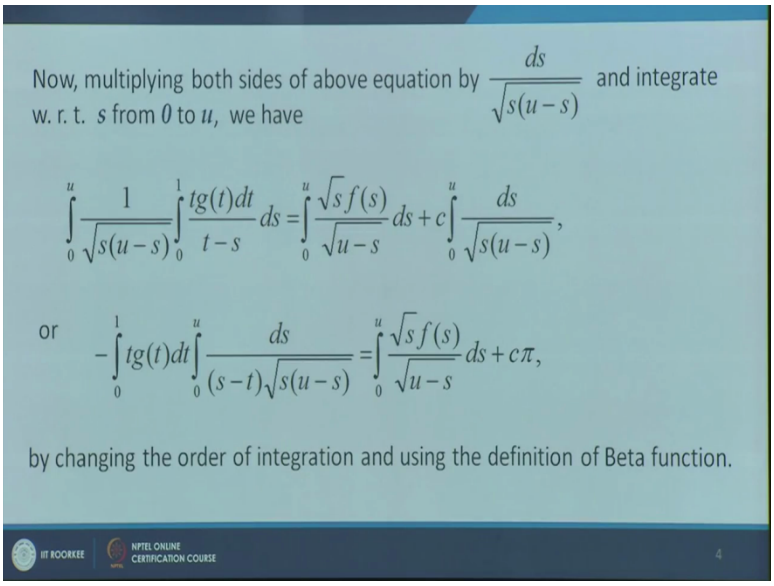

Now multiplying both sides of this equation, let us multiply both sides of this equation by ds over square root s into u minus s and integrate with respect to s from 0 to u. So we shall have integral o to u 1 over square root s into u minus s, integral 0 to 1 tgtdt upon t minus s ds and then we have integral 0 to u root s into fs upon square root u minus s ds because this square root s will cancel with s here and will get square root s left, so square root s fs upon square root u minus s ds plus c times integral 0 to you ds upon square root s into u minus s.

Now s and t are varying independently here, so we can change the order of integration and write this also as minus integral 0 to t, tgtdt than 0 to u ds over this t minus s we are writing as s minus t and put a negative sign here, so s minus t square root s into u minus s and this is same as here integral 0 to u root s fs ds upon square root u minus s the value of this integral is equal to pi.

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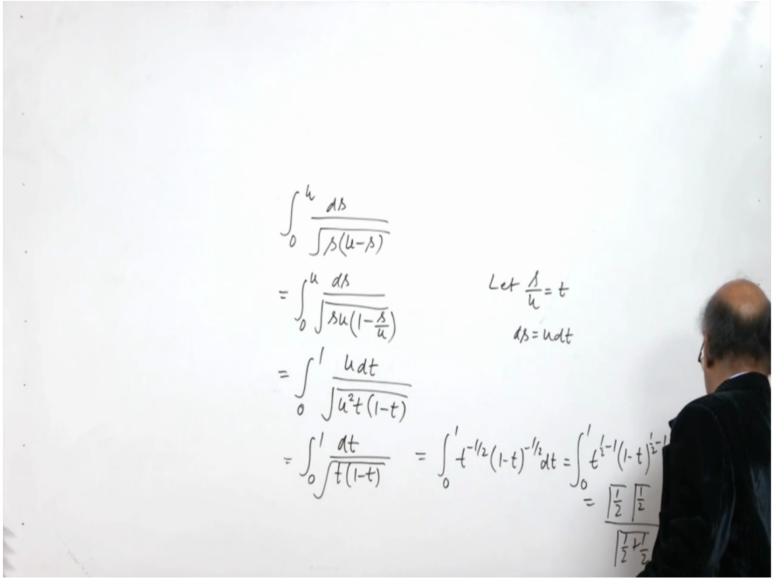

Let us see how it is pi? So integral 0 to u, integral 0 to u ds upon square root s into u minus s, we can write it also as integral 0 to u ds upon s into u 1 minus s by u. Now let us put s by u equal to t. So that ds is equal to udt then this will be equal to when u is when s is 0 t is 0 and when s is equal to u, t is equal to 1. ds is equal to udt and what we will be having the denominator? s is equal to u into t, so u square into t 1 minus t.

So this will be nothing integral 0 to 1, this u square, square root of u square is u, so it will cancel with this and we will get dt upon t times 1minus t square which is equal to integral 0 to1 t to the power minus half, 1 minus t raise to the power minus half dt or we can evaluate it by beta function 0 to 1 t to the power half minus 1, 1 minus t raise to the power half minus 1 dt. So we shall have gamma half divided by gamma half plus half and this is gamma 1, gamma 1 is equal to 1 gamma half is equal to square root pi, so square root pi whole square, so it is pi. So we get c into pi here, so this expression comes after we change the order of integration here and put the value of this integral by using beta function.

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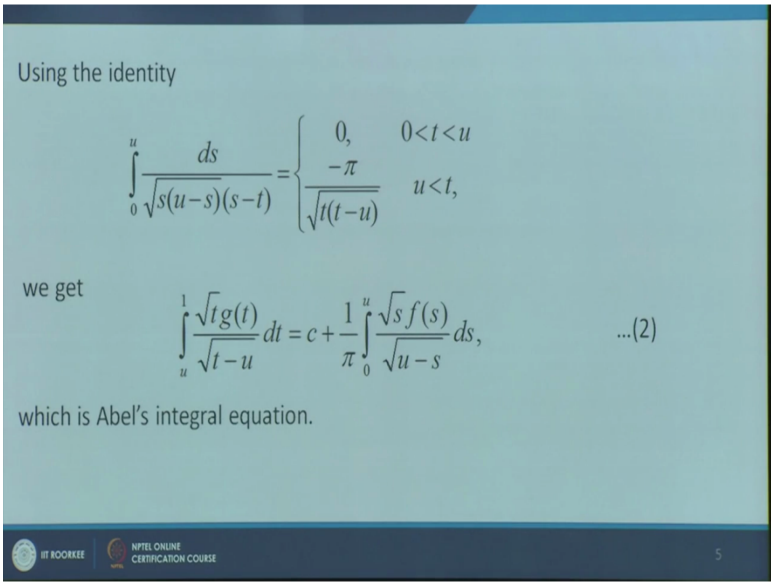

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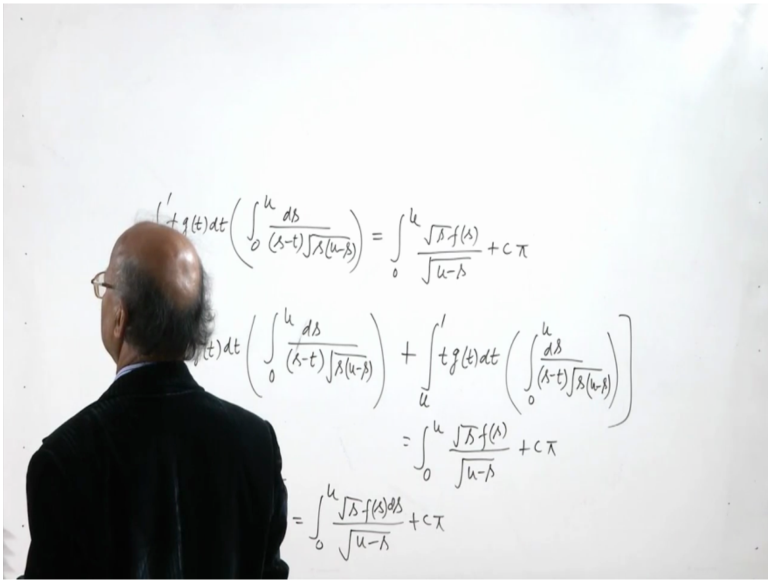

Now let us make use of this identity which we have proved earlier. When you use this identity how we get the expression in equation 2? So let us go back to the equation this one where we have minus integral 0 to 1 tgtdt inside we have integral 0 to u ds upon s minus t square root s into u minus s is equal to 0 to u square root s into fs divided by square root u minus s plus c pi.

Now we go to the identity which we proved, so integral 0 to u ds upon, now what we will do here? Yes the integral is the outer integral is with respect to t where t varies from 0 to 1 we can break this into 2 parts 0 to u and u to 1, so this can be written as minus integral 0 to t tgtdt integral 0 to u ds over s minus t, square root s into u minus s plus integral t to 1, 0 to u not 0.

0 to u and here u to 1 tgtdt integral 0 to u ds over s minus t square root s into u minus s equal to right side. 0 to u square root sfs divided by square root u minus s plus c pi.

Now when we use this identity, when we use this identity then this part integral 0 to u, here t varies from 0 to u, so this expression becomes 0 and therefore this part ventures and here we will put the integral 0 to u ds upon s minus t   root s into u minus s from here because t varies from u to 1, so this will be reduced to when you put the value this will become our integral u to 1 tgtdt, minus integral 0 so this will become pi upon root t into t minus u equal to 0 to u root s fs ds divided by square root u minus s plus c pi and we can write it as , okay.

We can we can further simplify this root t will get cancelled here and we will get root t here, so integral u to 1, root tgtdt upon root t minus u equal to the can divide by pi. So 1 upon pi integral 0 to u root sfsds by root u minus s and then c, so this is how we get this equation and this equation is nothing but this equation is called as the Abels integral equation.

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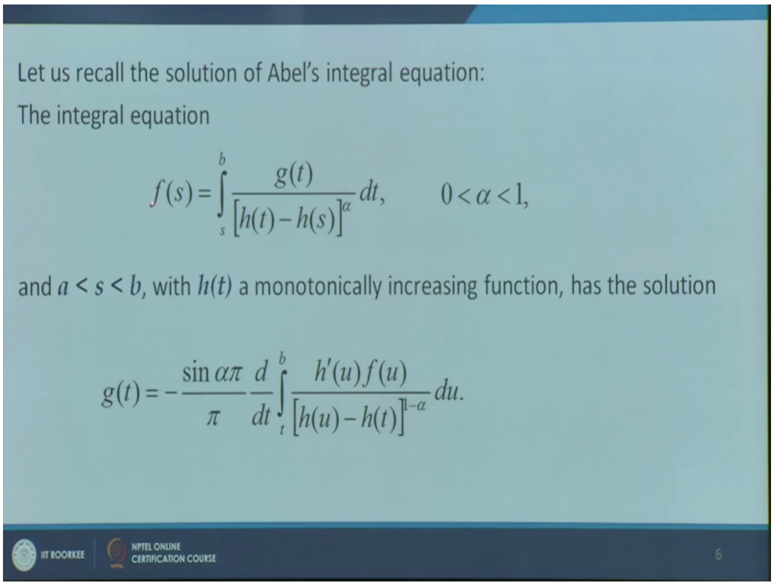

So now let us see how we solve this further? So let us recall the solution of the Abels integral equation, we have earlier studied Abels integral equation of this type, fs equal to integral s to b gt dt upon hy minus hs to the power Alpha where 0 less than Alpha less than one and a is less than s less than b and ht is a monotonically increasing function.

We have seen that it's solution is gt equal to minus sine Alpha pi over pi d over dt, integral t to b h dash u fu over hu minus ht to the power 1 minus Alpha du. So we will make use of this article to proceed further.

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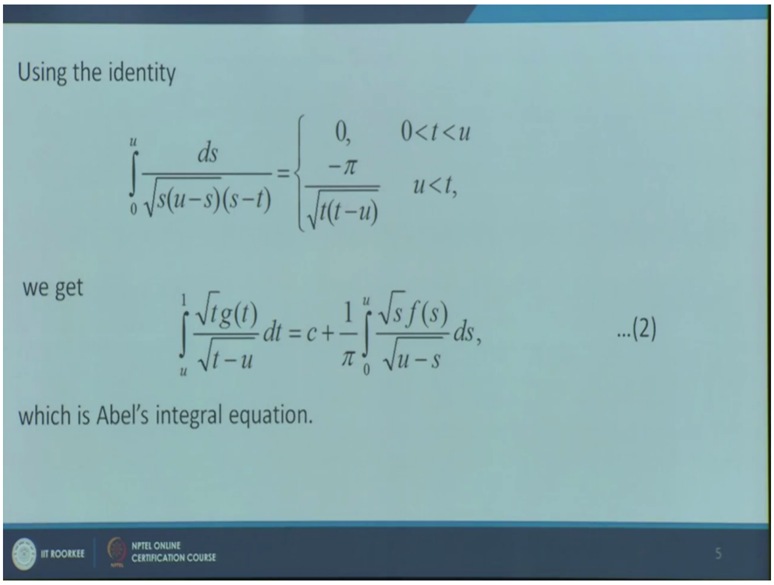

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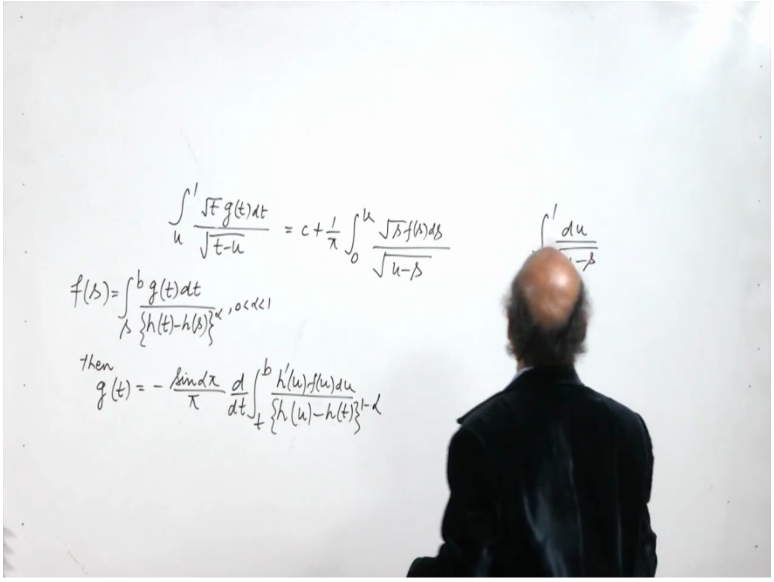

Now let us see what we have? This is our equation integral u to 1, u to 1 root t gt dt divided by square root t minus u this is equal to c plus 1 over pi integral 0 to u root sfs ds divided by square root u minus s and we will be using this results. So let’s see how we make use of this result?

So okay, so let us apply this result to this one. This is square root u to 1, so this side integral u to 1, root tgtdt upon square root t minus u when we solve using this equation, what we will get? we have fs equal to integral s to b gt upon gtdt this is integral equation ht minus hs to the power Alpha, 0 less than Alpha less than 1 when we have this equation then we get gt, gts, okay .

Then gt is equal to, alright. Minus sine Alpha pi divided by pi d over dt of integral t to b, h dash u fudu over htu minus ht to the power 1 minus Alpha, okay. Let us see how we make use of this? So here we will make use of, this integral equation is similar this integral equation is similar to this one instead of s here we have u, here we have one and this is our known function fs.

This is a known thing so this is fs this is fs and this is known to us which is of the integration gives you u. So actually what is happening here is that, the known function after integration gives you a function of u while here we are writing it as a function of s? So instead of s here we have u here and here what is happening? When we integrate, gt I known function that we have to find. So when you apply this what we will get?

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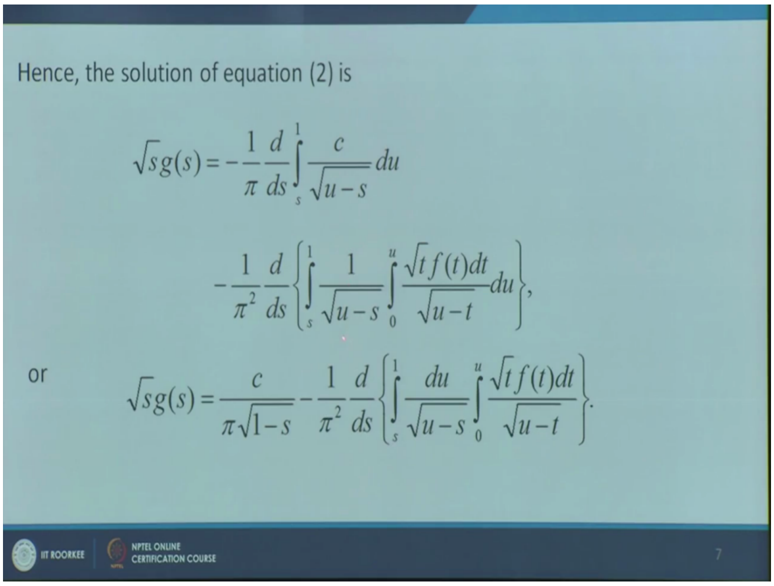

Okay, so when we make use of this solution of Abels integral equation then we will get integral square root s gs equal to minus 1 over pi d over ds integral s to 1, c over under root u minus sdu minus 1 over pi square d over ds integral s to 1, 1 over square root u minus s, 0 to u square root t ftdt over square root u minus t du.

Or we can say that this is square root us into gs c upon under root 1 minus s minus 1 by pi square d over ds, integral s to 1 du upon square root u minus s, 0 to u root t ftdt over root u minus s. So this you can get, how we come here?

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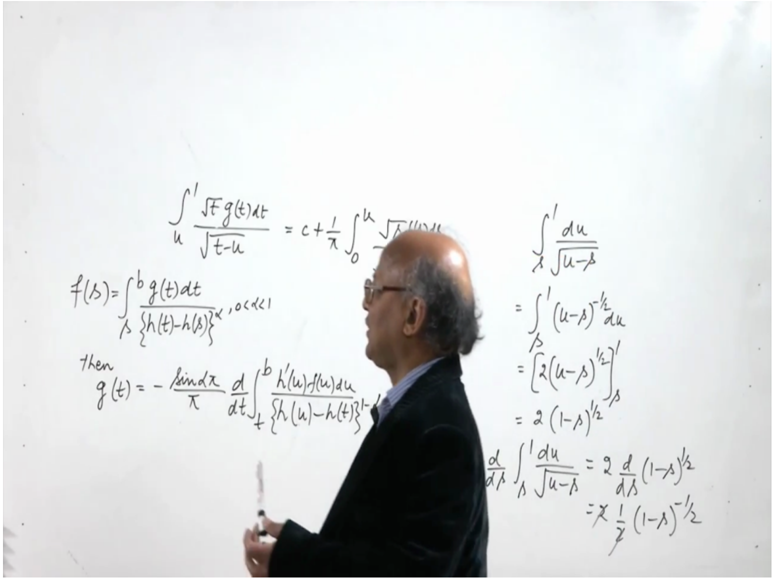

C upon pi root 1 minus you can integrate s to 1 du upon square root u minus s when we integrate, this is what? s to 1 u minus s to the power minus half du, so this gives you after integration u minus s raise to the power half divided by half, so that is 2 times this and then we get here let us put limits, so what we get? 2 times 1 minus s to the power half.

Now we can differentiate it with respect to s, so d over ds of integral s to 1 du upon u minus s square root is equal to 2 times d over ds of 1 minus s to the power half. So this gives you 2 times 1 over 2, 1 minus s to the power minus half. So we get this is how we come here c upon pi under root 1 minus s.

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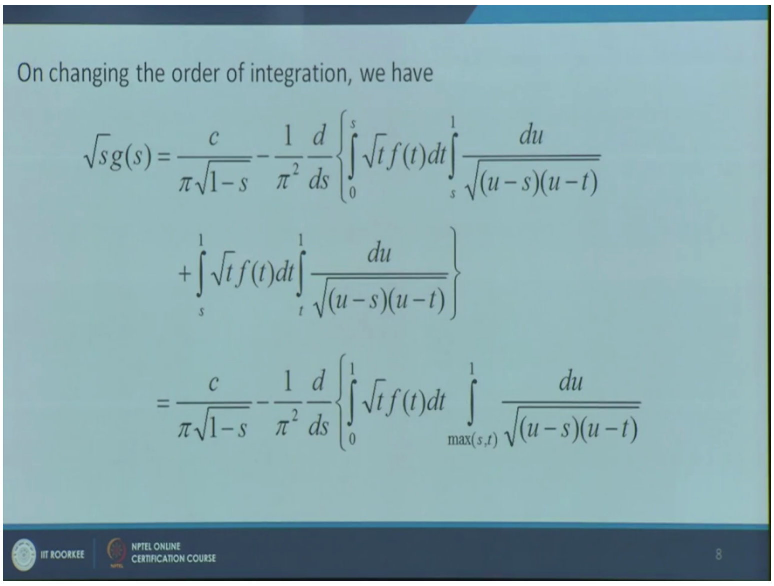

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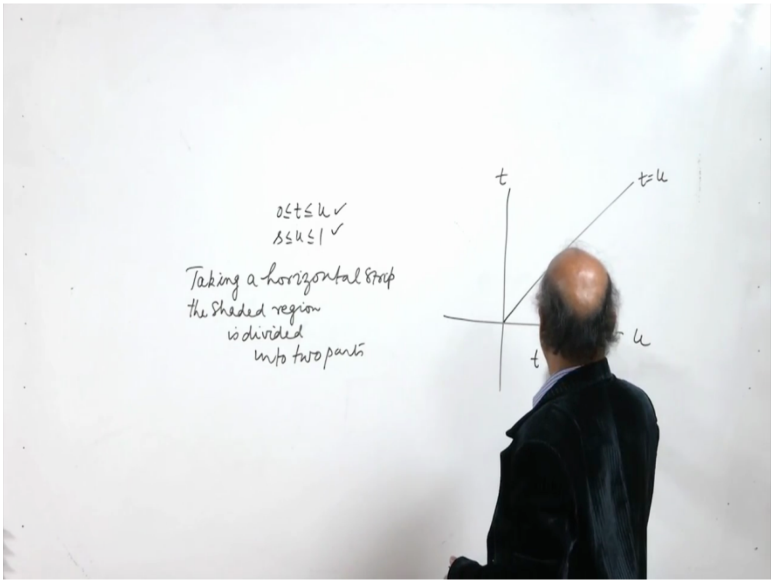

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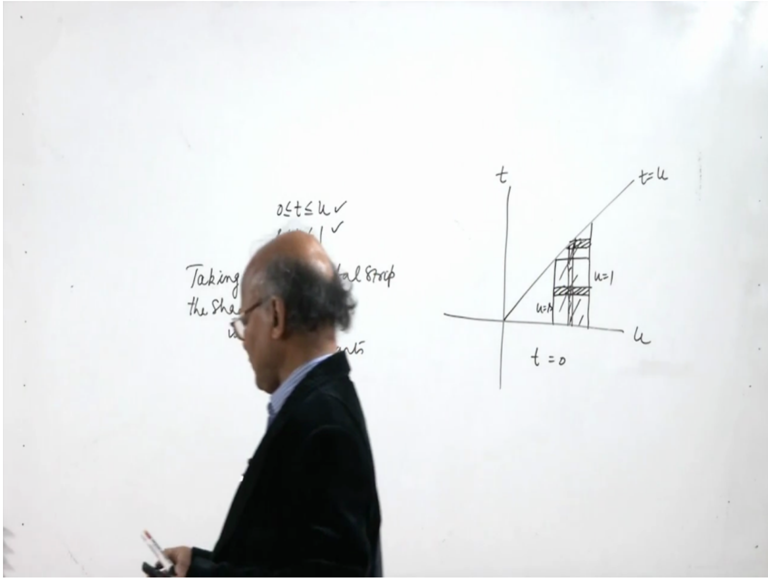

Now let us change the order of integration here. Let us change order of integration here and see what happens, okay. So let us now change the order of integration we have here limits of integration, like this t varies from 0 to u and u varies from s to 1. So if we draw the figure to change the order of integration then this is my t axis, this is u axis; this is the line t equal to u.

Now t varies from 0, so t 0 on the horizontal axis, t varies from 0 to u and u varies from s to 1. So let us say this is u equal to s line and this is u equal to 1. So this is the region over which we are integrating, in this limits of integration we are taking a vertical strip because when you take a vertical strip then for the vertical strip t varies from 0 to u and u varies from s to 1.

Now when we want to convert, when you want to change the order of integration we will have to divide this region into 2 parts and take the horizontal strips. So in this part when we change the when we take up the horizontal strip then for the horizontal strip you can see that u varies from s to 1 and t varies from 0 to t equal to s and when we take in the upper part, in this part then in this part u varies from u varies from t and u goes up to 1 and t varies from s and t goes up to 1 and t varies from s and t goes up to 1. So this is how we change the order of integration and then we have t upon pi square root 1 minus s minus 1 pi square d over ds.

Now here integral s to 1 and t is less than s because t varies from 0 to s, here t is more than s because t varies from s to 1. We can combine these 2 integral is and write integral 0 to 1 square root t ftdt maximum over s,t, maximum st 1 du upon square root u minus s into into u minus t because when you can break this 0 to 1 into 2 parts, 0 to s and s to 1 and then this expression will give you this one and this one on changing the order of integration and we can combine these 2 integral is in this form.

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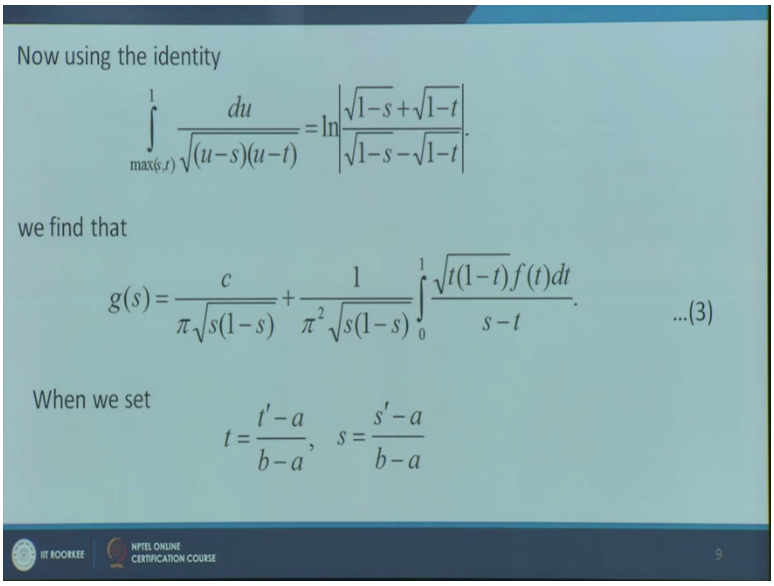

Now let us use the identity which we have proved earlier maximum st to 1 du upon square root u minus s, u minus t equal to this. We find that gs comes out to be c upon pi root s into 1 minus s, 1 by pi square into square root s into 1 minus s integral 0 to 1 square root t 1 minus t ftdt over s minus t.

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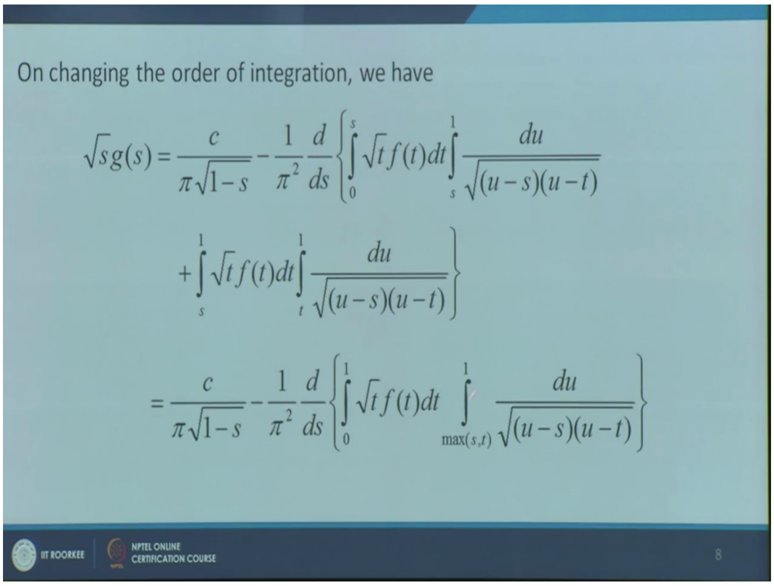

Now we will have to differentiate this expression with respect to s because of d over ds that we can do by considering 2 cases when t is greater than s and s greater than t. So by taking 2 different cases we can differentiate this expression with respect to s and we will get these quantities. So now what we do is we put t equal to t dash minus a over b minus a, s equals to s dash minus a over b minus a to change interval of integration from 0, 1 to a,b.

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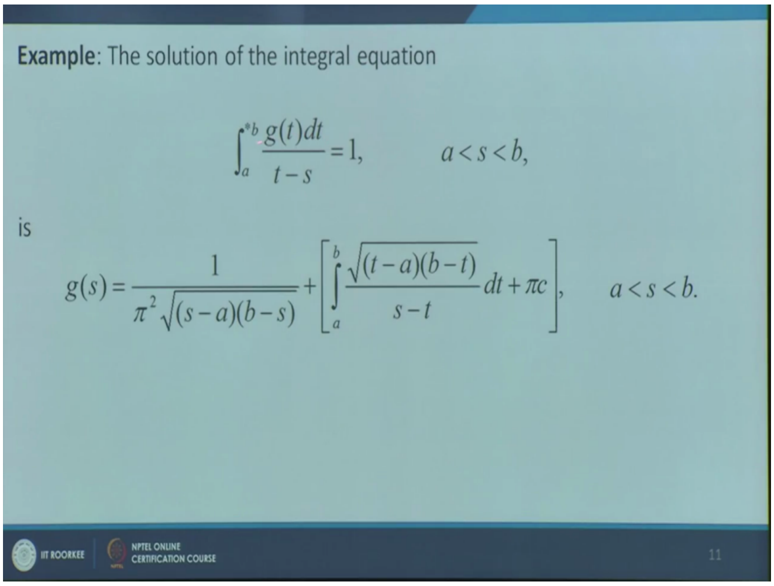

And then we get the solution to this equation, the principal value of integral 0 to 1 gtdt upon t minus s ft a less than s less than b to b gs equal to this expression. So this is a solution of the Cauchy integral equation of first kind. For example if we have this equation integral a to b gtdt upon t minus s equal to 1a less than s less than b then here the function fs is given to be equal to one.

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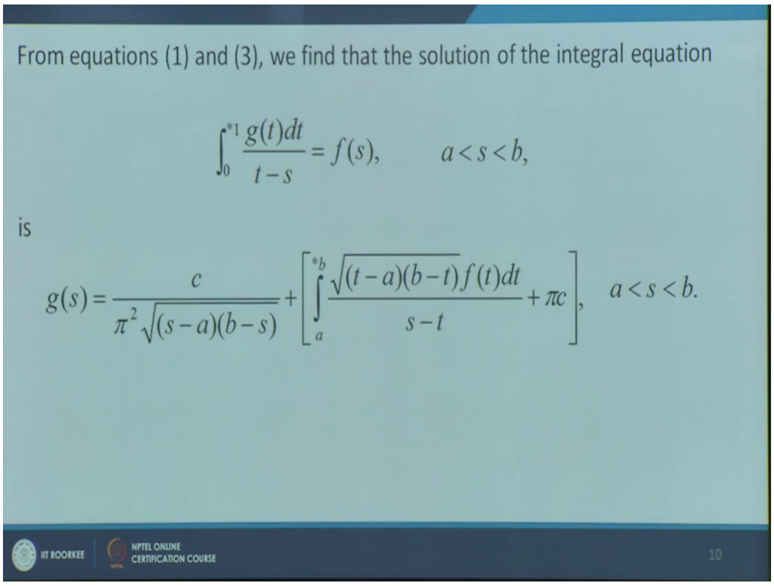

So making use of that we can put it as gs equal to c upon pi square, square root s minus a b minus s a is equal to 0, b equal to1 we are given. So c upon pi square under root s into 1 minus s 0 to 1 square root t into 1 minus t and ft is given to be equal to 1. So we have this dt integral, interval of integration is a,b okay sorry. Interval of integration is a,b so we will get this you can simply both fts1 here and will get the solution of Cauchy integral equation of first kind, with that I would like to conclude my lecture, thank you very much for your attention.