Integral equations, calculus of variations and their applications
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Indian Institute of Roorkee
Lecture 37
Hilbert transforms-1

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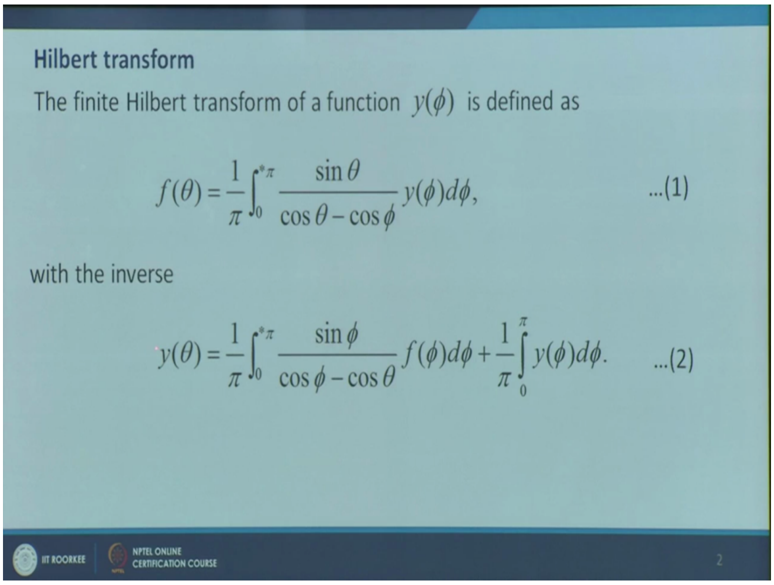

Hello friends I welcome you to my lecture on Hilbert transforms one, there will be 2 lectures on this, so this is one of the first 2 lectures. The finite Hilbert transform of a function yphi is defined as f theta equal to 1 upon pi, the principle value of the integral 0 to pi, sin Theta over cos Theta minus cos Phi yphi dphi and the inverse of this is given by y Theta equal to 1 over phi, 0 to pi principal value of the integral 0 to pi, sin Phi over cos phi minus cos theta fphi dphi plus 1 over pi, 0 to pi yphi dphi.

Now if we have an integral equation of this type we can find the solution of the integral equation by using this formula that is what the inverse means. Now so this is a pair of the 2 formulas, one gives the integral equation the other gives its solution. Now there are other alternate forms of these are which can occur in literature, so let us discuss the first alternative form of the Hilbert transform pair.

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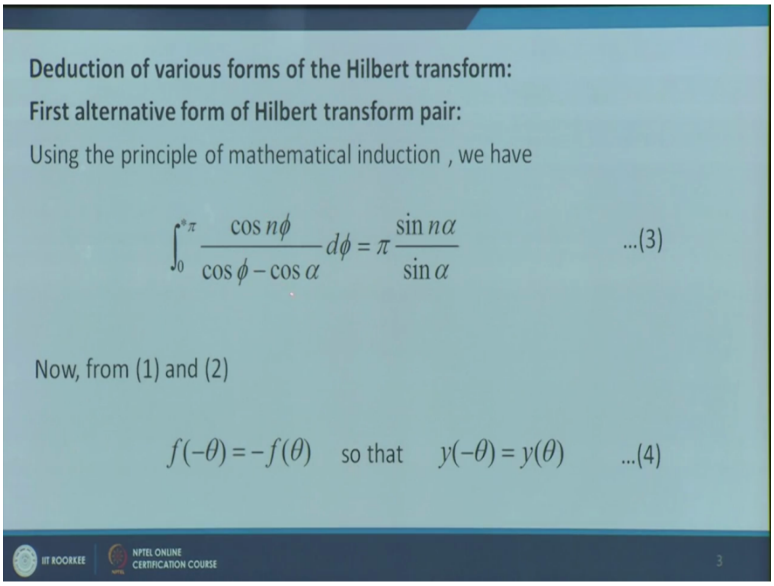

Now here we shall make use of a result which can be proved by the principle of mathematical induction on n and it says that the principal value of integral 0 to pi cos b phi over cos phi minus cos Alpha dphi is equal to pi times sin n Alpha over sin Alpha where n takes value 0,1,2,3, and so on, so n is a nonnegative integer.

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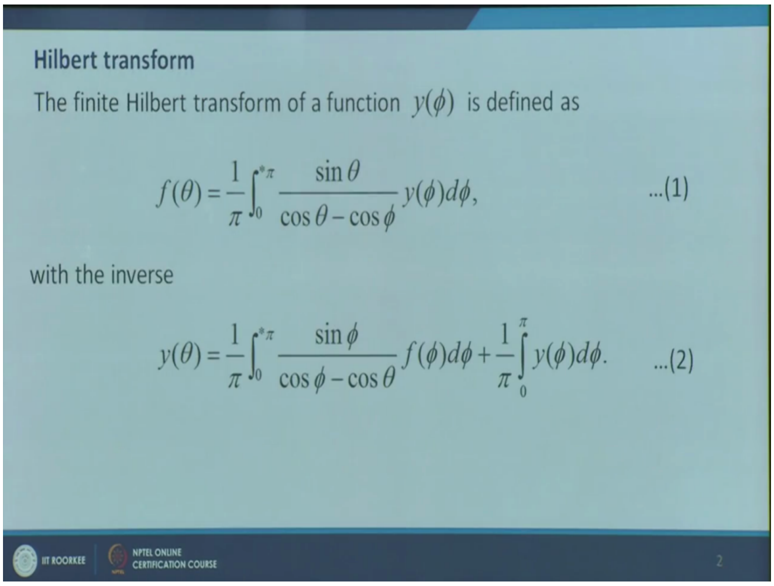

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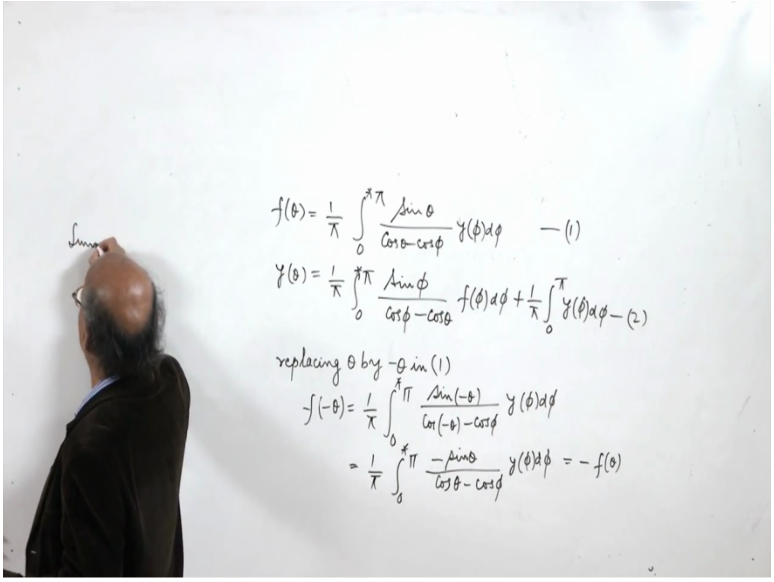

Now from the equations 1 and 2, let us see from the equations 1 and 2 we notice that, the equation 1 is f theta equal to 1over pi sin Theta divided by cos Theta minus cos phi yphi dphi and its inverse is given by y Theta equal to 1 upon pi, 0 to pi, sin Phi over cos Phi minus cos Theta fphi dphi equal to plus 1by pi, 0 to pi, yphi, dphi, okay. From these pair of Hilbert transform we note we observe that if we replace theta by minus Theta here, so then let it be, this is equation number 1 this is equation number 2.

So in 1 let us replace theta by minus theta, in equation 1 what do we get? f minus theta equal to 1 over pi, 0 to pi, sin minus theta over cos minus theta minus cos phi yphi dphi. Now sin minus theta is minus sin Theta, cos minus theta is cos Theta, so I can write it as 1 over pi which is equal to minus f theta.

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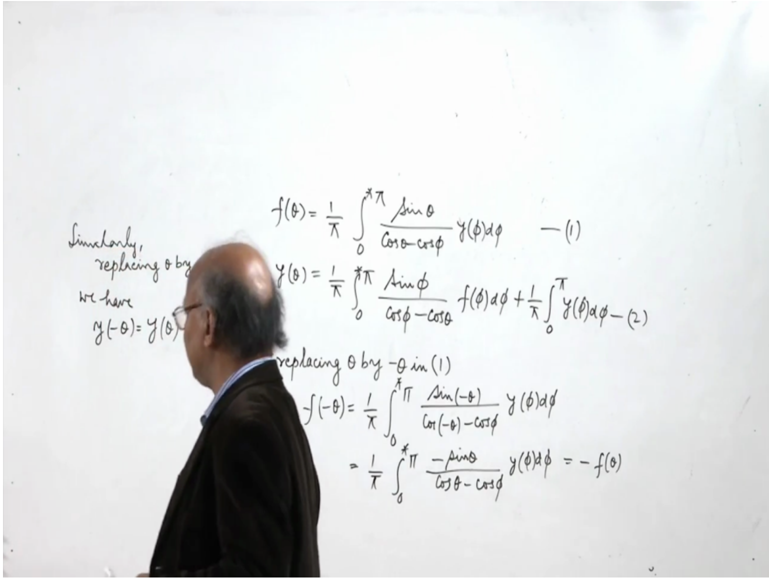

Similarly replacing theta by minus Theta in equation 2, what do we get? We have y minus theta equal to again when we replace theta by minus theta there is only one term here in theta which is cos theta and cos minus theta is cos Theta, so we will get y minus theta equal to y Theta. So from the given Hilbert transform pair with notice that f minus theta is equal to minus f theta and y minus Theta is equal to y Theta.

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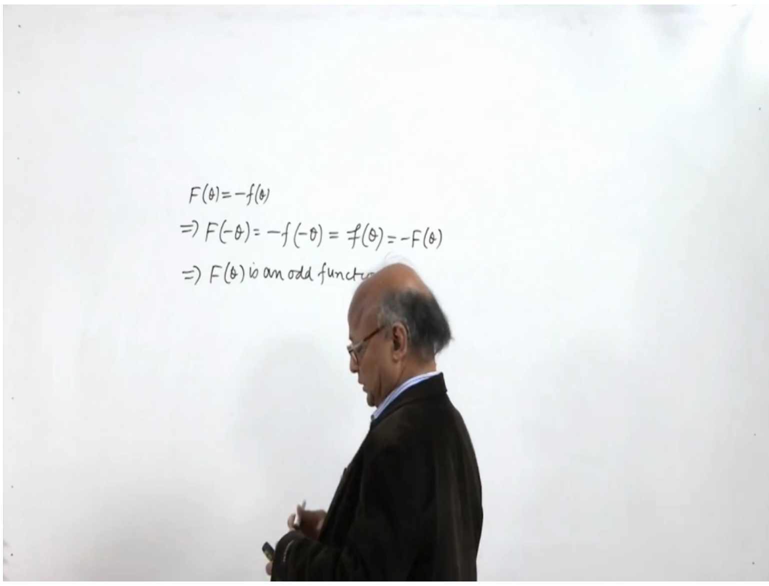

Now assume that f theta equal to minus f theta and if we make this assumption f theta equal to minus f theta then let us see whether f is an even function or odd function this implies F minus theta equal to minus f minus theta. Now we have earlier seen F minus Theta is equal to minus f theta, so this is equal to f theta and f theta is equal to minus F theta. So when we replace theta by minus theta we get F minus Theta equal to minus F theta, this means that F Theta is an odd function of Theta.

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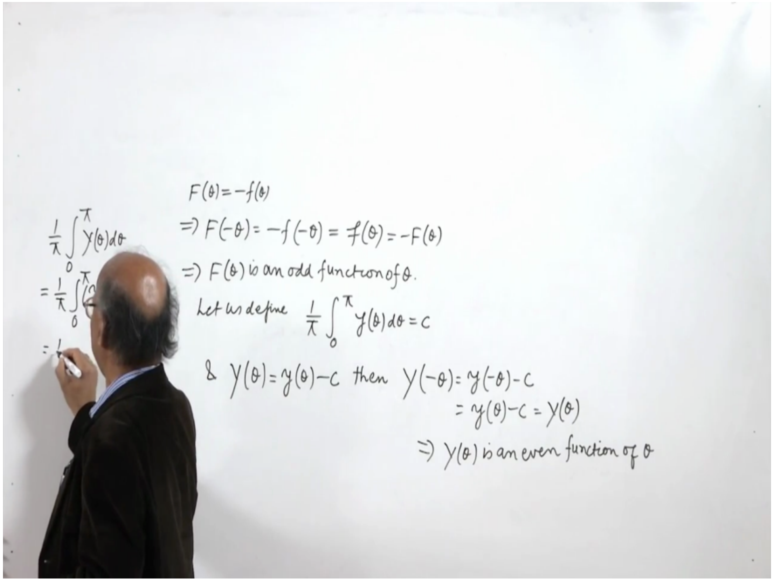

Now let us further define 1 over pi, integral 0 to pi y Theta d Theta equal to, see the unknown function as y Theta which we are trying to find but the unknown function y theta when integrated over 0 to pi and divided by pi the value divided by pi will give you a constant value, so we can denote it by C.

So let us say C denotes 1 over pi, integral 0 to pi, y theta d Theta and also let us define capital Y Theta equal to small y Theta minus this constant C then if you replace theta by minus theta in this definition what we get? y minus theta equal to y minus theta minus C and we have earlier seen when we replace y minus theta y minus theta is equal to y Theta, so this is equal to y theta minus C and y theta minus C is y theta. So this gives you y theta is an even function of theta.

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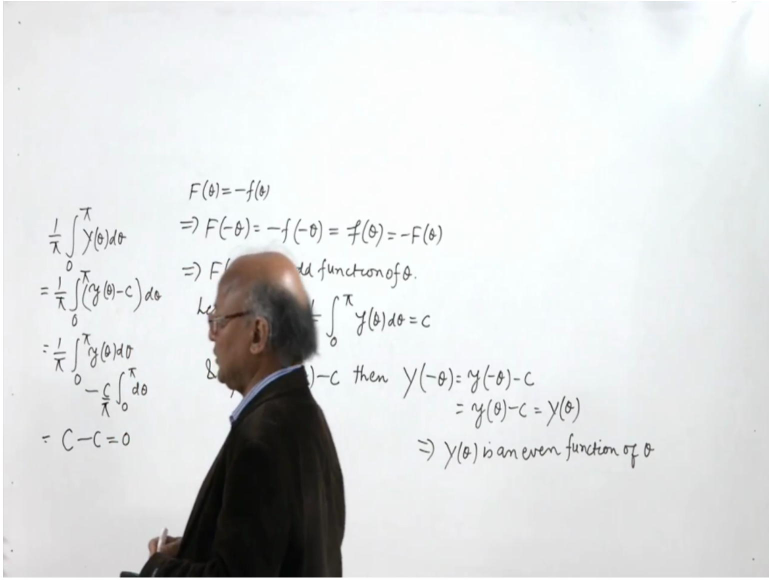

Further we notice that if we calculate this integral 1 over pi, 0 to pi y Theta, capital Y Theta d theta then if we make use of this definition of y theta, we have one over pi, 0 to pi, y theta minus C d theta and this is equal to 1 over pi, integral 0 to pi, Y theta d Theta minus C times integral 0 to pi, d Theta divided by pi and so this is equal to one over pi, integral 0 to pi, Y theta d Theta is equal to C, so we get C here and this is C over pi into pi, so we get C. So C minus C is equal to 0.

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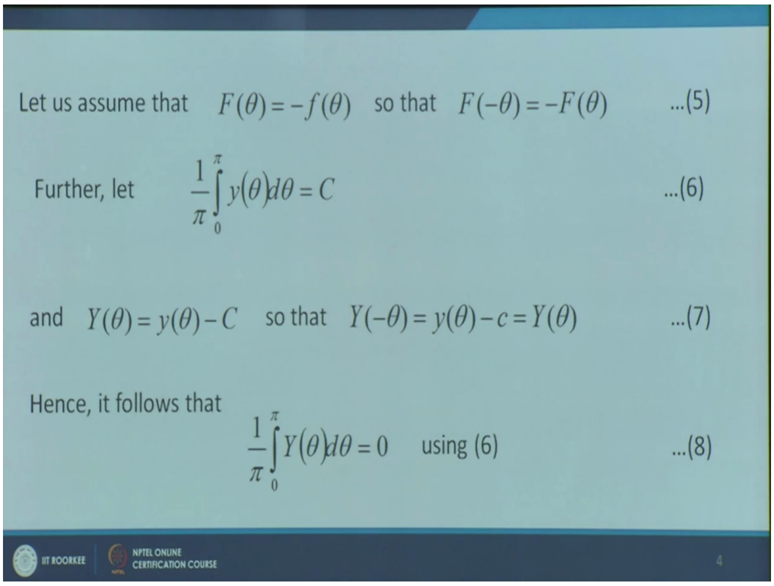

So using the definition of C, we notice that one over pi, integral 0 to pi, Y Theta d Theta is equal to 0.

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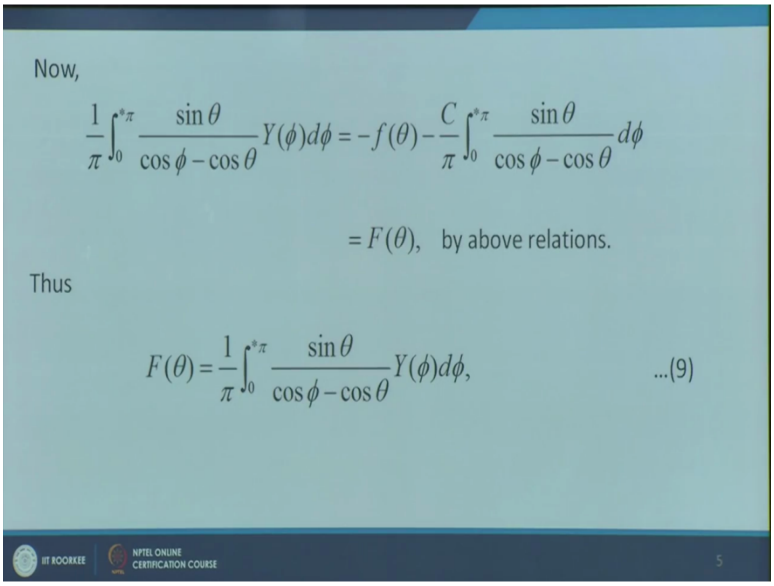

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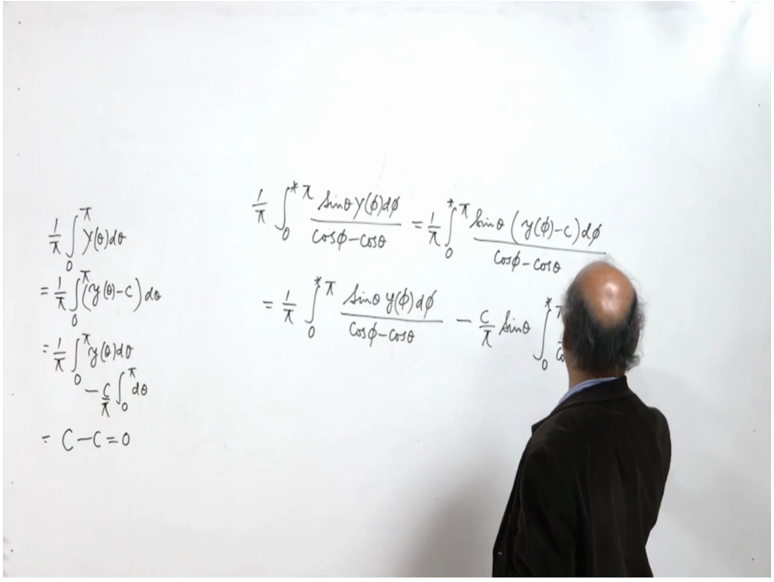

Now let us , let us evaluate this determinate 1 over pi, 0 to pi, sin Theta Y phi dphi divided by cos Phi minus cos Theta. I can write it as 1 over pi, 0 to pi, sin Theta into Y Phi minus C dphi divided by cos phi minus cos Theta and I can write it as 1 over pi, 0 to pi sin Theta into Yphi dphi divided by Cos phi minus cos Theta minus C upon pi into sin Theta, Theta is independent of Phi, so I can write it outside the integral 0 to pi, d phi divided by cos Phi minus cos Theta.

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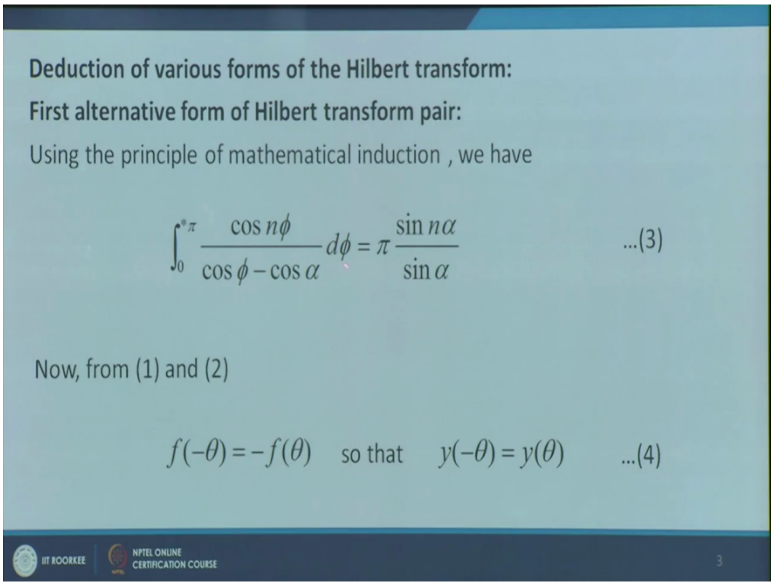

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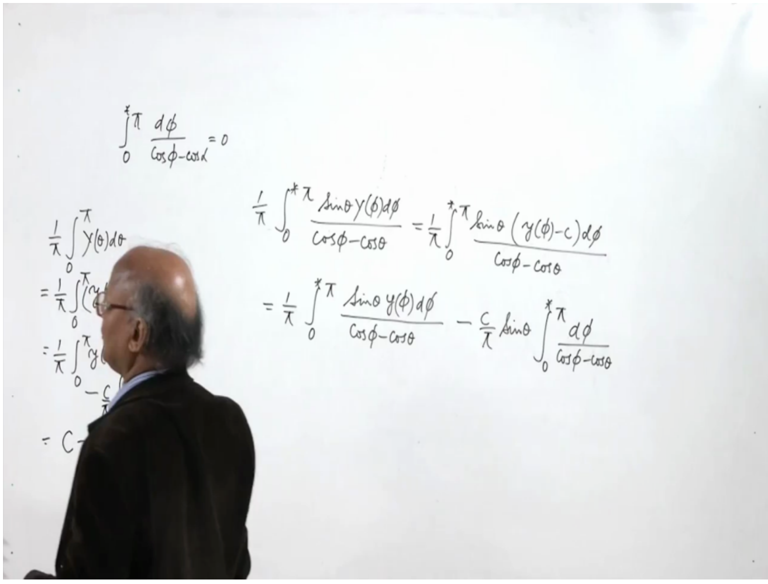

Now let us go back to this formula, in this formula we have integral 0 to pi, cos and Phi over cos phi minus cos Alpha d Alpha equal to pi times sin n Alpha over sin Alpha and it is valid for all nonnegative integral values of n. So let us put n equal to 0 in this and then we get integral 0 to pi, d phi over cos Phi minus cos Alpha equal to 0, putting n equal to 0. So we shall make use of that here and when we use that instead of alpha there, here we have theta. So this integral 0 to pi dphi over cos Phi minus cos Theta will be equal to 0. So this term vanishes and this first-term can be written as from the definition of f theta, f theta is 1 over pi, 0 to pi, sin Theta over cos Theta minus cos phi Yphi dphi, so from here I can write this as minus F Theta, so I can write it as minus F theta. So this is minus F Theta, this as I said earlier this is equal to 0.

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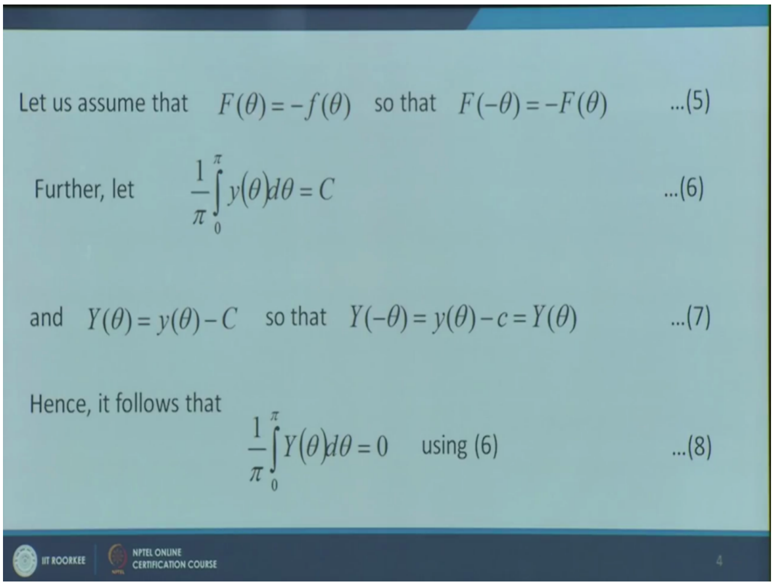

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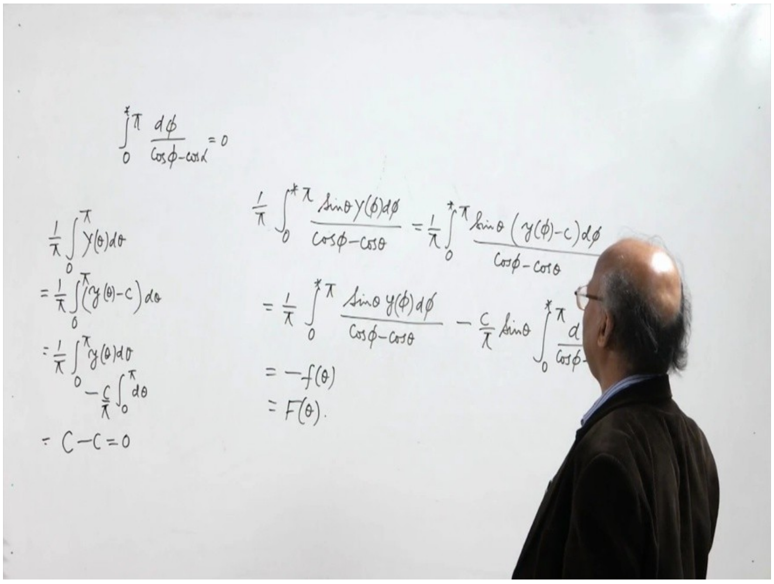

And minus F data, you can see minus F theta is equal to capital F Theta. So what I get is this integral 1 over pi, principal value of the integral 0 to pi, sin Theta Yphi dphi over cos Phi minus cos Theta turns out to be capital F Theta. So thus we get this F Theta equal to 1 over pi, 0 to pi, sin Theta over cos phi minus cos Theta Yphi d phi, okay. So this is out of the pair of forms, pair of Hilbert transforms this one equation, the other equation we have to still get.

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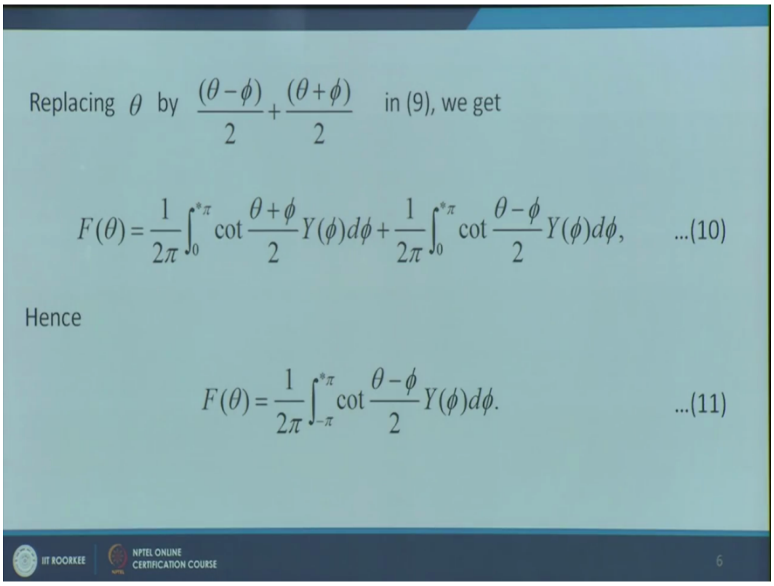

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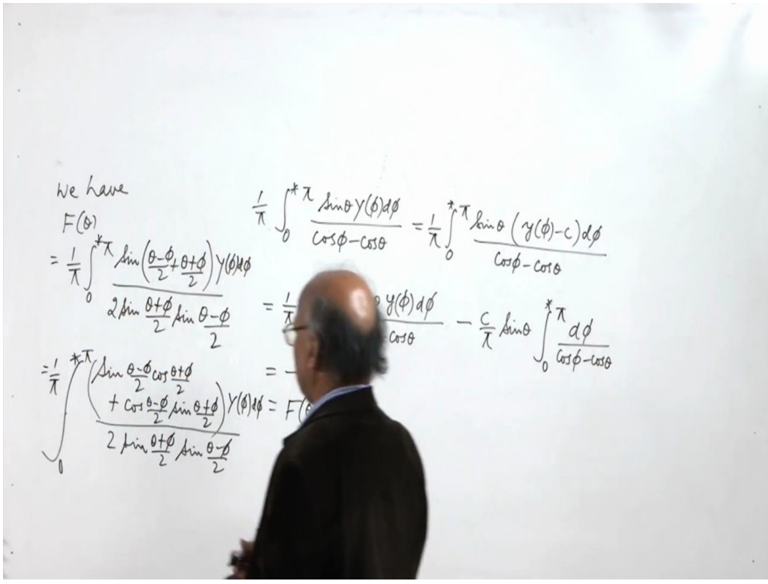

Now replacing, this can be further explained like this replacing theta by Theta minus Phi by 2 plus Theta plus Phi by 2. F theta could be written as we have, F Theta equal to 1 over pi, integral 0 to pi sin Theta can be written as Theta minus Phi y2 plus Theta plus phi by 2 and then Yphi dphi divided by cos Phi minus cos Theta can be written as twice sin Theta plus phi by 2 into sin phi minus Theta by 2 into Theta minus phi by 2 Theta minus phi by 2 and we can express this as 1 over pi, 0 to pi sin Theta minus sin a cos b, sin a plus b equal to sin a cos b plus cos sin b.

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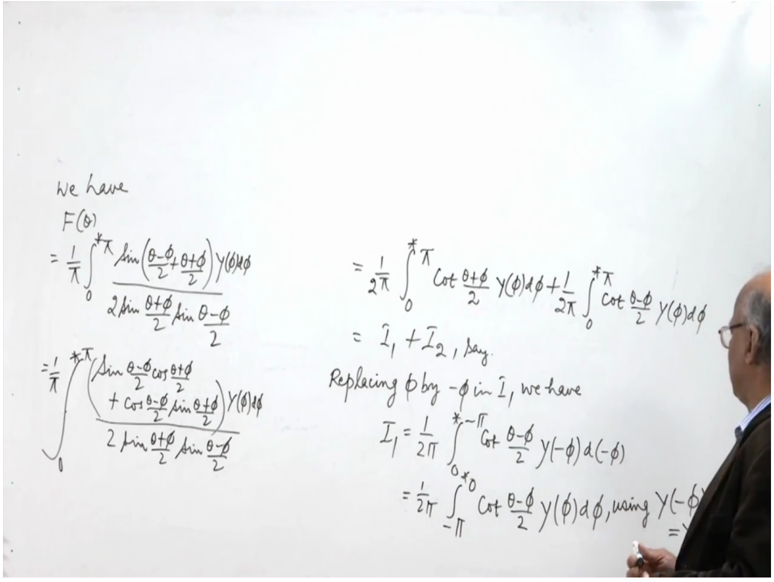

So sin Theta minus phi by 2, cos Theta plus Phi by 2, sin a cos b, cos a sin b this divided by Yphi dphi over 2 sin Theta plus Phi by 2, sin Theta minus Phi by 2. Let me take this, okay. So we can write it in this form, this is equal to 1 over pi, so cos Theta plus Phi by 2 over sin Theta plus Phi by 2, I can write it as 2 pi here and then we have got Theta minus Phi by 2 Yphi dphi plus 1 over 2 pi, 0 to pi in this term we get, no this first-term gives us cot Theta plus Phi by 2 and the 2nd term gives us cot Theta minus Phi by 2.

Now let us say this is I1 and I2 these 2 integrals, let us take I1 and I2. So if I call them I1 plus I2 then in the integral I1 let us replace phi by minus phi, okay. So replacing phi by minus phi, in I1 we have I1 equal to 1 over 2 pi, 0 to minus pi cot Theta minus phi by 2 by minus phi, d minus phi.

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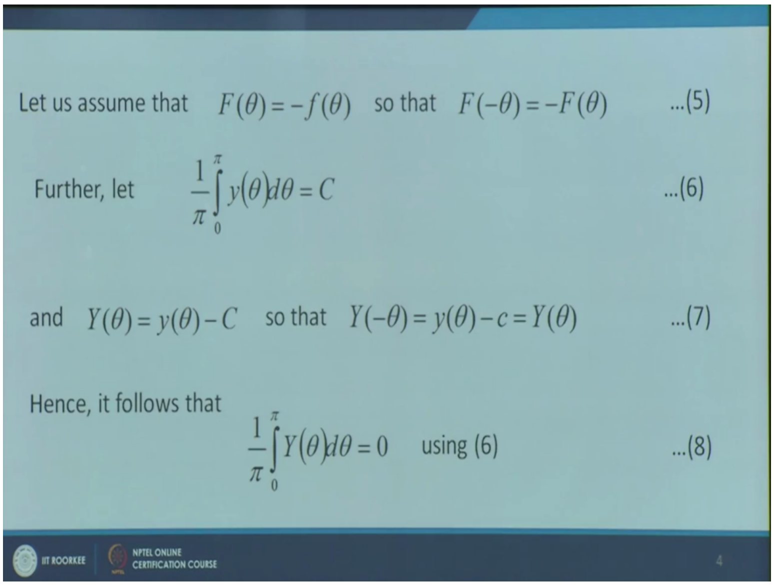

Now let us see we have earlier seen that Y Theta is an even function of Theta, so Y minus phi is Y phi and so what we will get? this d minus phi will become minus d phi with the minus dphi with the negative sign from minus d phi we can change the limits of integration here and then what we have is, this is 1 over 2 pi minus pi to 0, cot Theta minus phi by 2 yphi dphi using y minus phi equal to yphi.

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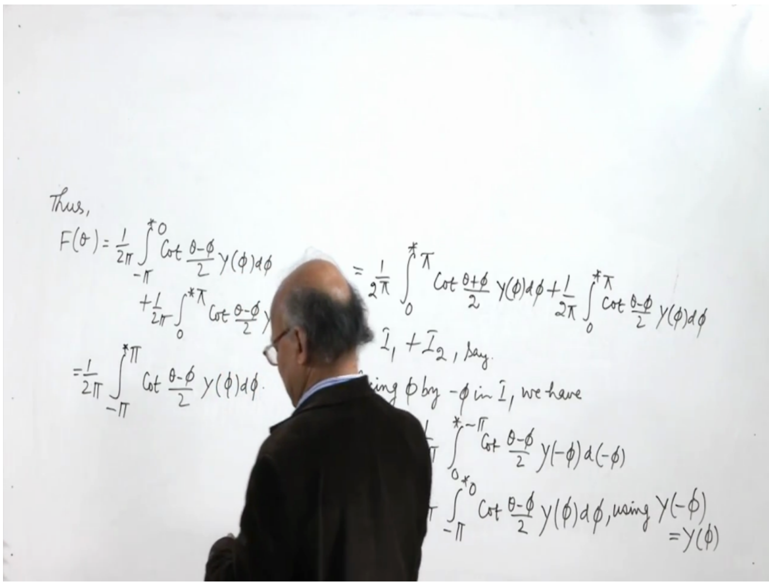

Now let us replace the value of I1 here, so then we will get F Theta equal to, so thus F theta which is the sum of I1 and I2 will become 1over 2 pi, we can combine the 2 integrals and write it as 1 over 2 pi minus pi 2 pi cot Theta minus phi by 2 Yphi dphi. So this is alternative form of F Theta, F Theta equal to 1 upon 2 pi, integral over minus phi to pi, cot Theta minus Phi by 2 by phi.

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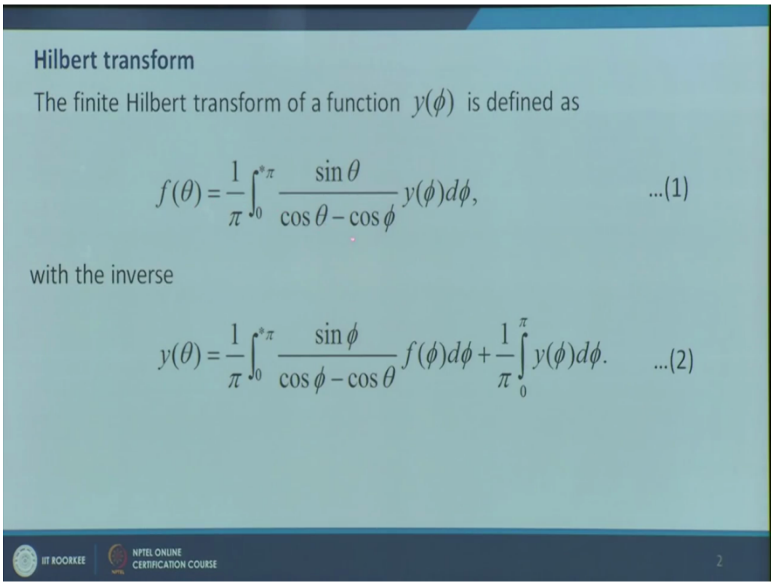

So the equation 1, this equation 1 can be rewritten as this, in this form. It can be expressed in this form (()) (20:31) F Theta equal to one over 2 pi minus pi to pi, cot Theta minus Phi by 2 yphi dphi. Now let us see how we can rewrite equation 2. Equation 2, these 2 equations form a together river transform pair. Now here also we will get a pair of equations.

So one equation we have got this equation which is the alternate form of equation 1. Now let us get the alternate form of equation 2. The other equation which we will get corresponding to the equation 2 that equation together with this, equation which we have just now found they will make the first alternate transform pair, Hilbert transform pair. So that the alternate form of equation 2, let us derive.

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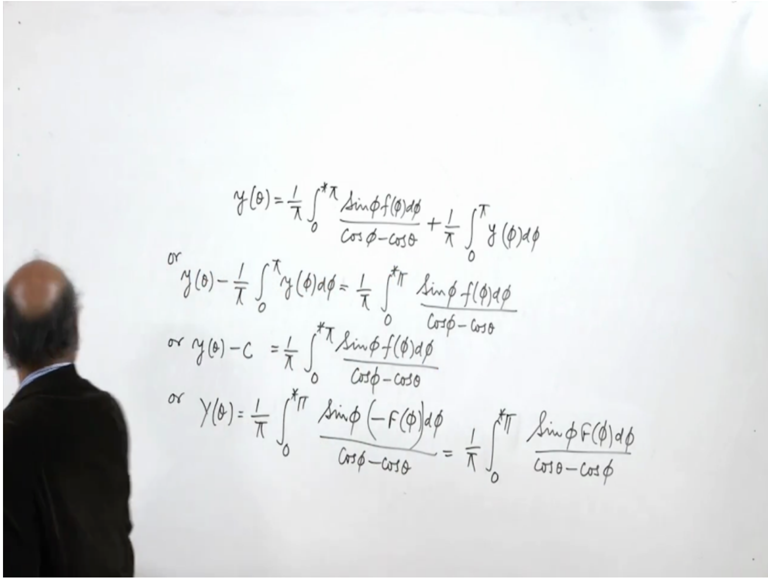

So we have equation 2 as y Theta equal to 1 over pi, 0 to pi, sin Phi f phi d phi divided by cos Phi minus cos Theta plus 1 over pi, 0 to pi, y phi d phi. We can write it as y Theta minus 1 over pi, 0 to pi y phi d phi equal to 1 over pi, 0 to pi, sin Phi, F phi, d phi divided by cos phi minus cos Theta.

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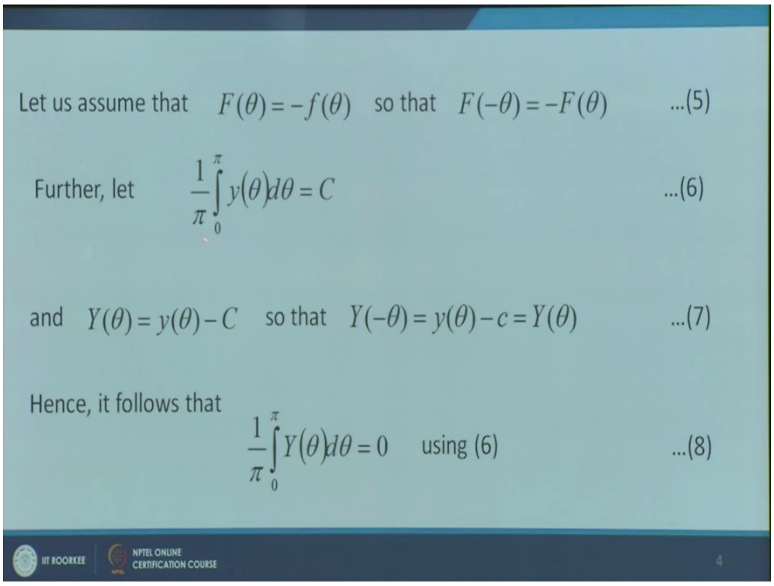

Now we have denoted   the integral 0 to pi, y theta, d theta by pi, divided by pi equal to C, so 1 over pi, 0 to pi y phi d phi will be equal to C and what we will get then? Y theta minus C equal to 1 over pi, 0 to pi, sin Phi f phi d phi over cos phi minus cos Theta we will get. Now y Theta minus C, we have denoted by capital Y Theta. So let us write her capital Y Theta.

Phi can write as, Phi minus Theta by 2 plus phi plus Theta by 2, so we can write like this sin Phi minus Theta by 2 plus Phi plus Theta by 2. We will have F phi, F phi we will have to convert to here F phi before this I can do the following because we have to convert this to the second alternative form, so this is small F Phi has to be converted to capital F phi. So F Theta is equal to minus F Theta.

So I write here minus F phi. So 1 over pi, 0 to pi sin Phi and then minus F phi d phi. Let us put f phi equal to minus F phi first, cos Phi minus cos Theta, so that I will get minus 1 over pi 0 to pi, okay. What we do is, this minus we can absolve here, I write it as 1 over pi, 0 to pi, sin Phi, f phi d phi divided by cos Theta minus cos Phi, let us write it like this.

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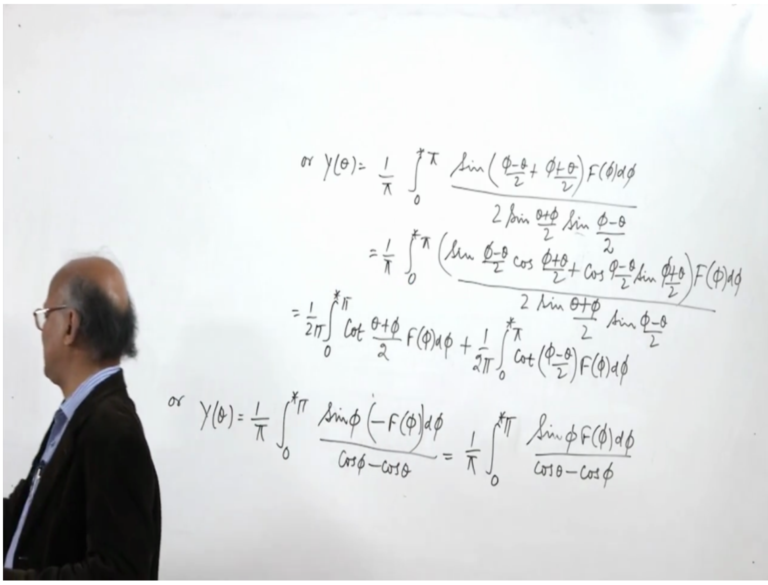

And then we will write phi s, phi minus theta by 2 plus phi plus Theta by 2, so that this will give you, so y Theta will be equal to one over pi sin Phi minus Theta by 2 plus Phi plus Theta by 2 into F phi d phi divided by twice sin Theta plus Phi by 2 into sin phi minus Theta by 2.

Now we may write it as 1 over pi, 0 to pi, sin a plus b, sin a cos b, cos a sin b. So sin Phi minus Theta by 2 into cos Phi plus Theta by 2 plus cos phi minus Theta by 2 into sin Phi plus Theta by 2 into F phi d phi divided by 2 sin theta plus phi by 2 into sin phi minus Theta by 2.When we will divide, you will get y Theta equal to, so 1 over 2 pi, sin Phi minus Theta by 2 will cancel with sin Phi minus Theta by 2, we will get cos Theta plus Phi by 2. So 0 to pi, cot Theta plus Phi by 2, F Phi d phi, okay. The second integral will be 1 over 2 pi, 0 to pi, here sin phi plus Theta by 2 will cancel and we will get cot Phi minus Theta by 2 F phi d phi.

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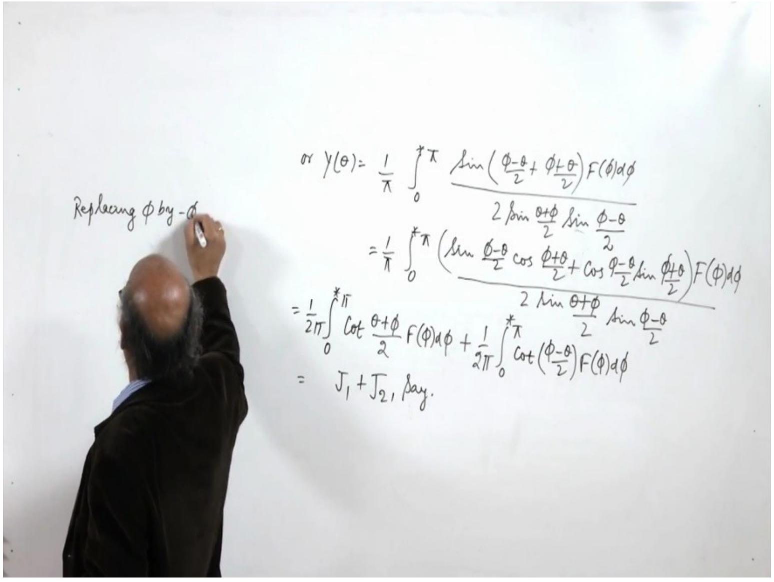

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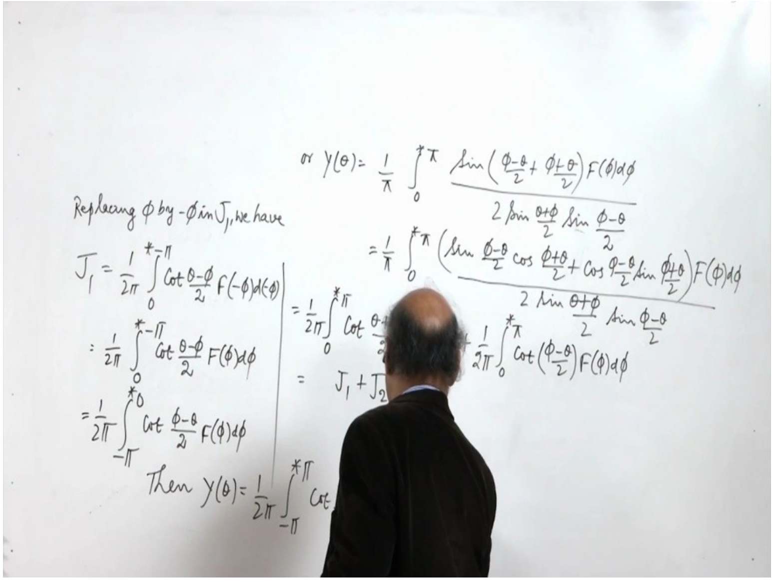

Now as we did in the case of the first alternate form, here also we can combine the 2 integrals. So replacing, let me call it as I1 and I2 or say J1 and J2 then replacing theta by replacing phi by minus phi in J1, okay. So replacing phi by minus phi in J1, we have J1 equal to 1 over 2 pi cot Theta minus phi by 2, F minus phi d minus phi. Now F minus phi, we can see here F minus Theta is minus F Theta.

So F minus Phi is minus F of phi, so what we will get? So this is 1 over 2 pi 0 to minus phi cot Theta minus phi by 2 and F minus phi is minus F phi this is minus d phi, so I get F phi d phi. Here I change the limits of integration with there will be a negative sign that minus sign we can absolve in cot of Theta minus Phi by 2. So I write it as 1 over 2 pi minus pi to 0, cot this will become Phi minus Theta by 2 F phi dphi, okay.

Now so their J1 is equal to this and this value when we put her for J1 y Theta will become, so then y Theta becomes one over 2 pi minus pi to pi, cot Phi minus Theta by 2, F phi d phi. So this is what we get, yes. So y Theta equal to 1over 2 pi, integral over minus pi to pi cot Phi minus Theta by 2, F phi d phi.

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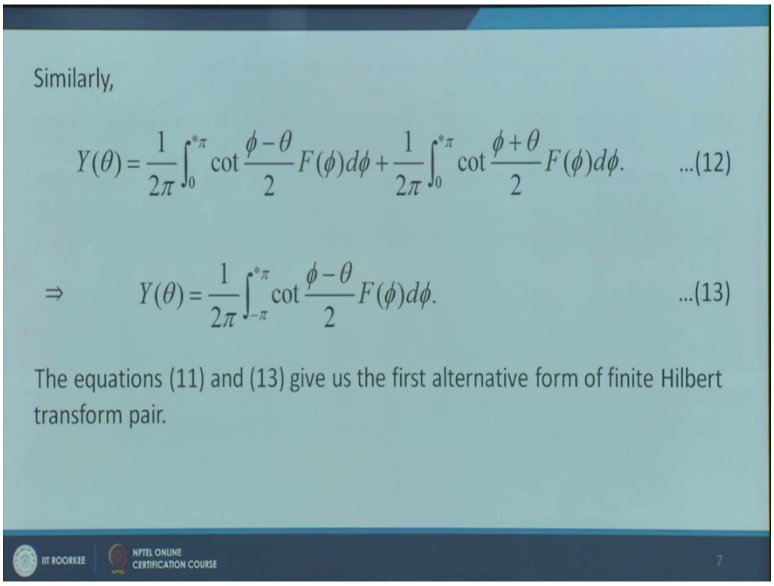

So this is the alternate form of equation 2.

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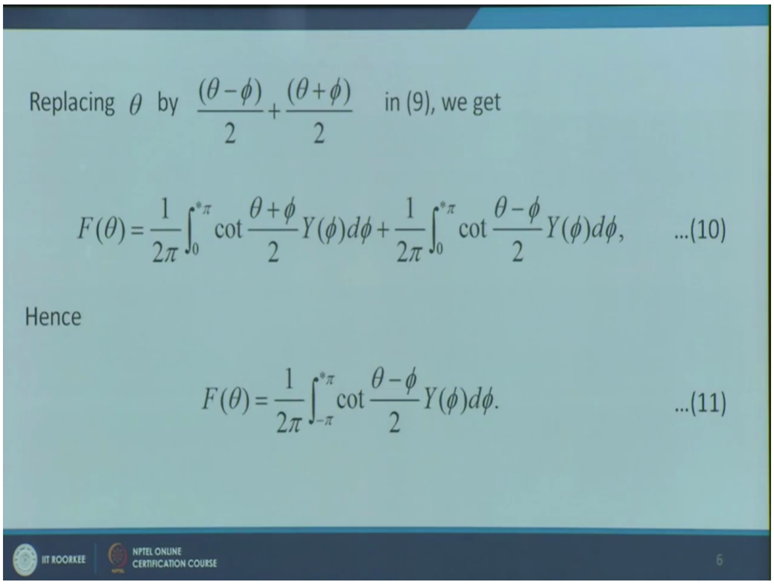

Alternate form of equation 1 we have earlier got this is F Theta equal to one over 2 pi minus pi to pi, cot theta minus phi by 2 y phi d phi. So equations 11 and 13 together give us the first alternative form of the finite Hilbert transform pair, there are 2 more alternative forms of this finite Hilbert transform pair and then there is infinite Hilbert transform that we will discuss.

So the 2 other alternative forms of finite Hilbert transform pair and the definition of the in finite Hilbert transform pair and the example we shall discuss in our next lecture, in this lecture we will now conclude I thank you very much for your attention.