Integral equations, calculus of variations and their applications
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Lecture 34
Cauchy type integral equations-4

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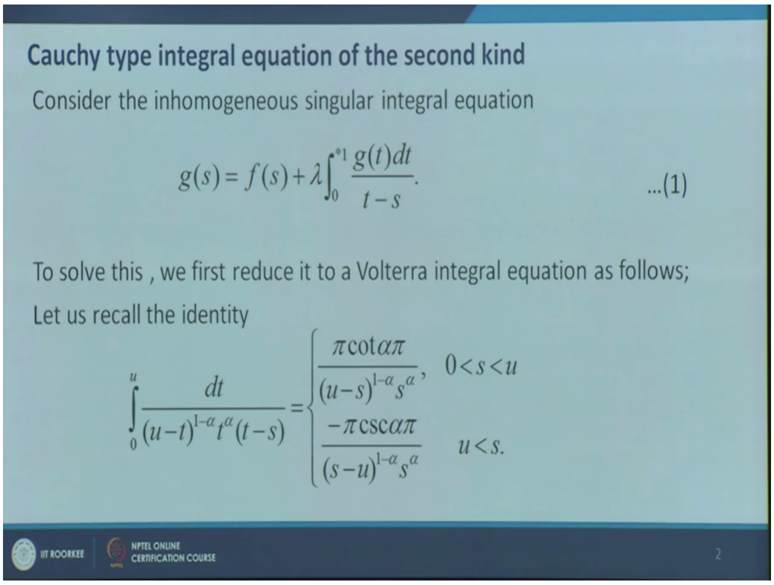

Hello friends welcome to my fourth lecture on Cauchy integral equations. So we will be solving Cauchy type integral equation of the second kind. Let us consider the non-homogeneous similar integral equation gs equal to fs plus lambda times integral 0 to 1 gtdt over t minus s. Let us recall that by the Star we mean that we are considering the Cauchy principal value of the integral 0 to 1 gtdt over t minus s.

Now in order to solve the singular integral equation we shall reduce it to a Volterra integral equation, how we do it? Let us see, we will need this identity which we have shown in the last lecture. So integral 0 to y dt upon u minus raise to power minus Alpha T to power Alpha into t minus s equal to pi cot Alpha pi over this and then minus pi cosec alpha pi over s minus u to the power 1 minus Alpha s to the power Alpha.

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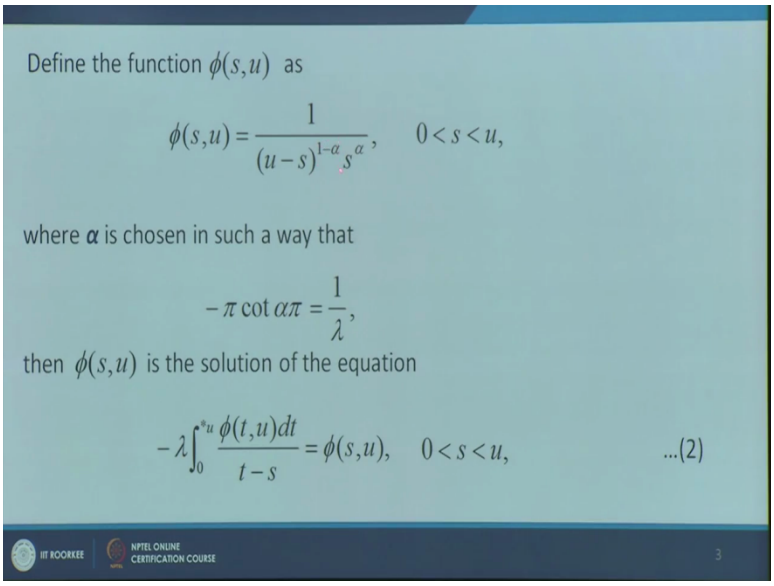

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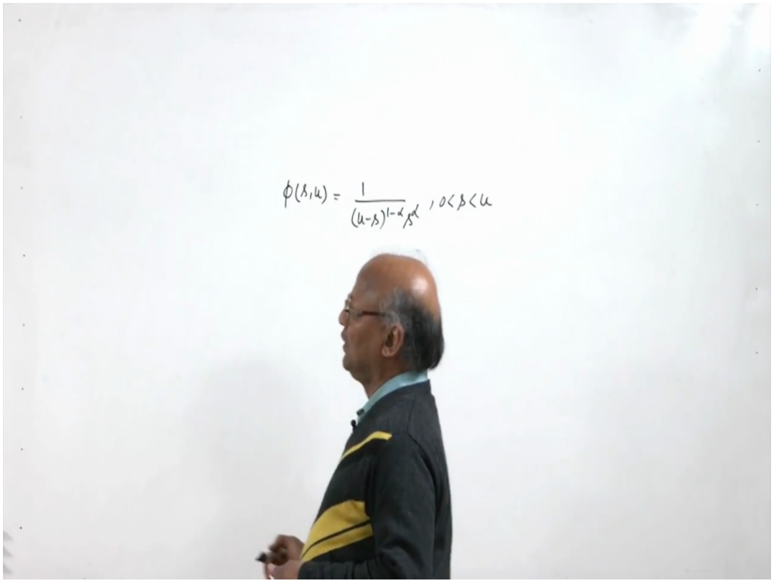

So we may need this identity, now what we do is let us defined a function Phi s u as phi s,u equal to 1 minus u to the power 1 minus alpha s to the power Alpha, where 0 is less than S less than u. Phi s,u equal to 1 over u minus s to the power Alpha where 0 is less then s less than u. Phi s,u equal to 1 over u minus s to the power 1 minus Alpha, s to the power Alpha when 0 is less then s less than u.

And let us choose alpha in such a way that minus pi cot Alpha pi equal to 1 by lambda, where lambda is the parameter in the given integral equation. Now then Phi s,u is the solution of this equation minus pi integral 0 to u phi t,u dt upon t minus s phi equal to phi s, u. So minus lambda integral 0 to phi t,u dt over t minus s equal to phi s,u 0 less than s less than u and when s is greater than u minus lambda 0 to u, phi t, u dt over t minus s equal to minus pi cosec Alpha pi divided by s minus u to the power 1 minus Alpha, s to the power Alpha t, u less than s.

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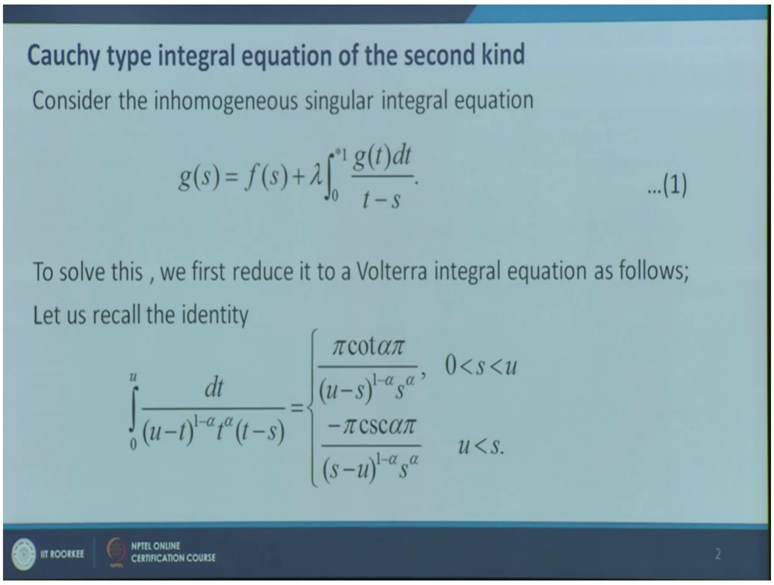

Now how we are saying this? This follows from the identity, we have seen that integral 0 to u dt upon u minus s to the power 1 minus Alpha, t to the power Alpha t minus s equals to this quantity.

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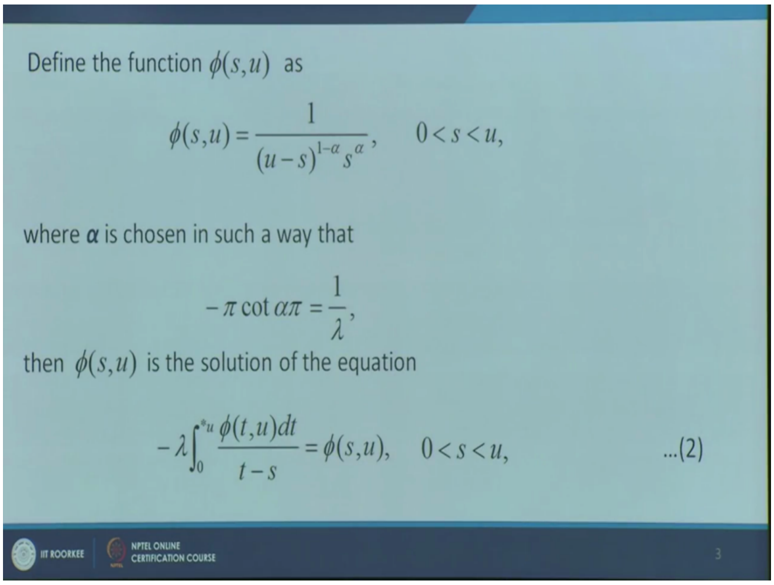

So if you choose your alpha like this minus pi cot Alpha pi equal to one over lambda then and then you use this identity then you use this identity for the case 0 less than u less than 1, 0 less then u less than u then you get this one, this equation.

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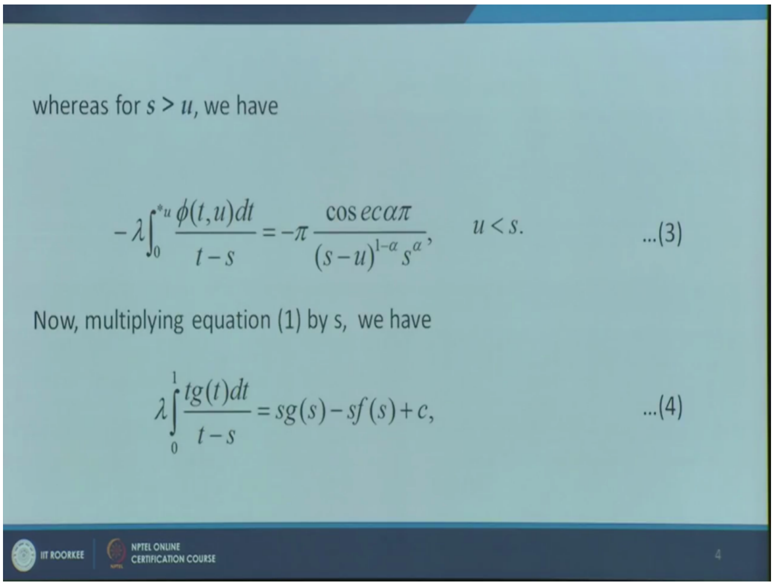

And when you take s to be greater than u then you get this equation.

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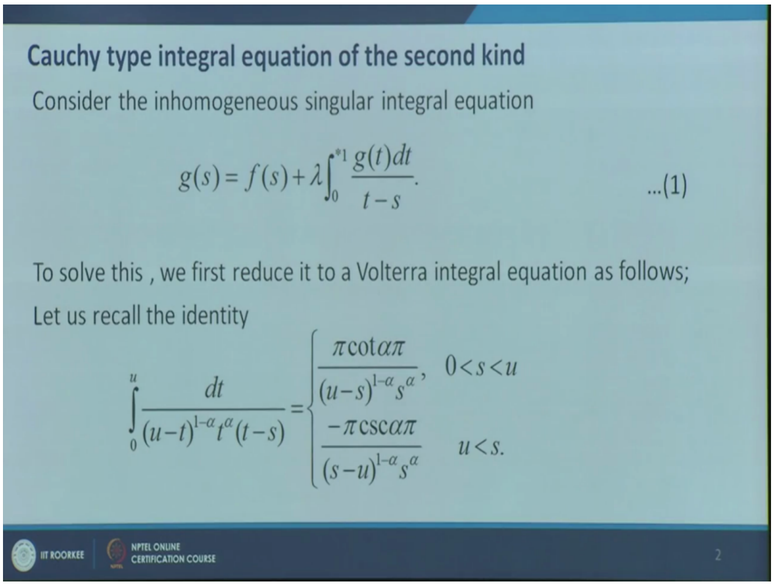

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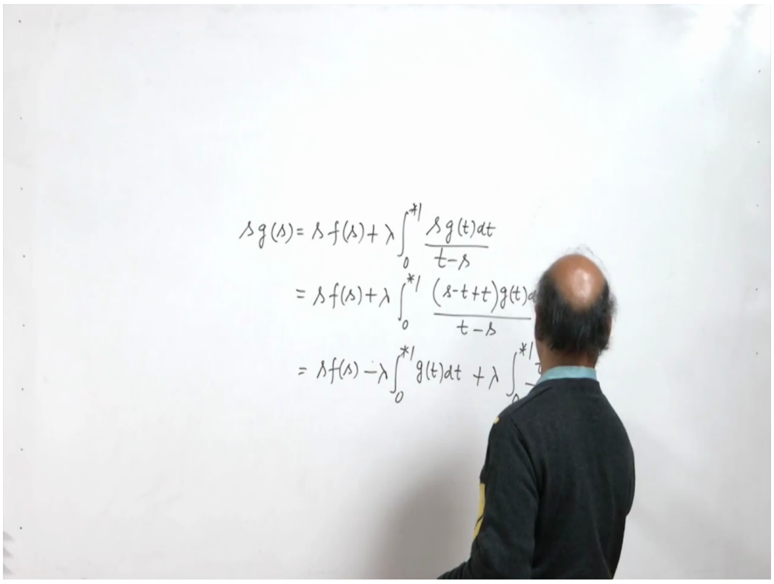

Now what we do is, let us multiply equation 1 by s, so this is our equation 1, in this equation we are multiplying by s, so we get sg s equal to s fs plus lambda times integral 0 to 1, sg tdt divided by t minus s, I can also write this as s fs plus lambda times integral 0 to 1, s minus t plus t into gt dt divided by t minus s which is also equal to s fs plus, okay. S minus t we can cancel with t minus s with the negative sign, so minus lambda 0 to 1 gt dt plus lambda times integral 0 to 1 t gt dt over t minus s, okay.

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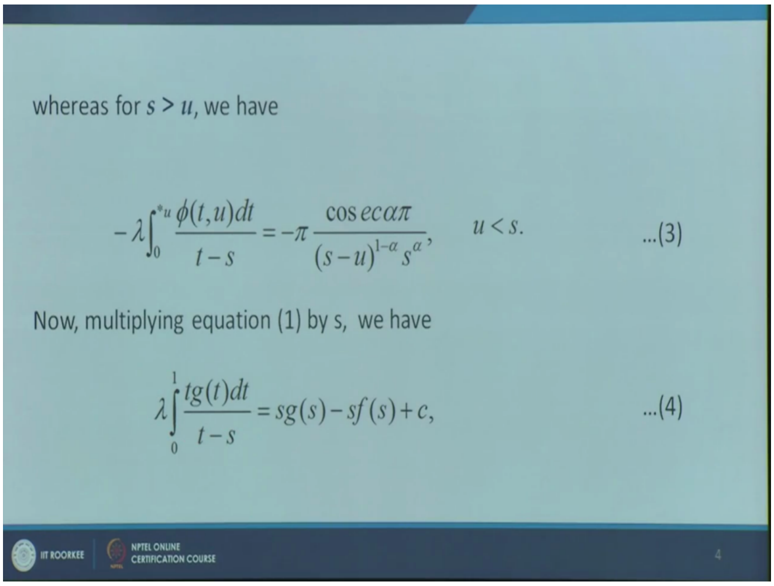

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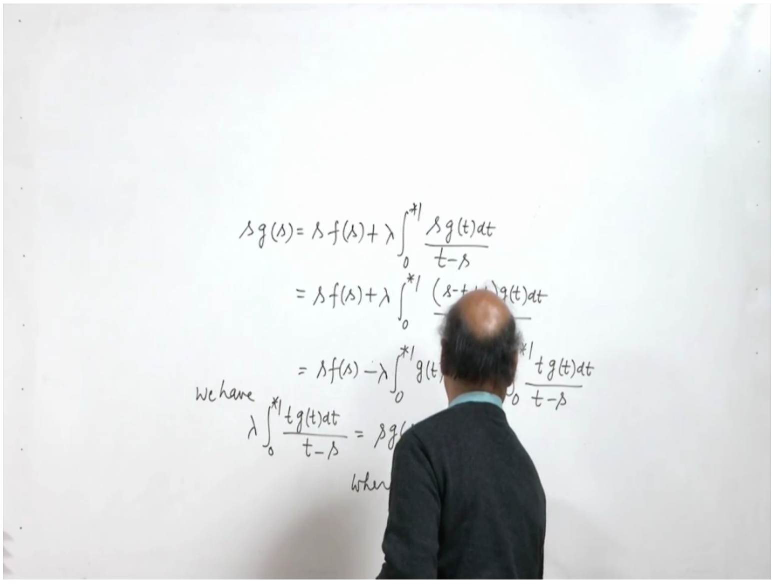

So we multiply equation 1 by s and then we have this, so lambda times you can see lambda times integral 0 to 1 t gt dt divided by t minus s equal to s gs minus s fs and then we write plus c, where c is lambda times integral 0 to 1 gt dt, so we have lambda times integral 0 to 1 t gt dt divided by t minus s is equal to s gs minus s fs plus c where c is equal to lambda times integral 0 to 1 gt dt, okay so this is what we have.

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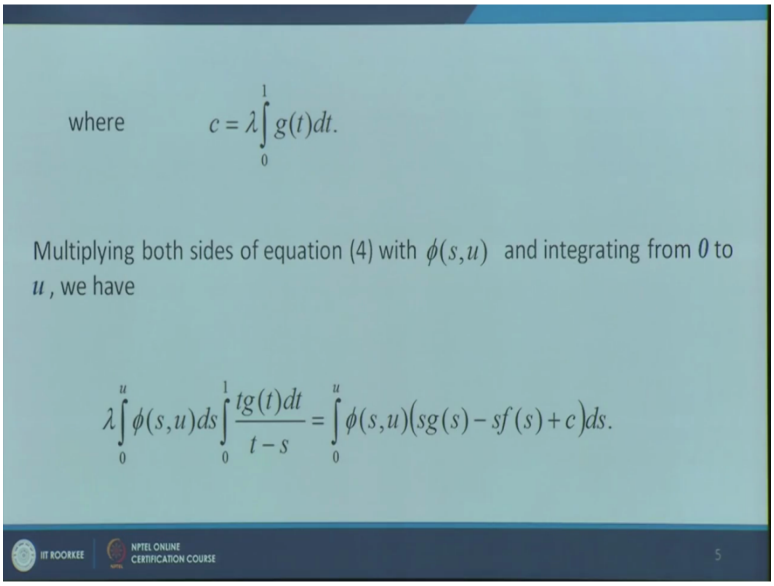

Now let us multiply both sides of equation 4 this equation, so this equation we are getting. So let us multiply both sides of this equation by phi s,u and integrate from 0 to u, let us see what we get? So the question is lambda times integral 0 to 1 gtdt upon t minus s we multiply by phi s,u and integrate with respect to u from 0 to u, so we get this lambda times integral 0 to you phi s,u ds integral 0 to 1 tg tdt upon t minus s equal to integral 0 to u phi s,u into sgs minus sfs plus cds.

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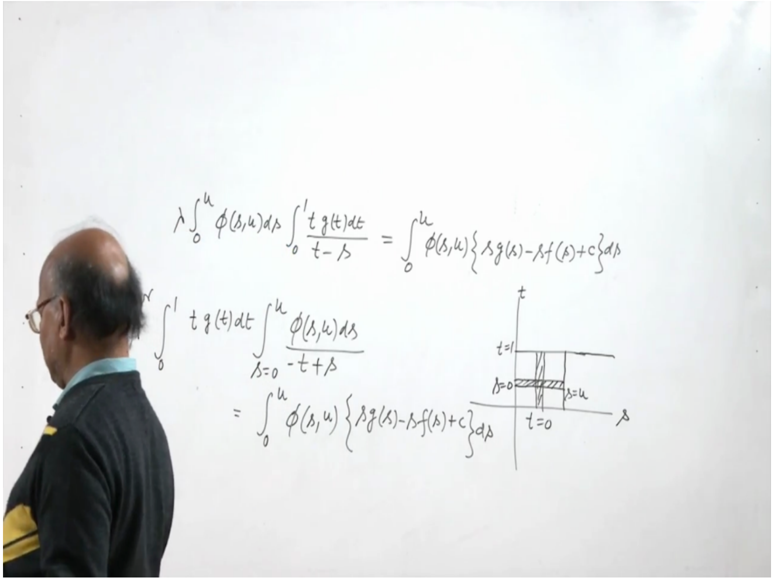

Now let us change the order of integration. So let us see how we change the order of integration? So we have lambda times 0 to u phi s,u ds 0 to 1 tg tdt divided by t minus s equal to integral 0 to u phi s,u into sgs minus sfs plus cds, so we want to change the order of integration in the left inside, so let us see t where is from 0 to 1, so this is t equals to 0 this is t equal to one. T varies from 0 to 1 and s varies from 0 to u.

So this is x axis this is t axis, s is 0 here say s equal to u here. So we are integrating first with respect to T which means that we are taking a vertical strip in this region, for the vertical strip t varies from 0 to 1 and vertical strip starts from s equals to 0 and ends at s equals to u, so s varies from 0 to u. Now we will take a horizontal strip in this region, so when we take a horizontal strip in this region, we shall write the limits of integration for s first, so varies from 0 to u and t varies from 0 to 1.

So we will have or lambda times integral 0 to 1 t gtdt integral 0 to t varies from 0 to 1 and s varies from 0 to u, so s varies from 0 to u phi s,u ds divided by t minus s equal to integral 0 to u phi s,u sgs minus sfs plus c. So for changing the order of integration, now what we do is, the integration with respect to t is broken into 2 parts from 0 to u and from u to 1.

So we write it as sum of 2 integrals, integral over 0 to u where t varies from 0 to and then the other integral where t varies from u to 1. So we have minus lambda integral 0 to you t gtdt integral over 0 to u phi s,uds over s minus t I can we can bring here negative sign outside and write here s minus t. So we have here minus lambda times this and then which is equal to integral over right inside can be written as integral over 0 to u phi s,u into s gsds minus integral over 0 to u sfs phi suds and c times integral over 0 to u phi s,u ds.

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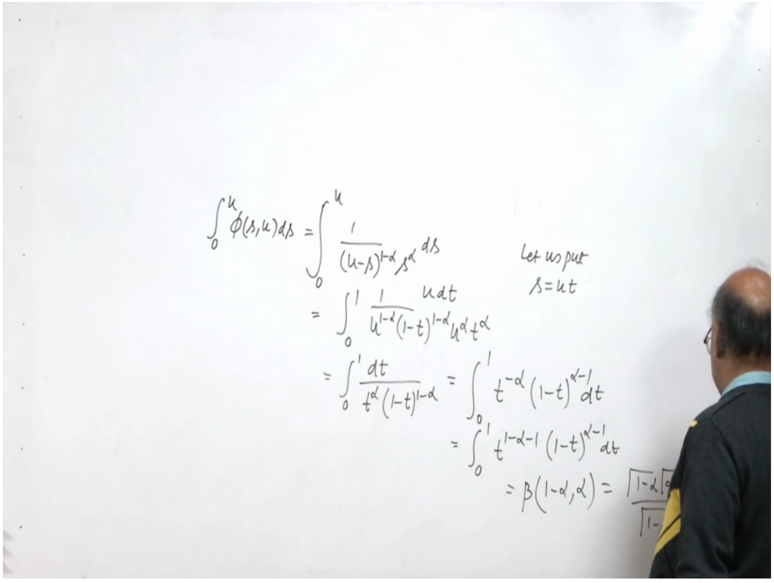

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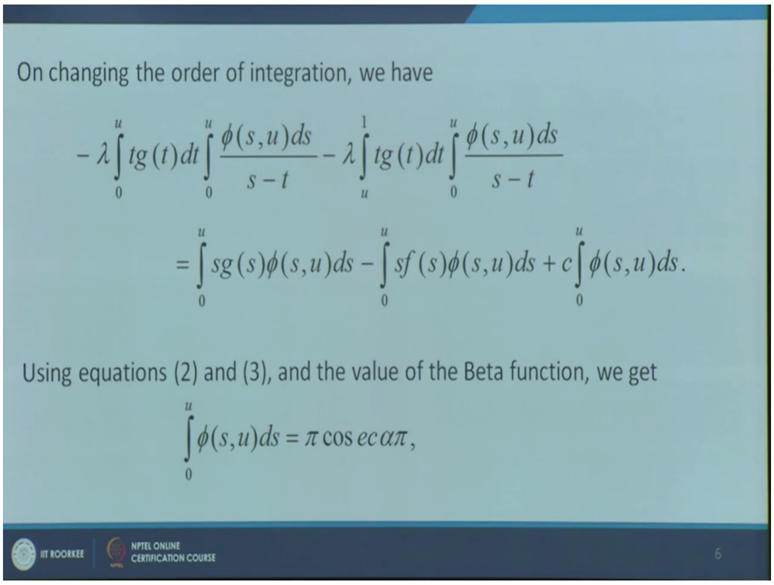

Now we know that integral, let us first evaluate integral over 0 to u phi s,u ds. So integral over 0 to you phi s,u ds we have assumed to be equal to 1 over u minus s to the power 1 minus Alpha into s to the power Alpha, this is our 0 to u and ds. As we have done earlier also you can put here u equal to st let us put u equal to say s into t then we will have du equal to, no.

We are writing it s minus s, no u minus s; it is not s minus u. So this is 1 over u minus s to the power 1 minus Alpha s to the power Alpha ds. So let us put s equal to ut then this will change to 0 to 1 because when s is 0, t is 0, when is u, t is 1 and 1 over u to the power 1 minus Alpha, 1 minus t to the power 1 minus Alpha, t to the power Alpha ds will be udt. So we will get udt here and this will be integral 0 to 1 dt upon t to the power Alpha 1 minus t to the power 1 minus Alpha t to the power minus Alpha 1 minus t to the power Alpha minus 1 dt.

So this is integral 0 to 1 t to the power 1 minus Alpha minus 1, 1 minus t to the power Alpha minus 1 dt this is beta function 1 minus Alpha Alpha. So this is gamma 1 minus Alpha, gamma Alpha, gamma 1 minus Alpha plus Alpha, Gamma 1 is equal to one and when Alpha lies between 0 and 1 gamma Alpha and gamma 1 minus Alpha is Pi over sine Alpha pi. So we have, so integral over 0 to u, phi s,u ds is pi over sine alpha pi.

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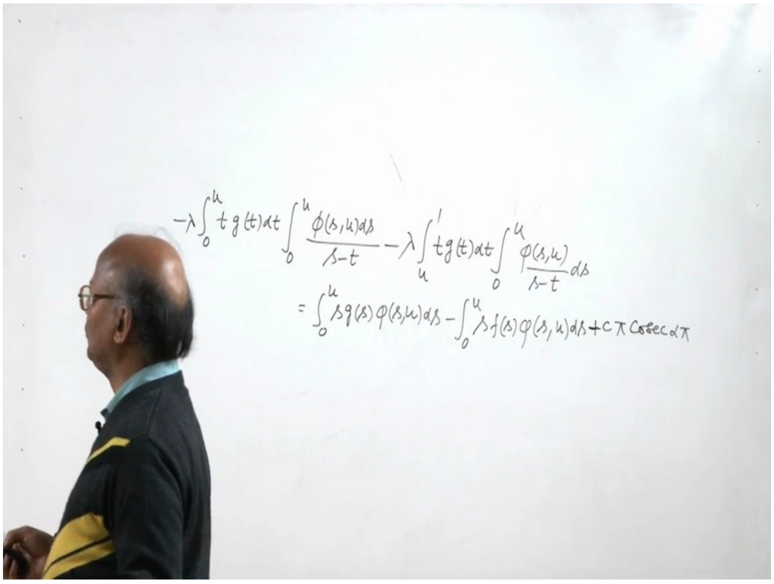

Now let us see how we use equations 2 and 3, what we have is minus lambda integral 0 to u t gt dt integral 0 to u, phi s,u ds over s minus t minus lambda u to 1 t gtdt integral 0 to u, pi s,u ds s minus t right hand side we have calculate at the value of 0 to u phi s,u ds, so right-hand side is 0 to u s gs phi s,u ds minus 0 to u s fs phi s,u ds and then plus c times pi into cosec Alpha pi.

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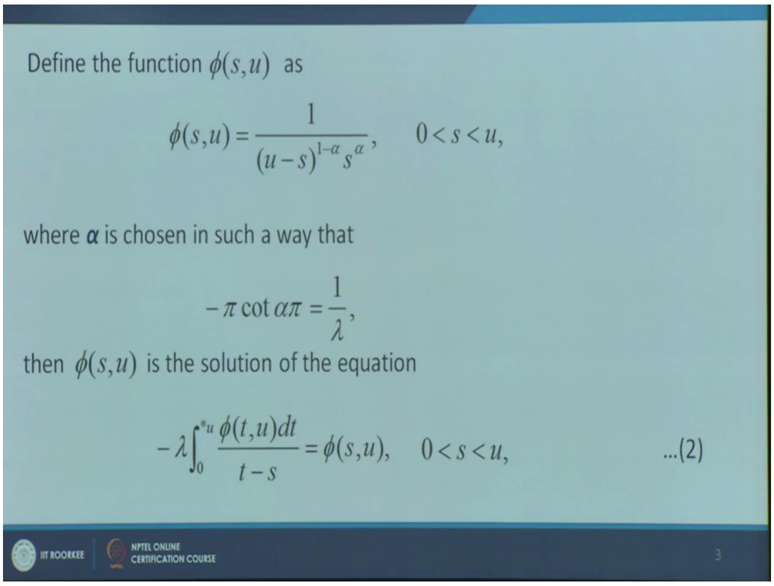

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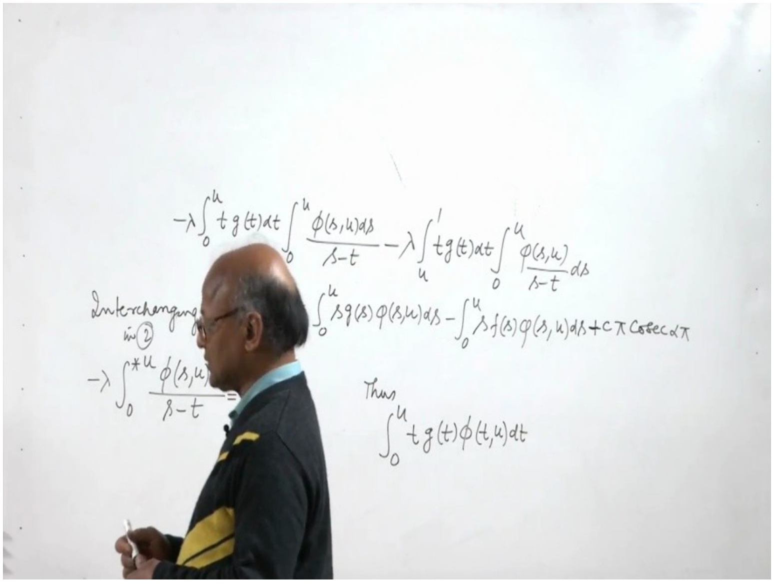

In the left hand side we will use equations 2 and 3, so let us see how we use equations 2 and 3? Now in the equation 2 we have integral over minus lambda 0 to u phi t,u dt over t minus s equal to phi s,u. So let us interchange s and t here. When you interchange s and t here, when you interchange s and t in equation 2, interchanging s and t in equation 2, we will have minus lambda integral 0 to u phi s,u ds divided by s minus t equal to phi t, u when t lies between 0 to u.

So when t lies here between 0 to u the value of minus lambda into 0 to u phi s,u ds upon s minus t can be written as phi t, u. So the first term, so thus we have integral 0 to u tgt into phi t,u dt, the first term becomes this.

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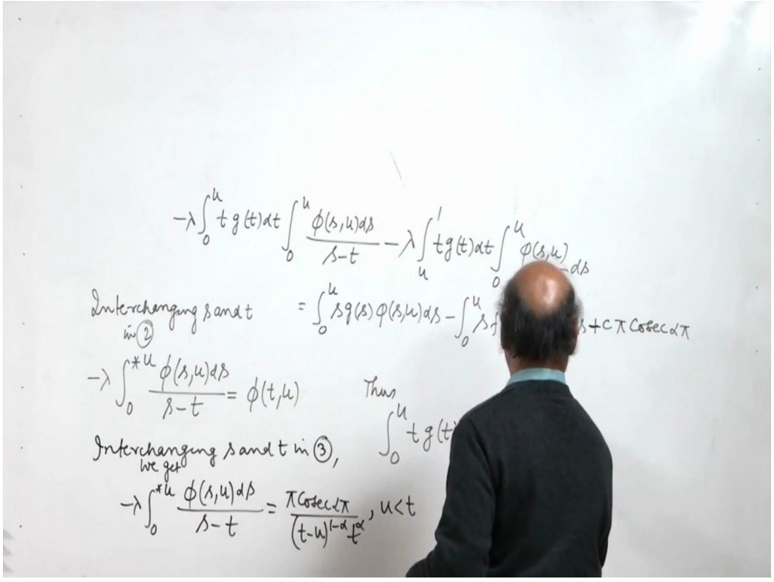

Now let us see the second term, so in case when s is greater than u we have this identity. So let us again change interchange s and t here, so interchanging s and t, we have s and t in equation 3 we get minus lambda phi s,u ds divided by s minus t equal to pi cosec Alpha pi divided by s minus u, so t minus u, t minus u to the power 1 minus Alpha t to the power Alpha u is less than t.

So interchanging s and t in equation 3, interchanging s and t in this equation, we arrive at minus lambda 0 to u, phi s,u ds divided by s minus t equal to pi cosec Alpha pi over t minus u to the power 1 minus Alpha t to the power Alpha when u is less than t and then we put this value here. so what we get is tgt and then pi cosec Alpha pi divided by t minus u to the power 1 minus Alpha t to the power Alpha dt and the right-hand side is integral 0 to u sgs phi s,u ds plus minus integral 0 to u sfs this and then c times pi cosec Alpha pi.

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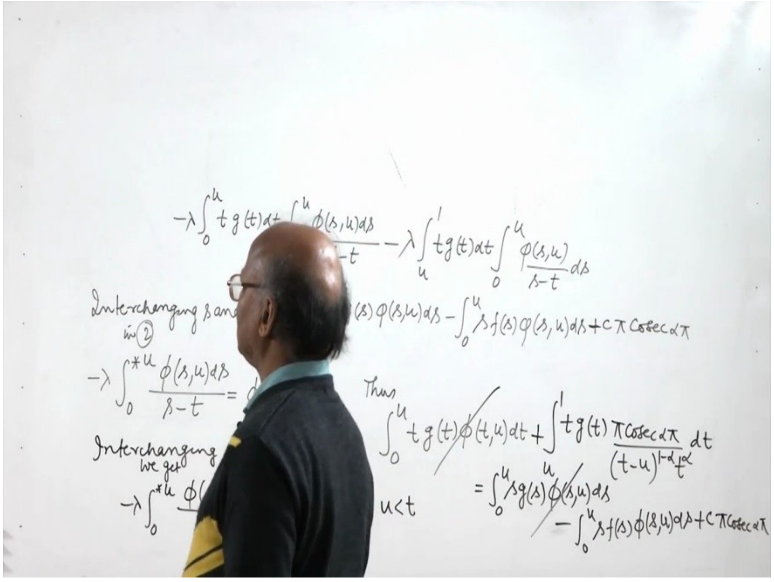

Now let us see this first-term and this first-term they cancel only change is in the variable of integration. Here we have t, here we have the variable of integration s, so in the definite integral the variable of integration doesn’t matter , so this integral is the same value as this integral therefore they cancel out and so what we get is this equation lambda pi cosec Alpha pi integral 0 to 1 t to the power 1 minus Alpha is we can write t to the power alpha as t to the power 1 minus Alpha gt dt upon t minus u to the power 1 minus Alpha dt equal to minus 0 to u sfs phi s,u ds plus c pi cosec Alpha pi.

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Hence, we obtain

$$\lambda\pi \csc \alpha\pi \int_u^1 \frac{t^{1-\alpha} g(t)}{(t-u)^{1-\alpha}} dt = -\int_0^u sf(s)\phi(s,u)ds + c\pi \csc \alpha\pi . \quad ...(5)$$

This is an Abel type integral equation.

Let us recall that the integral equation

$$f(s) = \int_s^b \frac{g(t)}{[h(t)-h(s)]^\alpha} dt, \qquad 0<\alpha<1,$$

and $a < s < b$, with $h(t)$ a monotonically increasing function, has the solution

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$$g(t) = -\frac{\sin \alpha\pi}{\pi} \frac{d}{dt} \int_t^b \frac{h'(u) f(u)}{[h(u)-h(t)]^{1-\alpha}} du.$$

Hence the solution of (5) is

$$\lambda t^{1-\alpha} g(t) = \frac{\sin^2 \alpha\pi}{\pi^2} \frac{d}{dt}\left[\int_t^1 \frac{du}{(u-t)^\alpha} \int_0^u sf(s)\phi(s,u)ds\right] + \frac{c\sin \alpha\pi}{\pi(1-t)^\alpha},$$

$$\lambda t^{1-\alpha} g(t) = \frac{\sin^2 \alpha\pi}{\pi^2} \frac{d}{dt}\left[\int_t^1 \int_0^u (u-t)^{-\alpha} (u-s)^{(\alpha-1)} s^{1-\alpha} f(s) ds du\right] + \frac{c\sin \alpha\pi}{\pi(1-t)^\alpha}, \quad ...(6)$$

$$0<t<1.$$

Now this integral equation is of Abel's type, let us recall that when we discuss the solution of this Cauchy integral equation fs equal to integral s to b gtdt upon ht minus hs to the power Alpha where 0 less than Alpha less than 1 and s lies between a and b with ht a monotonically increasing function we have the solution as this. Gt equal to minus sine Alpha pi over pi d over dt, so we are making use of this solution to arrive at the solution of this equation, let's see how we get?

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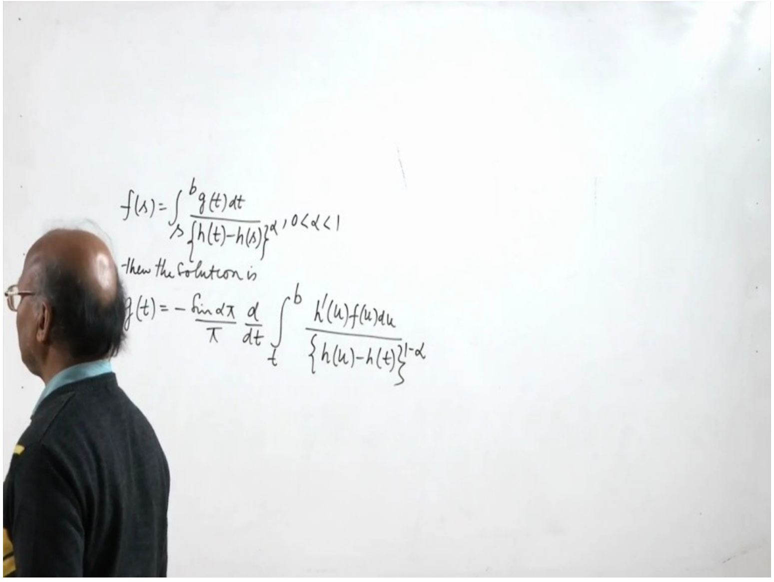

Okay, so fs is equal to when fs is equal to integral over s to b gtdt divided by ht minus hs to the power Alpha 0 less than Alpha less than 1, when we have this integral equation where ht is a monotonically increasing function then the solution is this gt equal to minus sine Alpha pi over pi d over dt integral t to b h time u fu du divided by hu minus ht raise to the power 1 minus Alpha.

Now let us compare, so our integral equation is this we want to know the function gt we want to solve this equation means we want to know the unknown function gt this is known function. We can divide this known quantity by c pi cosec Alpha pi and then the left hand side would be integral u to 1, t to the power 1 minus Alpha gt over t minus u to the power 1 minus Alpha t minus u to the power 1 minus Alpha dt.

So when we compare with this, what we notice is that here we have in place of s we have u here and so hs is nothing but hu, hu is equal to u ht is equal to t, when ht equal to t it is a monotonically increasing function because x prime t is equal to one, so ht is strictly increasing and so we can apply this solution of this integral equation. So what we have is integral u to 1, t to the power 1 minus Alpha gtdt upon t minus u to the power 1 minus Alpha when we compare with this in place of, so in place of s we have u and in place of b we have 1 here.

So what we have? In place of gt here we have t to the power 1 minus Alpha gt, so we will have this solution t to the power 1 minus Alpha gt we can have lambda also, this lambda also together with this known function, so lambda times t the power 1 minus Alpha gt is our unknown thing.

So we can say lambda times t to the power 1 minus Alpha gt and then we can divide by this quantity pi cosec Alpha pi here on the right side, so then we will get sine square Alpha pi upon pi square d over dt of then integral t to 1 because b is 1 here and then h prime u, hu is equal to u, h prime u is equal to 1. So then we have du in place of fu we write this function the known function and then we have u minus t to the power here we will have, yeah.

Here we had Alpha but here we have 1 minus Alpha. So Alpha will be replaced by 1 minus Alpha and therefore we have here u minus t raised to the power Alpha because in the solution Alpha will be replaced by 1 minus Alpha, so this is what we have and then see sine Alpha pi over pi times 1 minus t to the power Alpha or we can write it further as lambda times t to the power 1 minus Alpha into gt sine square Alpha pi over pi square d over dt of integral t to 1 0 to u, u minus t raised to the power minus Alpha, u minus s to the power Alpha minus 1 s to the power 1 minus Alpha fsds du 0 to u, u minus t we are putting the value of phi s,u here.

So when you put the value of phi s,u and combine the 2 integrals we have this. So phi s,u we are putting phi s,u is equal to 1 over u minus s to the power 1 minus Alpha, s to the power Alpha. So we are putting that value and arrive at this solution of the integral equation, Cauchy integral equation of second kind.

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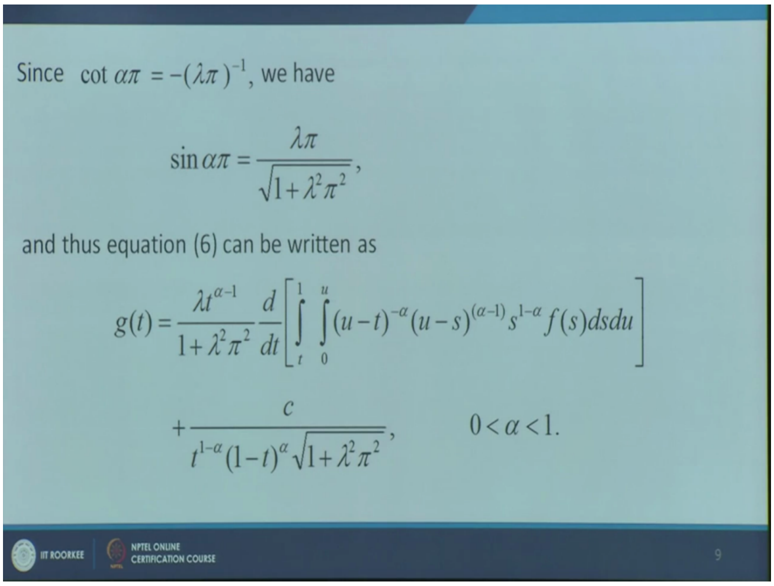

Now we can further write it as, since cot Alpha pi is equal to minus lambda pi to the power minus 1, sine Alpha pi is equal to lambda pi over under root 1 plus lambda square pi square, so this equation the this equation can also be rewritten as this lambda t to the power Alpha minus 1 over 1 plus lambda square pi square d over dt of this integral plus c over t to power 1 minus Alpha, 1minus t to the power Alpha under root 1 plus lambda square pi square.

So this is the solution of the Cauchy integral equation of the second kind. Now let's take an now here in the limits of integration, in the Cauchy integral equation we have taken to be 0,1 but they can be replaced by the real numbers a and b we will define the translation and discuss explanation the way we defined in the case of Cauchy integral equation of first kind we will write t dash equal to t minus a over b minus a and s thus equal to s minus a over b minus a and we will get the solution for the case where the in the integral equations the limits are and b, instead of a and 1.

Now let us consider this integral equation, Cauchy integral equation of second kind. So gs equal to s plus lambda times integral 0 to 1 gtdt over t minus s. So in place of fs we have s here, so fs is known to us and we want to know what is the unknown function g? So let us go back to the solution, this is a solution of the Cauchy integral equation of second kind here fs is known function, so replace fs by s and then we have s to the power 1 minus Alpha. So we will be getting s to the power 2 minus alpha.

So we replace the value of fs here and get the solution of the given Cauchy integral equation of second kind as gt equal to this plus this expression with that I would like to conlude my lecture, thank you very much for your attention.