Integral equations, calculus of variations and their applications
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Lecture 35
Cauchy type integral equations-5

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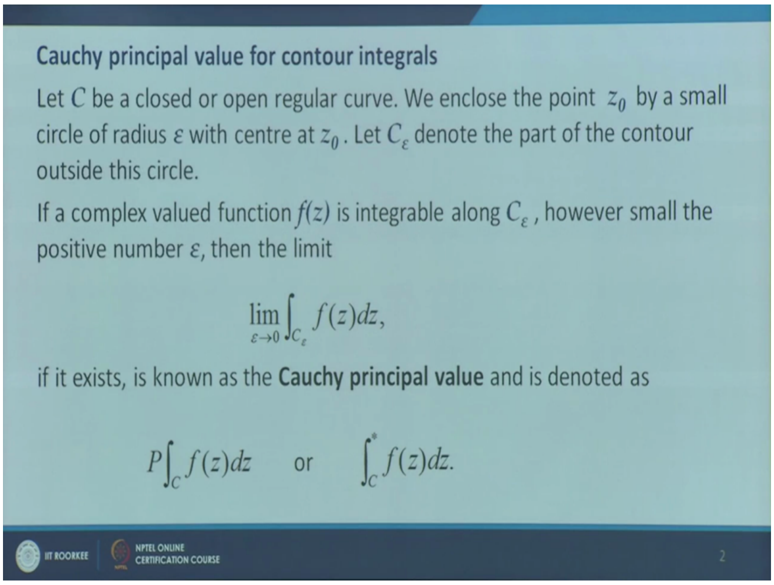

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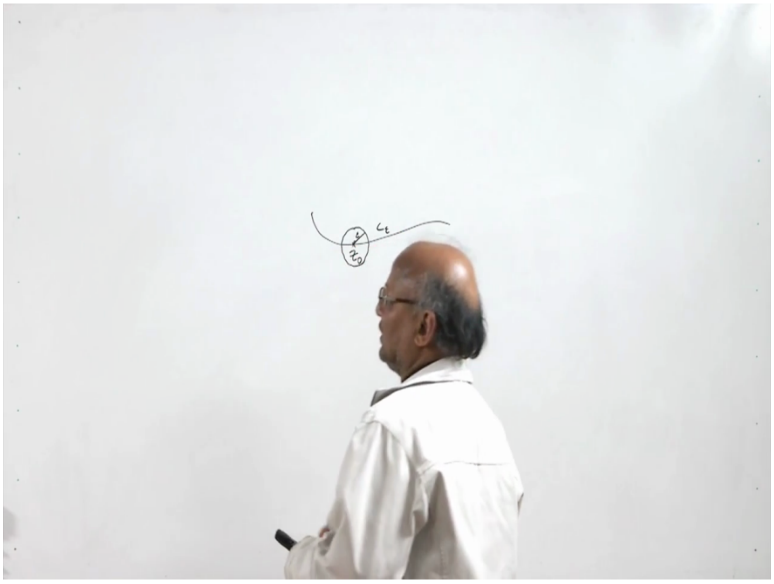

Hello friends this is my 5th lecture on Cauchy type integral equations here we will first define Cauchy principle value for contour integrals, so let C be a close or open regular curve and we enclose the point z not by a small circle of radius Epsilon. So let us say we have a circle a curve like this where z not is a point on this curve, we enclose it by a small circle of radius Epsilon.

So then let's see Epsilon denote the part of the curve outside this circle, if the complex valued function fz is integrable along C Epsilon, however small Epsilon we take then the limit, (limit Epsilon tends to 0 integral over see Epsilon fzdz, if it exists it is called the principal value and as we have defined denoted earlier P integral C fzfdz denotes the principle value of this Cauchy integral or we also denoted by integral over C star fzdz.

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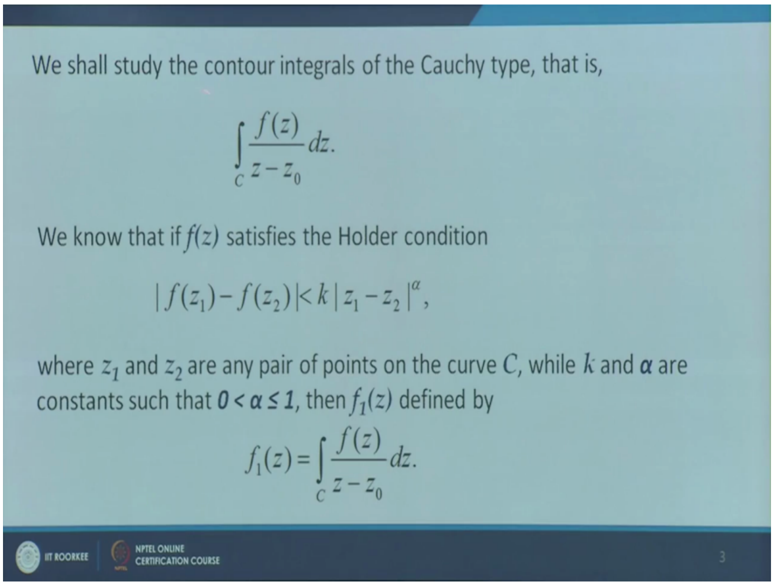

We will study the contour integrals of the Cauchy type we have earlier considered Cauchy type integral equations on a real line. Now we are going to define study Cauchy integral equations in a complex plane. So let us look at this Cauchy type contour integral of the Cauchy type, integral over C fzdz over z minus z not.

If fz satisfies the Holder condition mode of fz1 minus fz2 less than k times mod of z1 minus z2 to the power Alpha where z1, z2 are any pair of points on the curve see, k and Alpha are constants such that 0 less than Alpha less than or equal to 1 then f1z defined by integral over C fzdz over z minus znot, okay is also Holder continuous which causes is the same properties as possessed by the corresponding real functions.

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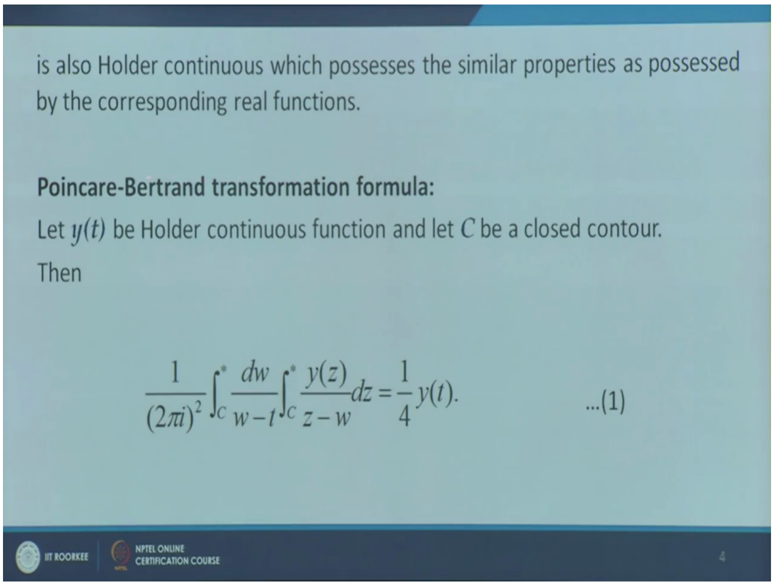

Now here varies there is a very important formula which we know by the name Poincare Bertrand transformation formula we will not prove this, we will simply use this formula. So let y(t) be a holder continuous function and let C be a closed contour then integral 1 over 2pi i whole square the principal value star means again let me remind you that it is the principal value of the integral over C dw over w minus t integral over C star yz over z minus W dz is equal to 1 by 4 y(t). So with this formula plays the crucial role in finding the solution of the Cauchy type integral equations in a complex plane.

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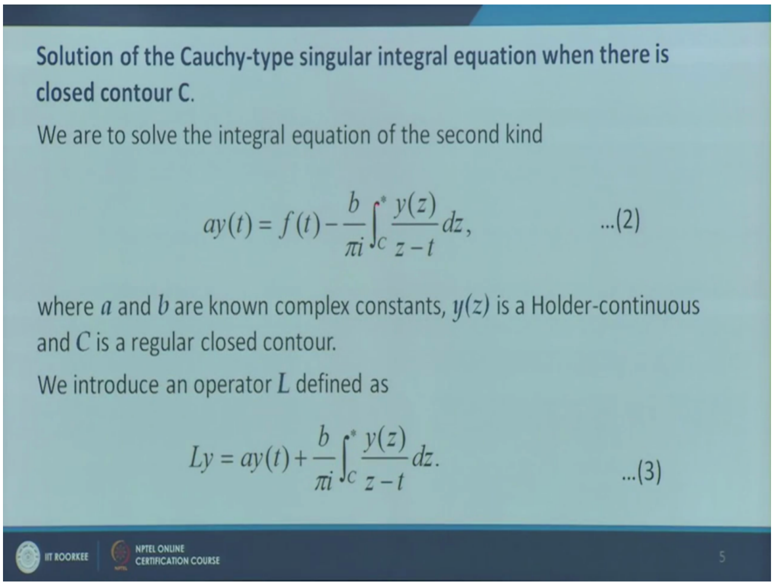

Now let us look at the Cauchy type integral equations when there is closed contour C, so we will consider the case of those Cauchy type singular integral equations where we are given a closed contour C. So we are going to solve the integral equation given by ayt equal to ft minus b over pi i principal value of the integral over C yz over z minus t dz where a and b are known complex constants yz is a Holder continuous function and C is a regular closed contour, so this is called as Cauchy integral equation of the 2nd kind.

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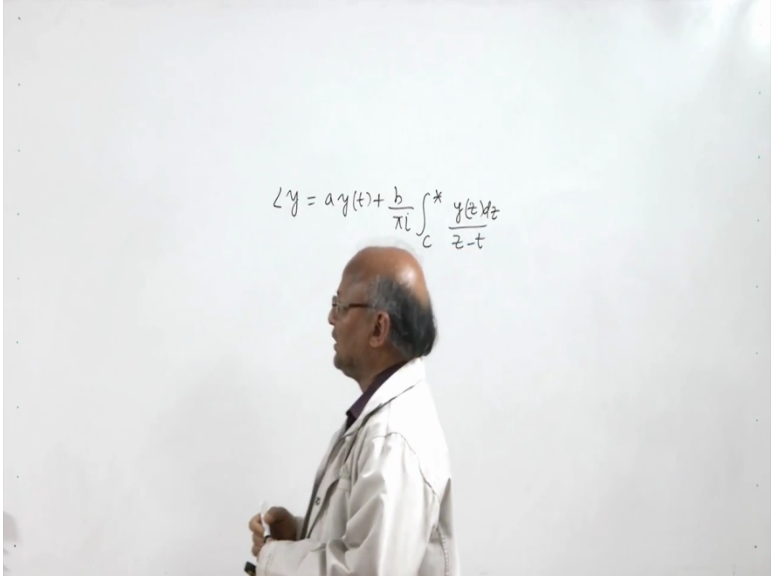

Later on we shall see that as a particular case we shall also be able to determine the solution for the Cauchy integral equation of the 1st kind. So to solve this equation we introduce an operator of let us define the operator as Ly equal to ayt plus b over pi i integral over C star yz over z minus t dz then, so if we define Ly equal to ayt plus b over pi i integral over C star yz over z minus t dz.

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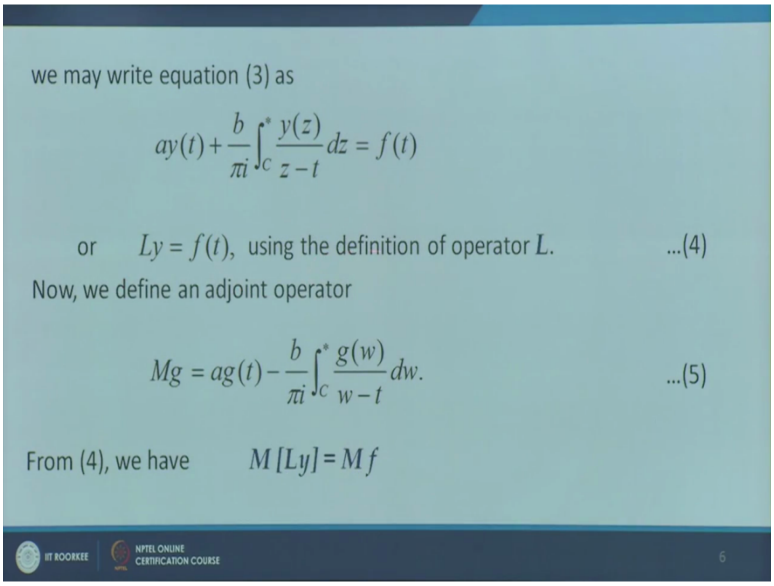

Then the given equation 2 can be written as, then 2 can be written as ayt plus b over pi i integral over C star yz over z minus t dz equal to ft r Ly the left hand side is Ly, Ly equal to ft. So by defining the operator L as Ly equal to ayt plus b upon pi i integral over C yz over z minus t dz, the equation 2 may be rewritten as LY equal to ft by using the definition of the operator L.

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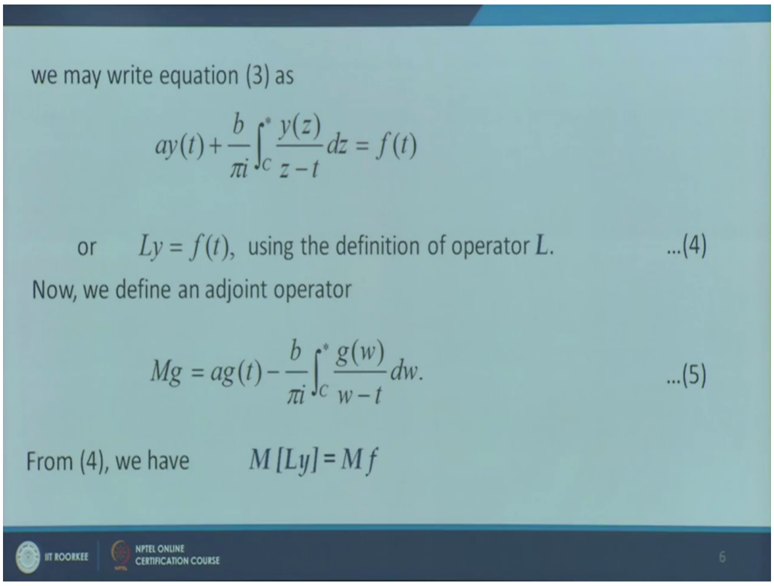

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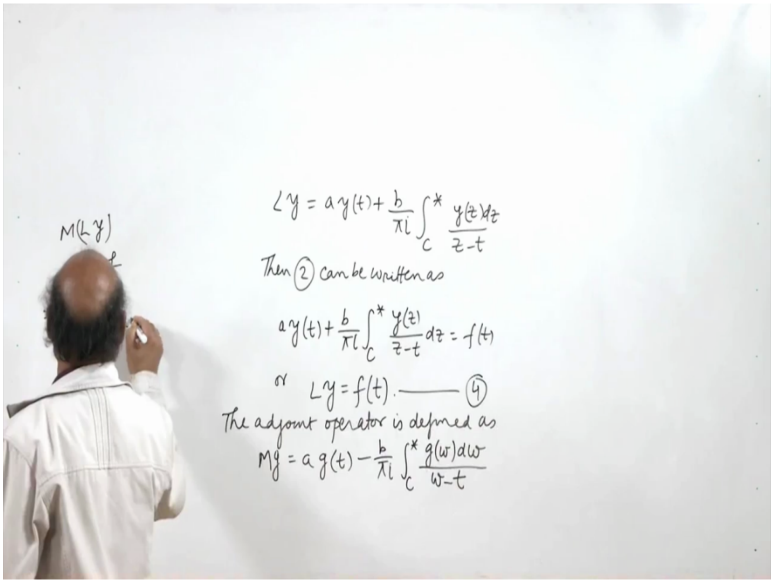

Now let us define and adjoint operator, so the adjoint operator we define as we define it Mg equal to agt minus b over pi i integral over C w minus t. So from 4, what do we notice? See from 4 we have Ly equal to ft and therefore, now let us recall this equation this equation number 4, so from 4 Ly equal to ft we then have M of Ly, Ly is equal to f, so we have Mf. So applying equation 4 or using equation for and definition of the adjoint operator Mg equal to agt minus b over pi i integral over C gw over w minus t dw, we can write M of Ly equal to Mf as Ly is f.

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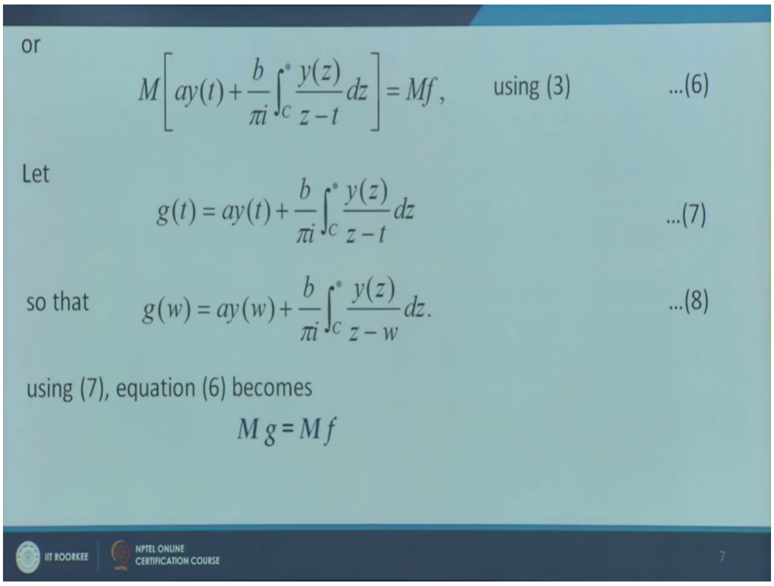

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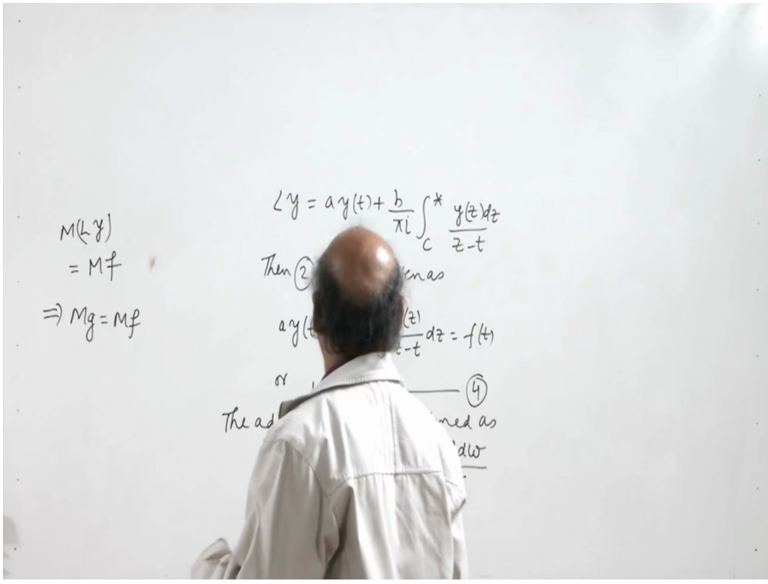

Now, so let us now substitute the value of Ly here , so Ly Ly is equal to ayt plus b over pi i integral over C yz over z minus t dz, so when we substitute the value of Ly here we get M of ayt plus b over pi i integral over C star yz over z minus t dz equal to Mf. ayt plus b over pi i integral over C yz over z minus t dz we denote it by , let us denote it by ayt plus b over pi i integral over C yz over z minus t dz equal to gt then we have gw equal to ayw plus b over pi i integral over C yz over z minus dw. So we will get this as Mg equal to Mf.

We are writing a ayt plus b over pi i integral over C yz over z minus t dz this is as gt, this as a function gt and then what we have is, this gt can be backed by changing t to w. We can write dw, gw equal to ayw plus b over pi i integral over C yz over z minus w dz and so Mg becomes Mf. So this is, this gives you Mg equal to Mf and r, now Mg is how much?

By definition of Mg we write Mg equal to agt minus b upon pi i integral over C gwdw divided by w minus t, this Mg and Mf is equal to aft's minus b upon pi i integral over C w minus t by using the definition of the operator. So now let us replace the value of gt here in place of gt we write ayt plus b over gt is equal to we are writing gt equal to ayt plus b over pi i integral over C yz over z minus t dz.

So a times ayt plus b over pi i integral over C g in place of gt, okay. In place of gt we are putting this, so ayt plus b over pi i integral over C yz over z minus t dz minus b over pi i integral over C1 over w minus t then gw is ayw plus b over pi i integral over C yz over z minus w dz dw and this is equal to right side that is aft minus b over pi i integral over C fw over w minus t dw. Now when we multiply this, this is a square by t plus ab over pi i integral over C yz over z minus t dz and here what we will have? If you multiply we will have minus ab upon pi i integral over C1 over w minus t yw and this integral will cancel.

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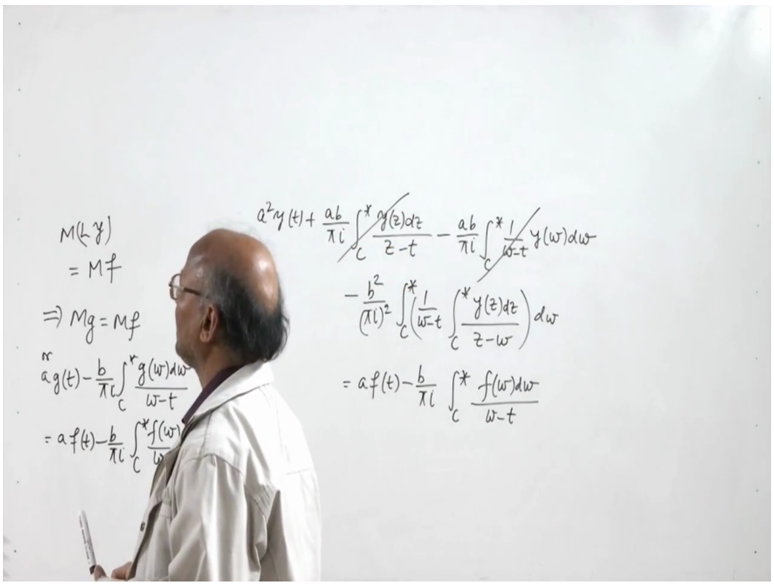

Let us multiply we get a square by t plus ab over pi i integral over C yzdz over z minus t and then minus ab over pi i integral over C1 over w minus t and we have ywdw and then we have minus b square over pi i whole square integral over C1 over w minus t yz dz divided by z minus w, dw equal to aft minus b upon pi i integral over C fwdw over w minus t. So you can see here ab upon pi i integral over C yzdz over z minus t will cancel with this because there is only a change of variable, here we have w, here we have z, so this will cancel with this.

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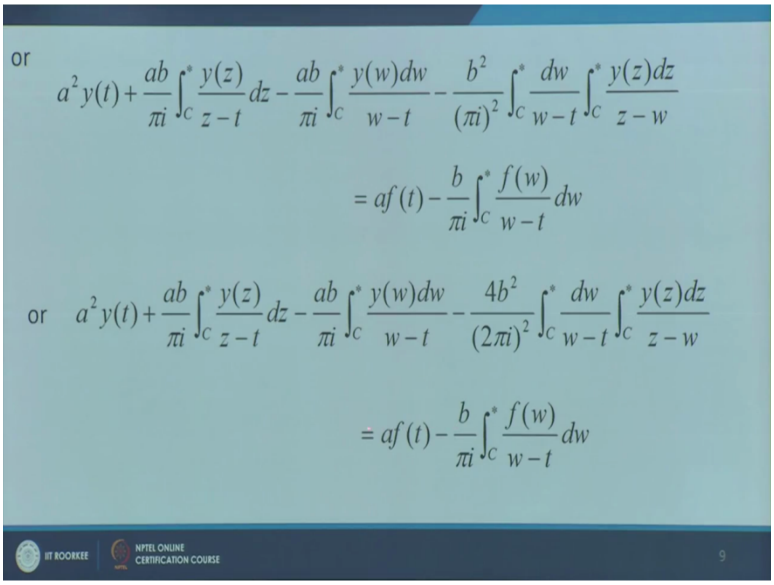

And what we will happen? So we will have a square by t, a square by t plus this this term will cancel with this term and we will have a square by t plus ab upon pi i integral over C yzdz over z minus t minus ab upon pi i integral over C yw dw over w minus t. Now this term will cancel with this term and here for this term we have used the Poincare Bertrand formula, so let us go back to that formula and see what is that formula?

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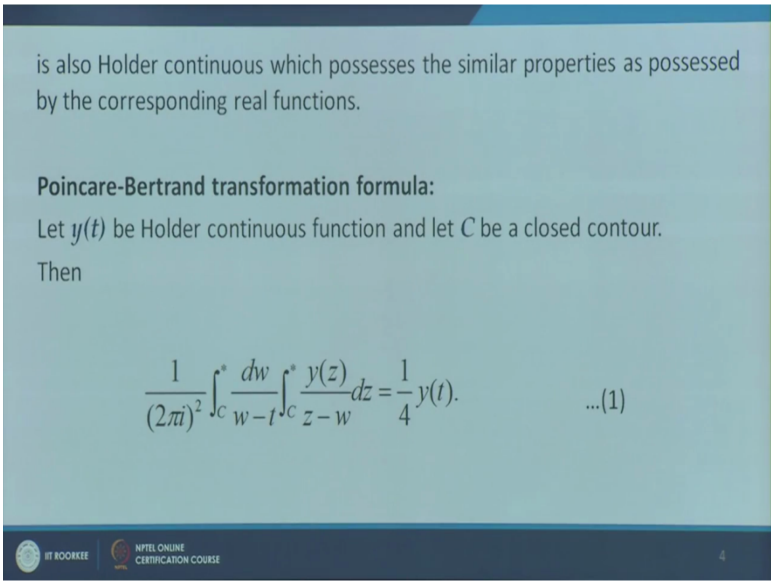

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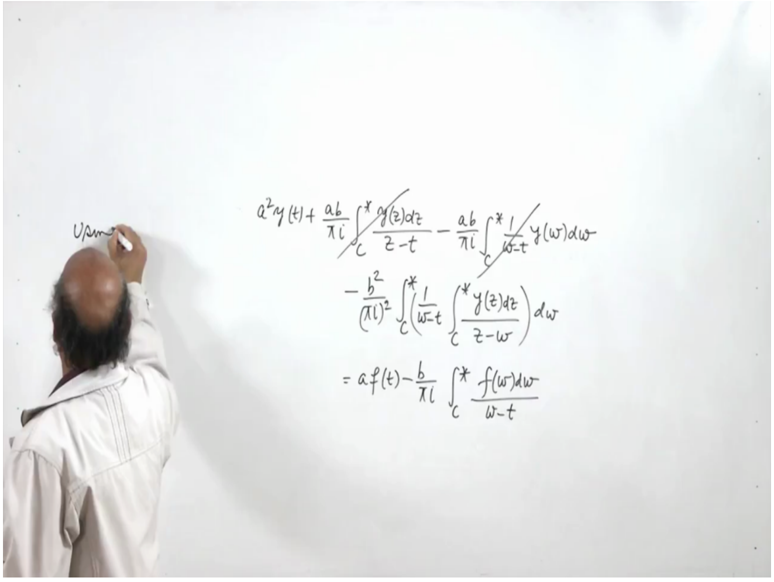

So this formula tells us that it is one over 2 pi i whole square, so by this formula we have using Poincare Bertrand formula, we have yes. So integral over C star dw over w minus t integral over C star yz dz over z minus w dz equal to 1 by 4 yt. So we will have a square yt minus b square upon pi i whole square equal to and then multiplied by 2pi i whole square divided by 4 yt equal to aft minus b upon pi i integral over C star this now 2 square will cancel with this 4 pi i whole square with this pi i whole square and we will get a square minus b square yt equal to, we get this.

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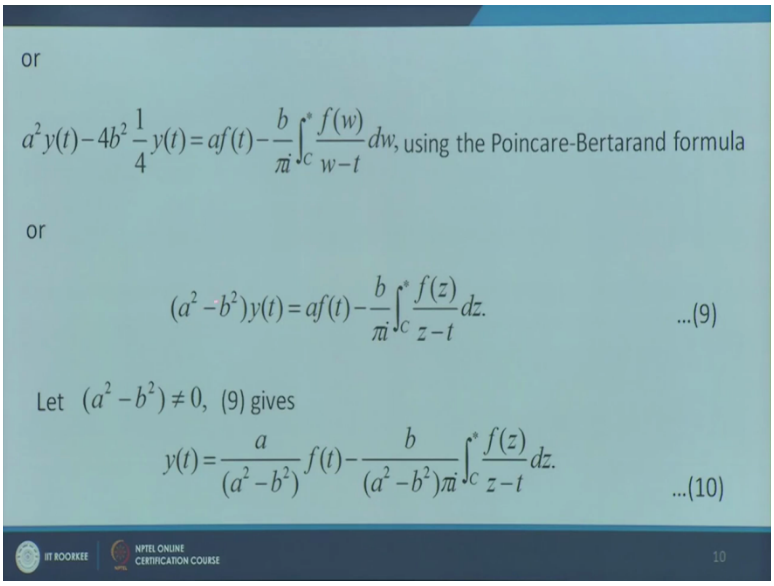

Now if we assume that a square minus b square is not equal to 0, so if we assume that so we come here, if we assume that a square minus b square is nonzero we can write the value of yt from here. So yt will be equal to a upon a square minus b square into ft minus b upon a square minus b square into pi i integral over C fz over fzdz over z minus t, we can replace w by z. So this will give us the solution of the Cauchy integral equation over a close contour.

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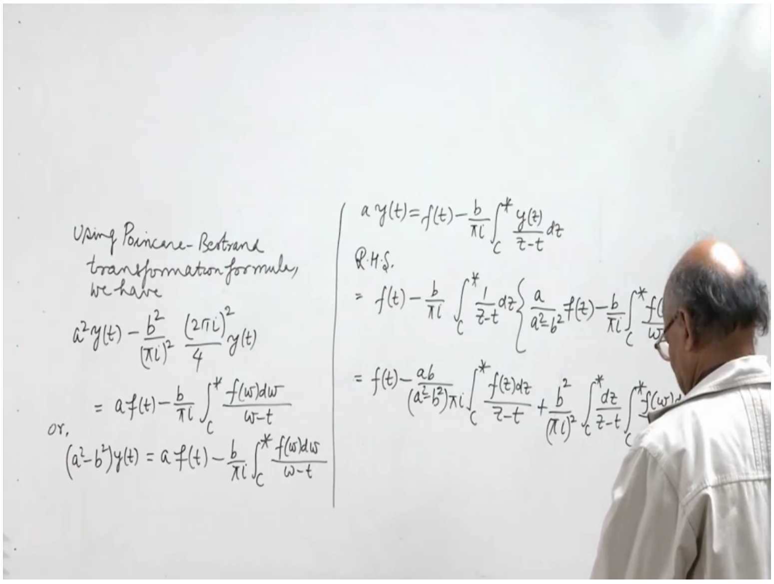

Now this if you substitute this back into the integral equation, it turns out that this yt satisfies the original equation we can see it, we can see how it satisfies, so let us go back to the Cauchy type integral equation whose solution we are trying to find. So this equation is ayt equal to ft minus b upon pi i integral over C star yz over z minus t dz this is the Cauchy type integral equation of the 2nd kind, we are claiming that this is the solution of this Cauchy integral equation.

So let us substitute it in this equation and show that it indeed satisfies this equation, so let us take the right-hand side. Right-hand side will be equal to ft minus b upon pi i integral over C star and then let us substitute the value of yz here. So yz will be equal to a upon a square minus b square fz minus b upon pi i integral over C star fw, we are writing the value, okay, so fwdw over w minus z.

So let us see what we get here? this is ft minus a, b upon a square minus b square into pi i integral over C star fzdz over z minus t, ft minus ab upon a square minus b square into pi i integral over C fzdz over z minus t dz and here what do we get? Plus b square upon pi i whole square integral over C star dz over z minus t then integral over C star fwdw divided by w minus z.

Now let us see we use this formula Poincare Bertrand formula. So when you use Poincare Bertrand formula integral over C dz over z minus t in place of w, we have z. So dz over z minus t integral over C in place of w, we have now z. So we have here fw, fw over w minus z dw, this will be equal to 1 over 4and in place of f we have t, so we have 1 by 4 ft, so this will be equal to, so b square upon pi i whole square and then we have 2 pi i whole square by 4 and we have ft here.

So what we will get then? So 2 square will cancel with 4 pi i whole square will cancel with pi whole square and what we get?

“Professor-student conversation starts”

Sir a square minus b square would be placed on the right-hand side

“Professor-student conversation ends”

We are writing, we are putting the value of yts 1 upon af aft upon a upon a square minus b square b upon pi i a square minus b square, so 1 a square minus b square we have basically here, so we let us correct it a square minus b square integral over C star fwdw divided by w minus t, so this a square minus b square is also here, here also a square minus b square.

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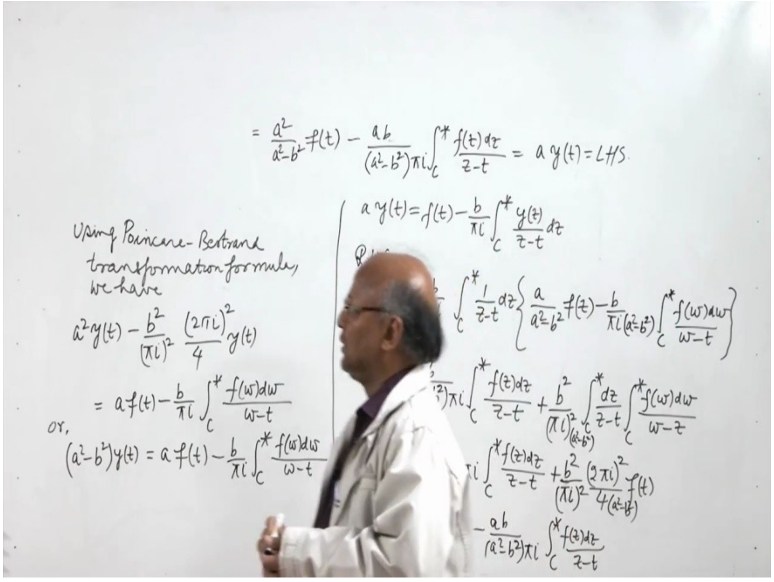

So now what we get? ft times 1 plus b square upon a square minus b square, this term plus this term minus ab upon a square minus b square pi i integral over C star fzdz over z minus t or we can say or ayt equal to, now a square minus b square plus b square will be a square, so a square upon a square minus b square ft minus ab upon a square minus b square into pi i.

So wait so right-hand side we are writing, sorry. Right-hand side we are writing right-hand side is equal, sorry this right-hand side is now equal to ab upon a square minus b square pi i integral over C star fzdz divided by z minus t and this is nothing but is equal to a into a into (()) (26:06) which is equal to because you multiply yt here is a upon a square minus b square ft minus b upon pi i a square minus b square integral over C fwdw over w minus t.

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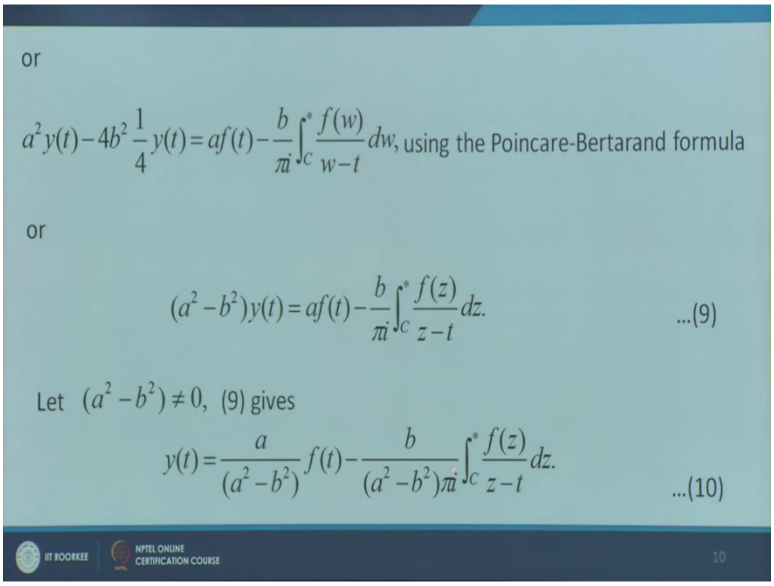

So you multiply that equation by a, you will get this equation. So this is equal to ayt and so we get the left hand side So this is how we show that the, yt equal to a upon a square minus b square ft minus b upon a square minus b square pi i integral over C fz over z minus t dz is given such a solution of the Cauchy integral equation along a closed contour C.

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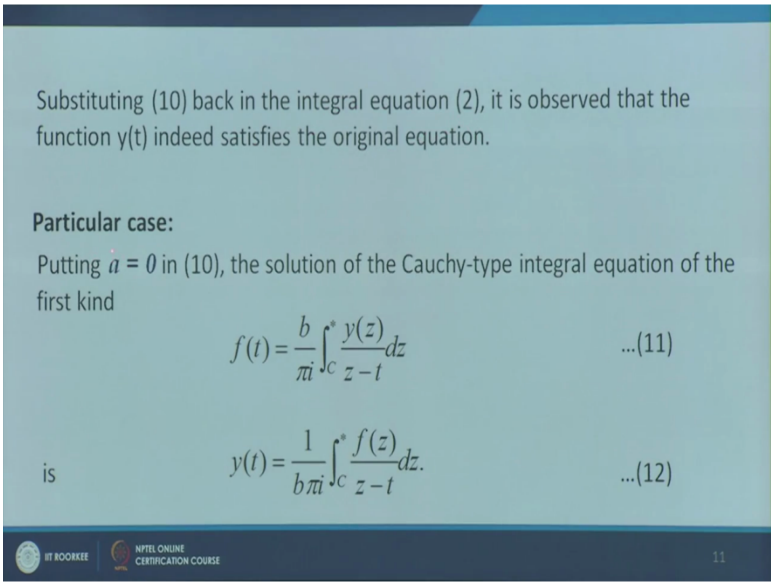

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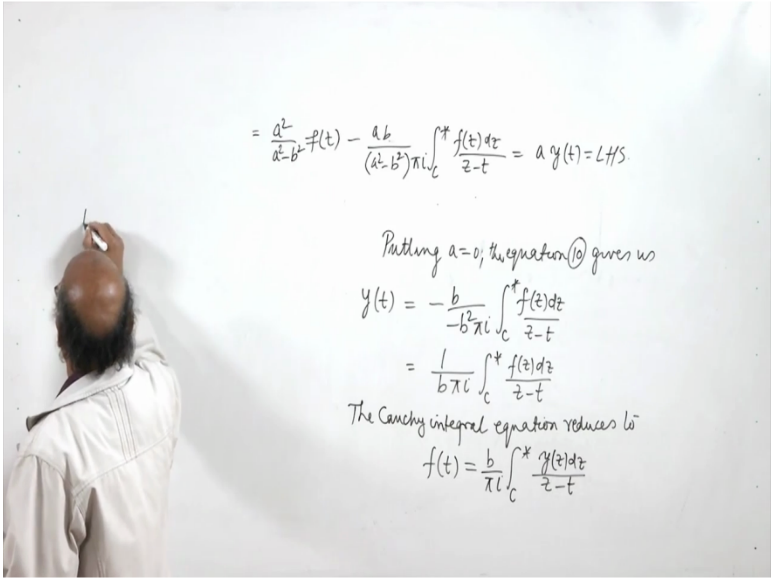

Now in particular if we put a equal to 0. If we put a equal to 0 in equation number 10. If we put a equal to 0 here, what we will get? y t equal to minus yt equal to putting a equal to 0 the equation 10 gives us yt equal to minus b upon minus b square pi i integral over C fzdz over z minus C or we can say yt equal to 1 upon b pi i integral over C star fzdz divided by z minus C, the Cauchy integral equation of 2nd kind reduces to on taking a equal to 0, it reduces to ft equal to b upon pi i integral over C yz over z minus t dz. The Cauchy integral equation ft equals to b upon pi i integral over C yzdz over z minus t. Now this is Cauchy integral equation of the 1st kind.

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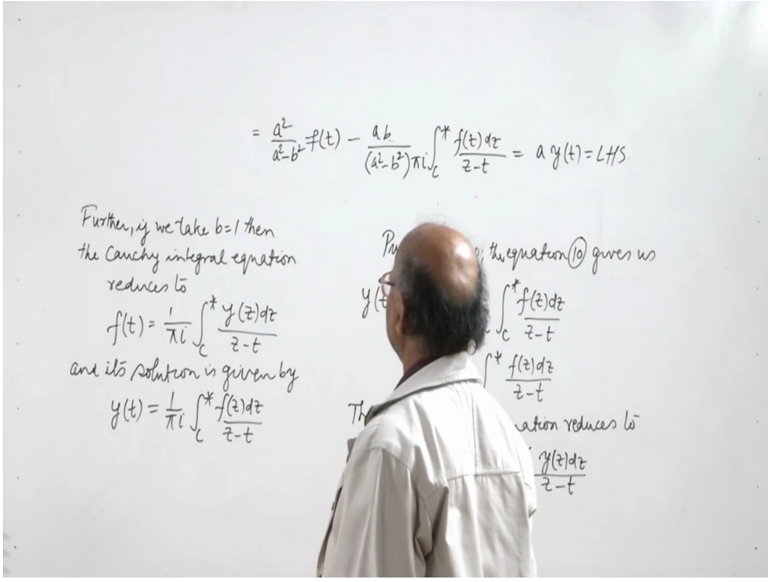

So when you take a equal to 0, the Cauchy integral equation of 2nd kind reduces to the Cauchy integral equation of the 1st kind and the solution of this Cauchy integral equation of the 1st kind is given by yt equal to 1 upon b pi i integral over C fz over z minus t dz. Further if we take b equal to1 then what will happen? Then we get the Cauchy integral equation S.

the Cauchy integral equation of the 1st kind reduces to ft equal to 1 over pi i integral over C yzdz over z minus t and its solution is given by yt equal to 1 upon pi i integral over C star fzdz divided by z minus t, so when the integration in the Cauchy integral equation of 2nd kind we take a equal to 0, b equal to 1 we get this Cauchy integral equation and its solution is given by yt equal to 1 upon pi i integral over C fzdz over z minus t.

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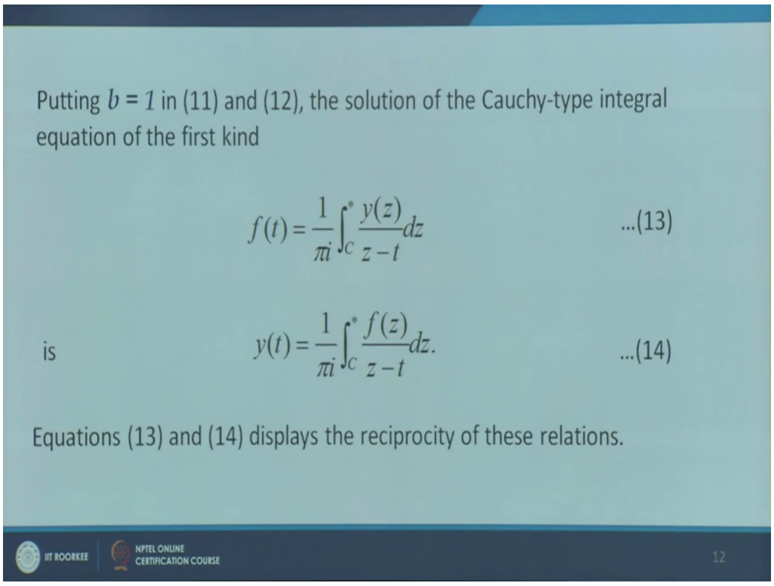

Now you can see these 2 equations, here in this Cauchy integral equation yz is the unknown function and 2 determine yz, what we do is, we simply replace yz by fz, so that the 2 equations from this given equation it is easy to write a solution , the solution is just replace yz by fz. So yt equal to 1 over pi i integral over C fzdz over z minus C. So that is why we say that equations 13 and 14 display the reciprocity of these relations.

So with this we will conclude our discussion on Cauchy type singular integral equations. In our next lecture we will see how to apply the Fourier transform to solve integral equations. We have already seen how to apply Laplace transform to solve integral equations. Now we will see the integral equations that can be solved by using Fourier transforms.

So with this I would conclude this lecture thank you very much for your attention.