**Integral equations, Calculus of Variations and their Applications**
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**Lecture 42**
**Basic Concepts and Euler's Equations**

Hello friends welcome to the today's lecture, so in this lecture we will continue our study which we have left in previous lecture. So if you recall in previous lecture we have discussed the concept of continuity. So continuity of a functional it is we have defined in the sense of m-th proximity and it is given by discontinuity of a functional.

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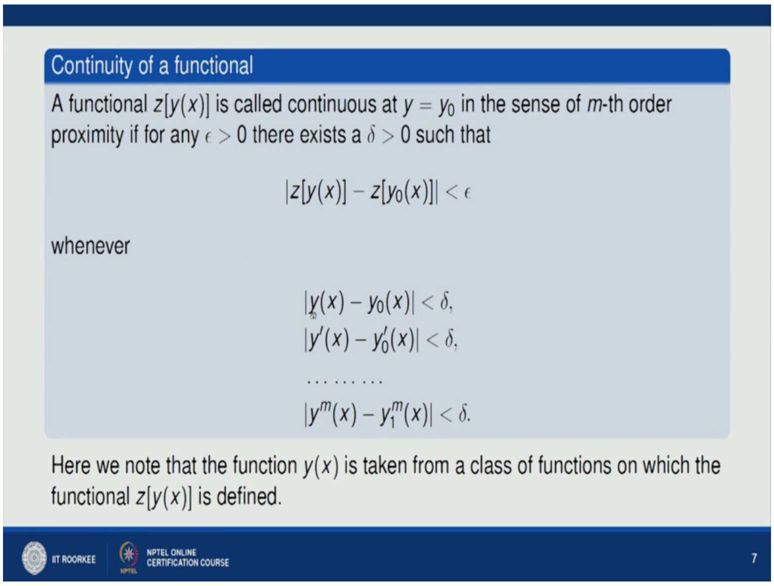


So here we say that a functional z yx is called continuous at y equal to y not in the sense of meth order proximity for any Epsilon greater than 0 there exist a Delta greater than 0 such that the absolute value of difference of the absolute value zyx minus z y not x is less then Epsilon provided modulus of yx minus y not x is less then delta, modulus of y dash x minus y not dash x is less than Delta till that modulus of ymx minus ymx is less than Delta and here all these yx are belonging to the class on which this functional is defined and after defining this functional, we have reached up to.

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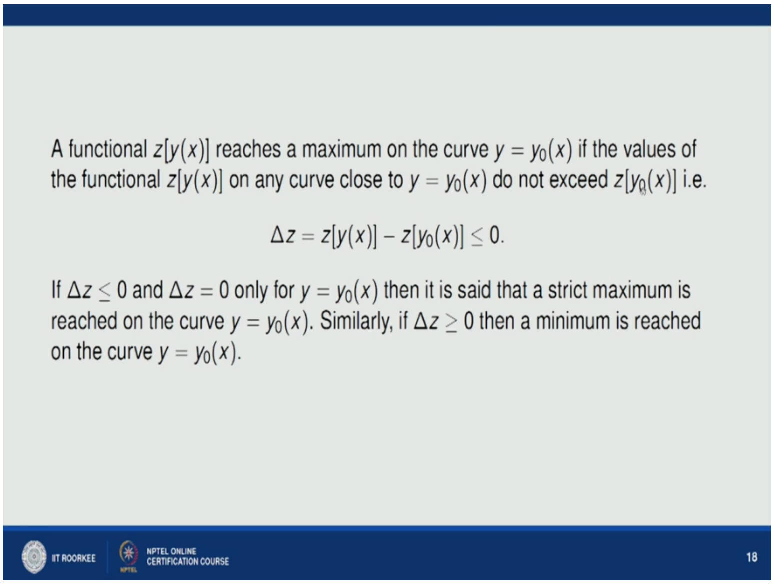


So with the help of increment in functional we can define now maximum and minimum value and we can say that a functional zy of x reaches a maximum on the curve y equal to y not x if the values of the functional z yx on any curve close to y equal to y not x do not exceed as the value of z on y not x. It means that if we find out this value increment Delta z which is defined as z of y of x minus z of y not x this has to be less than or equal to 0 and if the equality is achieved only for y equal to y not x then it is said that a strict maximum is reached on the curve y equal to y not x.

Now if we define, if we take delta Z as greater than or equal to 0 then this y not x is said to be the minimum of the functional z of y not of x and we say that a minimum is reached on the curve y equal to y not x.

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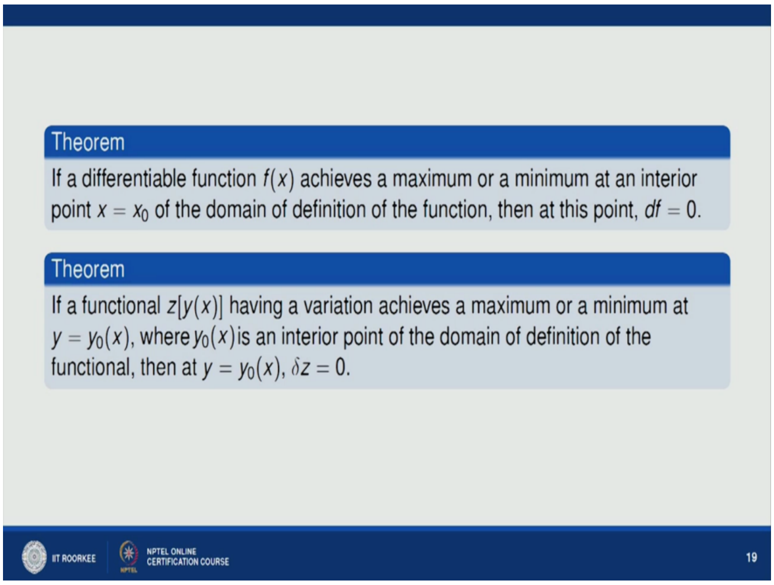


And now if you remember the theorem in to find out a maxima and minima for a function is given by this, that if differentiable functional f of x achieves a maximum or minimum at an interior point x equal to x not of the domain of definition of the function than at this point means at x equal to x not your differential is equal to 0.

So in a similar line we want to define a theorem which gives a maximum or minimum at the given curve y equal to y not x, we can say that if a functional z of y of x having a variation achieves a maximum or minimum at y equal to y not x where y not x is an interior point of the domain of the definition of the functional, then at y equal to y not x, your variation of Z is equal to 0.

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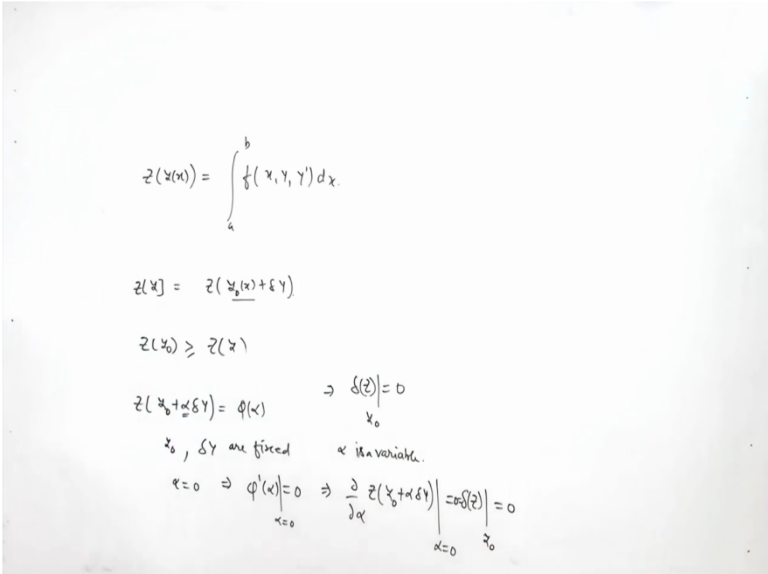


This we can explain it in this manner, so here it is given that z of y of y not of x is defined as say a to b, f of say x, y, y dash, d of x. Now to find out let us say Delta of z is basically what? You can say that delta of z is basically z of say y not x plus delta of y. Now here your delta y is a variable. So here we can say that if Delta y is a variable and we say that this z, let me write it here z of y not.

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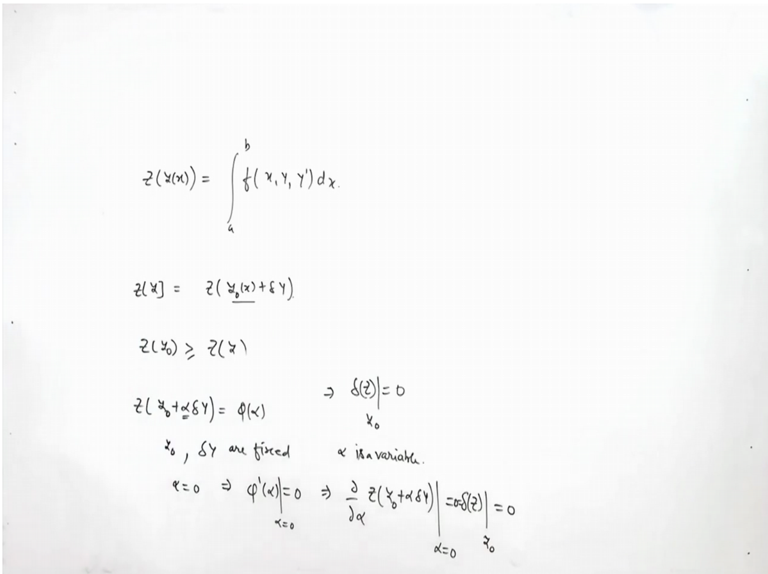


So this z of y not, let me write it here. So here we can say that z of y achieve maximum at the point y not here, so it means that z of y not is greater than r. If I assume that this y not it is a curve on which this functional achieves the maximum then z of y not is greater than or equal to z of all y in a neighbourhood of say y. So with the help of this we define a functional say z of y as follows y not plus Alpha, Delta, y.

So here in place of y I'm writing y not plus Alpha, Delta, y. Here we are assuming that y not is fixed which is already given and delta y is also fixed and only this Alpha is a variable quantity. So here we can say that alpha is a variable quantity. So we can say that if we consider this functional where y is replaced by y not plus Alpha, Delta, y keeping y not and Delta y as fixed then here we can say that this functional achieves maxima for alpha equal to 0.

So this will achieve a maximum for alpha equal to 0 and we can say consider that this as opposed this is a function of alpha here, so we can say that this functional is now considered as a function in terms of alpha, so if you vary this alpha you will see that this y not plus Alpha, Delta, y give a function in the domain of z and so for every alpha define a function on which this z is evaluated and we can say that at alpha equal to 0 this functional achieves the maximum value.

So it means that by the theorem which we have discussed for functions that if a function achieves an extrema at say any given point at that point particular point the derivative has to be 0. So here in this case this Phi alpha is achieving the maximum value at alpha equal to 0. So this implies that phi dash alpha has to be 0 at alpha equal to 0 and if you calculate this phi dash alpha had alpha equal to 0 it is coming out to be dabba by dabba alpha and z of y not plus alpha Delta y evaluated at alpha equal to 0 and this we have already seen that this is nothing but your variation delta z evaluated at y not and this is coming out to be 0.

So here using the theorem for functions we have shown that this variation this functional is equal to phi of alpha and at alpha equal to 0, since it is achieving the maximum thing because at alpha equal to 0 this is reduced to z of y not and z of y not is a maximum of this functional. So here we can say that phi dash alpha at alpha equal to 0 implies that this dabba by dabba alpha z of y not plus alpha Delta y evaluated at alpha equal to 0 is equal to 0 and this quantity is nothing but Delta z evaluated at y not equal to 0. So we can say that this implies that delta Z equal to 0 and which is evaluated at y not, so that is the theorem here.

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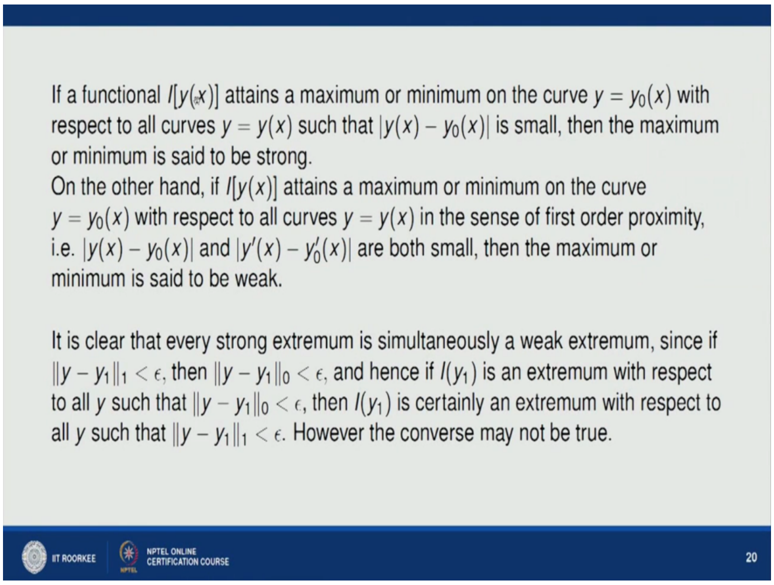


Next we can say that if a functional I y by x attains a maximum or minima on the curve y equal to y not x with respect to all curves y equal to y not x such that modulus of y of x minus y not x is a small, then the maximum or minimum is said to be strong because we have seen that whenever we define the neighbourhood of a given curve then we can define the neighbourhood was a close curves are any curve which is near to y of x in the sense of m-th order proximity. So if we define the closeness in the sense of 0 order proximity and in that case if your functional achieves a maxima or a minima in that case maxima or minimize is said to be strong.

So that is what we are trying to write here, that if your functional here I am denoting functional as I of y of x if this achieves a maximum or minima on the curve y equal to y not x with respect to all curves y equal to y of x such that this difference is small which is nothing but y of x is close to y not x in the sense of 0 order proximity then we say that this maxima or minima is said to be strong.

And similarly if we say that this functional attains a maximum or a minima on the curve y equal to y not x with respect to all the curves y equal to y of x which is close to y not x in the sense of first order proximity it means that this quantity y of x minus y not x and y dash x minus y not dash x are both small, than the maximum or minimum is said to be weak. So if maximum is attained in the sense of 0 order proximity we call it strong maxima or minima.

If maxima or minima of a functional is attained with respect to first order proximity then we call that maxima or minima as weak maximum minima and we can say that every strong extremum is simultaneously a weak extremum. It means that if we have a strong extremum then we can say that it is weak extremum as well. So here extremum means either Maxima or minima.

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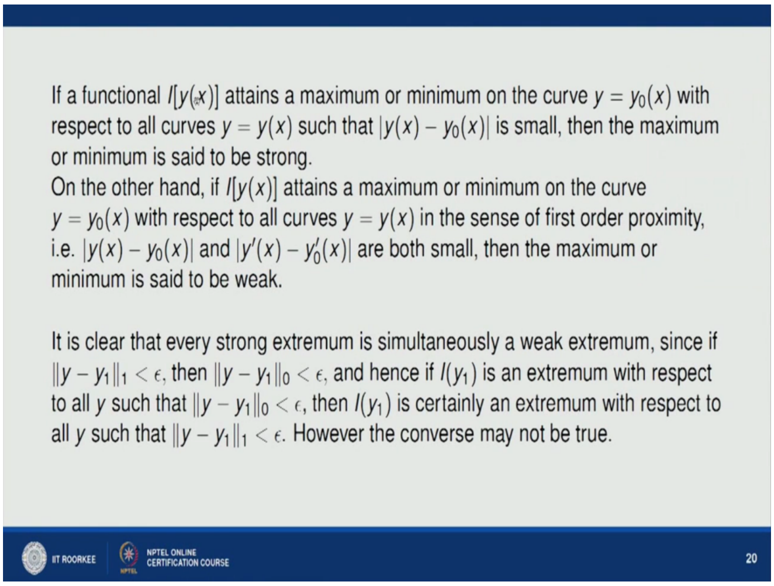


So here we can say that, how it is clear? Let us see this that since if say y is close to y1 in the sense of first order proximity. Here modulus of y minus y not less than or equal to 1 means that this both y dash y dash minus x minus y not dash x it is small and y of x minus y not, x is small.

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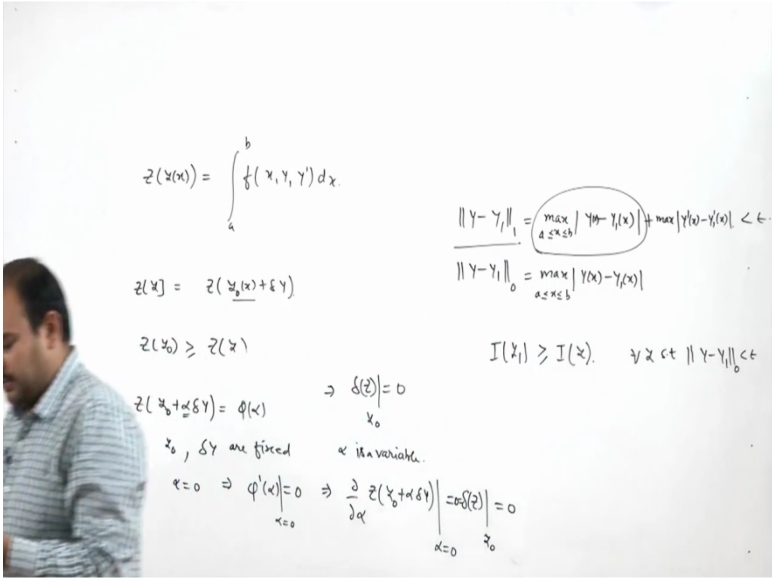

Here we have defined this norm y minus y1 is equal to say maximum of y, y of x minus y1 of x plus you can say that maximum of y dash, x minus y1 dash x.

So if I say that this quantity is less than small less than some number, if it small then these 2 are also less than is very small. So here we can say that if y and okay and similarly we can define the 0 norm like this y minus y1 in the sense of 0 norm, it is only that maximum y of x minus y1 of x here and x is lying between the range of the domain here. So it is in the domain of y.

So x is say a to b here, so it is in the sense of 0 norm it is in the sense of 1 norm, so if I say that y is close to y1 in the sense of 1 norm then it is also near to y1 in the sense of 0 norm because if this quantity is less than Epsilon than it is automatically that this quantity is less than Epsilon because if the bigger quantity is less than Epsilon then small quantity is certainly be less than Epsilon.

So if I say that y is close to y1 in the sense of first order proximity it is trivial to observe that y is close to y1 in the sense of 0 order proximity. So if I say that y1 is a strong maximum it means that I of y1 is greater than equal to I of y, for all y belong all y such that, that y minus y1 is 0 norm is less than Epsilon. So if y1 is strong maximum it means that I of y1 is greater than equal to I of y, for all y which is close to y1 in the sense of 0 norm.

Then we want to show that it is also a weak maximum because if we take all the curves which is close to y1 in the sense of 1 norm then it is also close to y1 with respect to 0 norm. So it means that we have already considered the weak extremum here. So it means that if we have a y1 as a strong maximum among all the curves which is closed to y1 with respect to 0 norm then it is also contains, all those curves which is close to y1 in the sense of 1 norms, okay.

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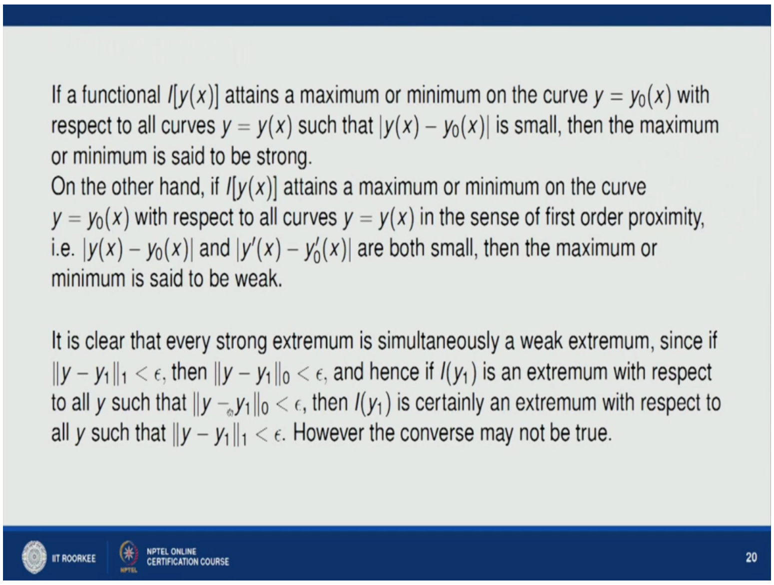


So let me summarize this and we can say that if I of y1 is an extremum with respect to all y such that y is close to y1 in the sense of 0 norm then I of y1 is certainly an extremum with respect to all y such that norm of y minus y1 with the sense of 1 is less than Epsilon but this converse may not be true.

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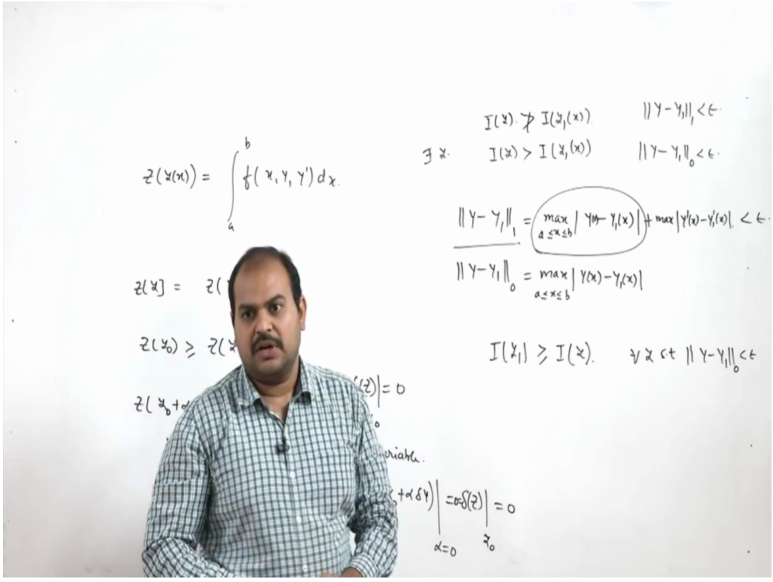

So it means that it may happen that if we consider the maxima or minima of this function I of y then we may find a curve in the sense of first order proximity that I of y is not equal to I of y1 of x it means that there is no curve y which is near to this y1 in the sense of y minus y1 is less than Epsilon but it may happen that there exist a curve y, such that I of y do greater than r equal to I of y1 of x in the sense of y minus y1 as in 0 less than Epsilon.

So it may happen that there is no y exist which is close to y1 in the sense of 1 norm and their value is not greater than I equal to I of y1 x or we can say that this y1 is the weak extremum of the function I but it may happen that if we consider all the curves in the sense of norm of y minus y1 is less than r equal to 0 in the sense of 0 order proximity, it may happen the value of functional at this particular on this function may be bigger than this.

So it means that there is a possibility that weak extremum may not be strong extremum but here we have shown that the strong extremum will be a weak extremum is it okay. So here strong implies weak but weak may not be strong and in fact we generally consider all the time weak extremum because the functional which we are considering is continuous with respect to first order proximity.

So though all this analysis all this discussion may not say help, may not disturb anything for the say finding the necessary condition for an extremum. So now we want to see what is the necessary condition to find out an extremum of a functional? So we want to find out that under what condition a given curve is an extremum of a given functional. So let us say that find the necessary condition, if you remember the necessary condition for maximum and minima of a function is that at the point of extremum the derivative has to be 0.

So here also we want to find out a similar kind of condition but here we are not bothering about, right now we are not bothering about the existence of extremum whether extrema exists or not or what kind of smoothness condition it should satisfy? Rather than we're trying to find out that if a function is giving you the extremum of a functional then what is looking like?

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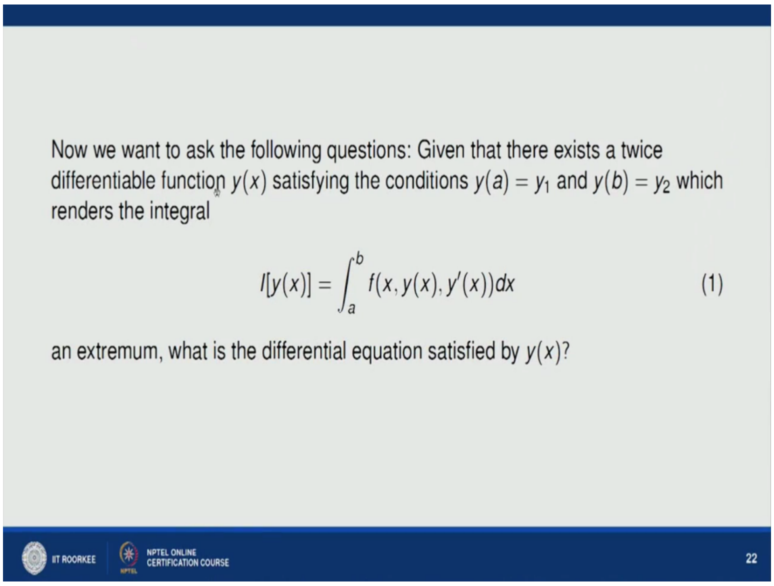


Or we can say that what kind of differential equation it should be satisfied? So here we may ask this question the that given that there exist a twice differentiable function y of x satisfying the condition y of a equal to y1 and y of b equal to y2 and which makes this integral I of y of x as an extremum. So it means that this y of x is the extremum of this functional and we want to find out what is a differential equation which is satisfied by this extremal function? Okay.

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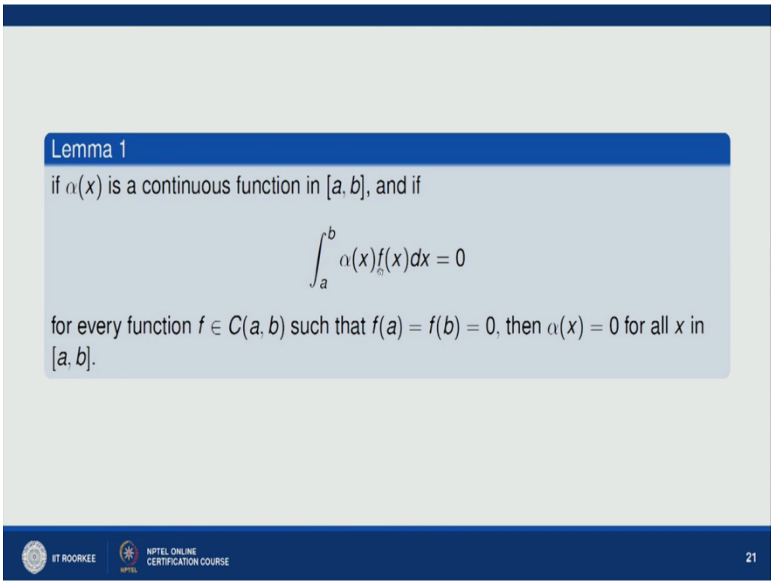


So before proceeding before answering this question we have to we are considering this a Lemma which says that if alpha x is a continuous function in close interval a, b and if a to b alpha of x, f of x, dx equal to 0 for every continuous function on a close interval a, b. Such that f of a equal to f of b equal to 0 then in this case your alpha x has to be 0, right? And has to be 0 for all x in close interval a, b.

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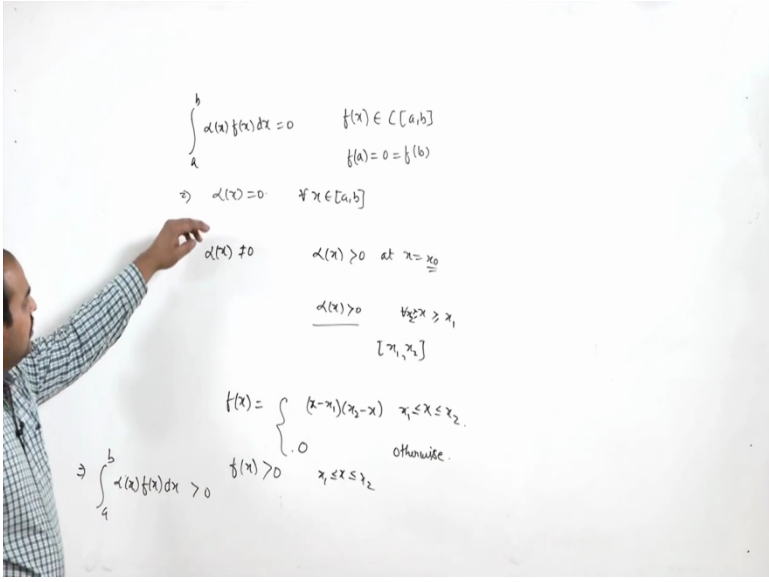

So there is a small proof of this, let us discuss this. So here we want to find out that if a to b, alpha of x, f of x, d of x is equal to 0, for all function f of x which is belonging to continue this is function space that is c a, b it means that function f of x is continuous on this interval satisfying the condition that f of a equal to 0 equal to f of b then we want to show that alpha of x is equal to 0.

So this we can prove in a sense that suppose that alpha x is not equal to 0 on this entire interval a, b, so it means that say that alpha x is not equal to 0. So we can consider that alpha x is say suppose some positive let us that alpha of x is positive say at x equal to say x not. Now since alpha x we are assuming this as a continuous function then we can say that we can always find a neighbourhood around x not such that alpha of x is positive for every x greater than or equal to some say x1 less than or equal to x2.

So we can say that alpha x is positive in the interval x1 to x2. So here it is positive in x1 to x2. Now once we know that alpha x is positive then we can define since this is true for every function f of x having this property then we can define our function f of x in this manner that x minus x1 and x2 minus x here. If you look at here this will satisfy the condition that f of a equal to 0 or equal to f of b because here we are finding that this is equal to this when x is lying between x1 and x2 and 0 otherwise.

So it will satisfy the condition that f of a equal to 0 equal to f of b and it is a continuous function between a to b and not only this, it is f of x is positive in this interval x greater than equal to x1 and x2. So if we assume f of x to be denoted as this then you can say that then this implies that a to b, alpha of x, f of x, d of x is going to be greater than 0, why? Because alpha x is positive in this reason and f of x by definition is positive in this reason then integrand is positive and this a, b is we are assuming that b is greater than a, so it is also positive.

So you can say that this is positive which contradict the assumption that it is equal to 0. So it means that this contradiction we are achieving because we have assumed that alpha x is not equal to 0 at some point inside this. So it means that alpha x cannot be a nonzero value, so it has to be 0 for entire interval a, b, okay. So using this lemma we are trying to find out say extremum of the function y of x.

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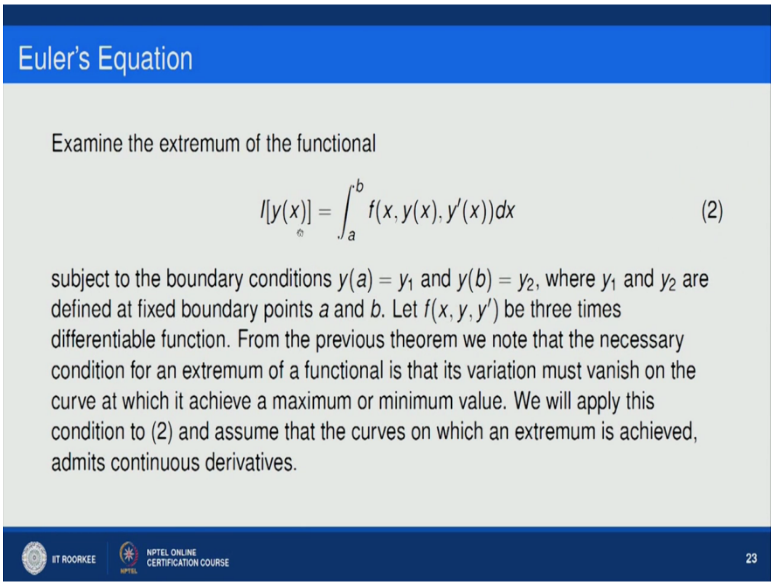


We want to find out the condition which an extremum function will satisfy, so moving this let us consider the Euler equation. This Euler equation here we are trying to find out the answer of this question that if y equal to y not x extremize the function I of y of x is given as a to b, f of x, y of x, y dash x, d of x satisfying the condition y of a equal to y1 and y of b equal to y2 then what kind of condition it should satisfy?

What kind of differential equation it should satisfy? So here we are assuming certain condition on this f, f here. So here we are assuming that f of x, y, y dash be 3 times differentiable function, in fact if we take 2 times differentiable function then it is quite okay, no problem and here we have seen that if we have an extremal of a functional then at this point, so we have seen it from the previous theorem we note that the necessary condition for an extremum of a functional is that its variation must vanish on the curve at which it achieves a maximum or minimum value.

So we will apply this condition to this i of y of x and assume that the curve on which an extremum is achieved admits the continuous derivative. So here we assume that if y is the curve of extremum then it has a continuous derivative.

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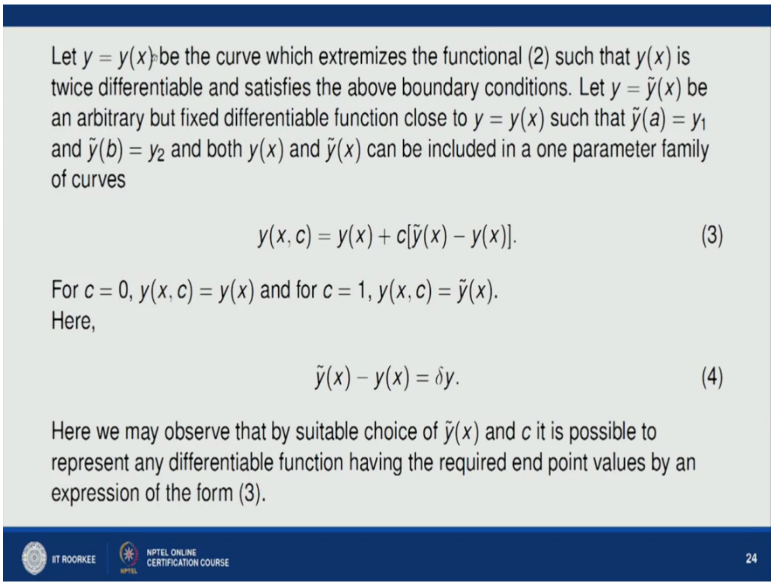


So let us assume that let y equal to y of x be the curve which extremizes the functional 2 and it also satisfy the boundary condition and it is twice differentiable and satisfy the boundary condition. So let me write it like this, let y of yx be the curve which extremizes the functional 2 such that yx is twice differentiable and satisfy the above boundary condition that is y of y of a equal to.

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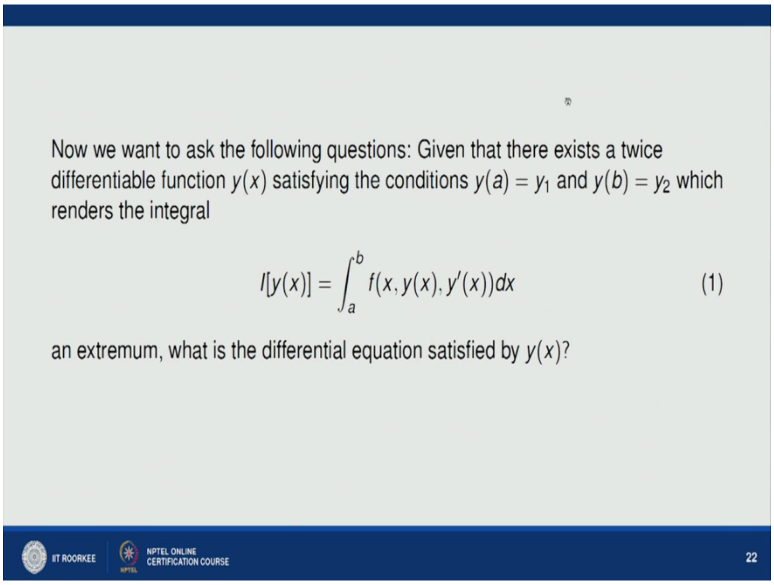


Here is the boundary y of a equal to y1 and y of b equal to y2 and let us take a arbitrary fixed differentiable function which is close to y of x, arbitrary in the sense that it can be any differential function satisfying the boundary condition y tilde a equal to y1 and y tilde b equal to y2 in the neighbourhood of y of x fixed in the sense that once it is taken it is fixed.

So and we say that we define with the help of y and y tilde x a new one parameter family of comparison curves, so here we define this family as y of x, c which is simply which gives you a rare combination between y of x and y tilde x and it is defined as y of x plus c y tilde x minus y of x. So if you look at your equation number 3 then at c equal to 0 this y of x plus c reduces to y of x and if you take c equal to 1 then this y of x, c reduces to y tilde x.

So we can say that as c vary then it y of x, c represents a differentiable function which is which can be written as a linear combination of y of x and y tilde x and here we denote this difference y tilde x minus y of x as delta y. So here we say that by suitable choice of y tilde x and c is possible to represent any differentiable function having the required endpoint values by an expression of the form 3.

Because if you look at here, if you evaluate y x, c at x equal to a, you will see that it is taking the value y of 1, so that is not very difficult to see you put x equal to a. So y of a, c equal to y of a plus c y tilde a minus y a, since y tilde a and y of a is same that is equal to y1. So this will cancel out and y of a, c is given by y of a that is y1.

Similarly you can verify that y of b, c is equal to y of b that is equal to y of 2. So it means that this y of x, c represents a differentiable function having fixed end point that is that y of a, c is equal to y1 and y of b, c is equal to y2 and this also contain the extremizing curve that is y of x at c equal to 0 and not only this, this also represents any differentiable function in the neighbourhood of y of x by suitable choice of y tilde x.

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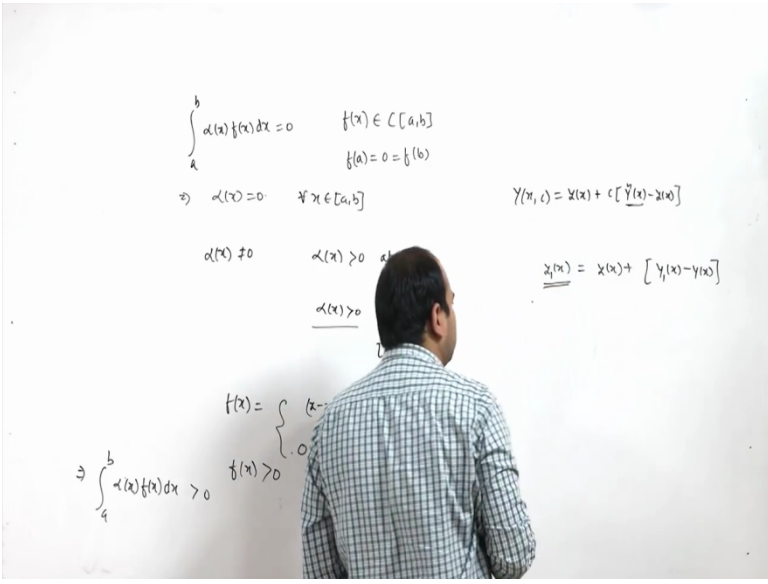

In fact if you want to represent for the time being let us see here, y of x, c is equal to y of x plus c, y tilde x minus y of x. So here if I want to represent in the neighbourhood y of x, let us say represent your y1x, right? So I can write y1x as y of x plus c time's c means you can take as 1. So here you can write it y1x minus y of x, so this y1x can be written as y of x plus y1x minus y of x.

So it means that this y1x can be written as if we take y tilde x as equal to y1x and c as 1 then this can also be written in the form of y of x, c. So it means that every differentiable function can be represented as the form n3. So it means and having the same property that y of a, c is equal to y1 and y of b, c is y2. So now if we use this as comparison function and find out say functional at this on in this comparison on this y of x, c then it becomes a functional, this functional becomes a function in terms of c.

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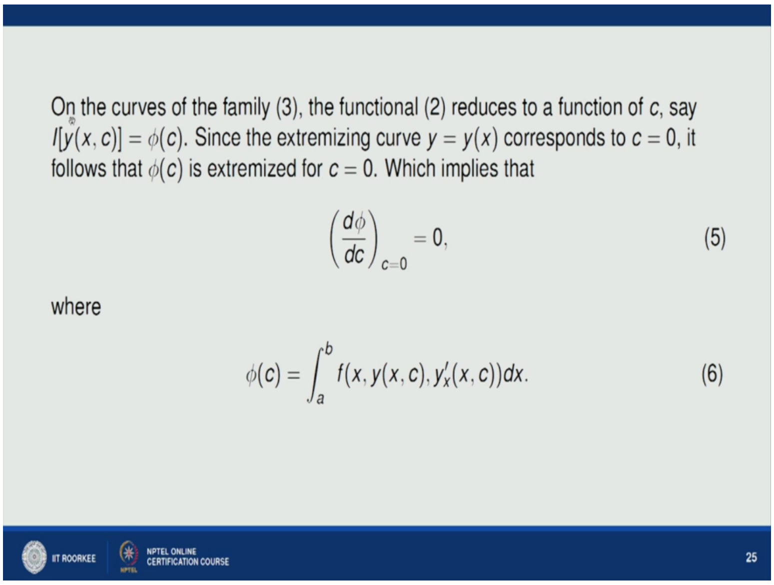


So here we can say that, on the curve of the family 3, the functional 2 reduces to a function of c. So it means that if we take this as a curve then on this curve this 2 converted to be a function of c, right? And we already know that since the extremizing curve y equal to y of x corresponds to c equal to 0.

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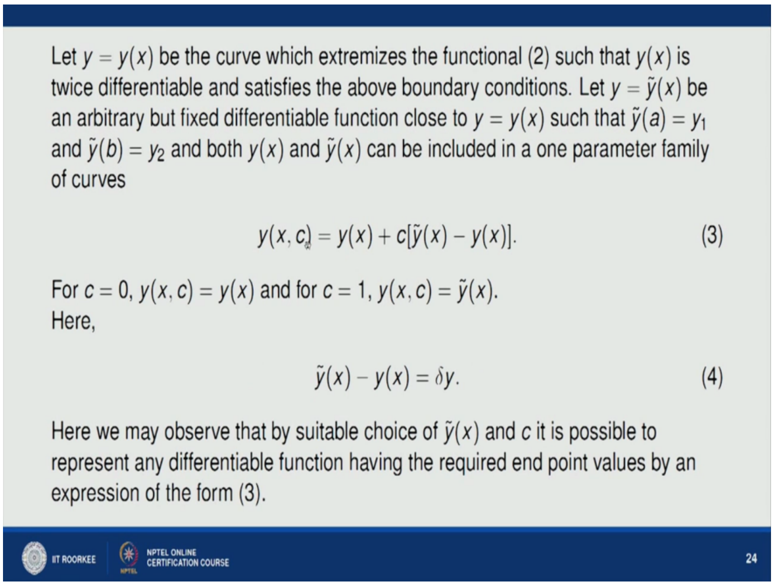


So here if you look at y of x, c we already know that if c, if we take c equal to 0 then this y of x, c is y of x which is an extremum function for functional.

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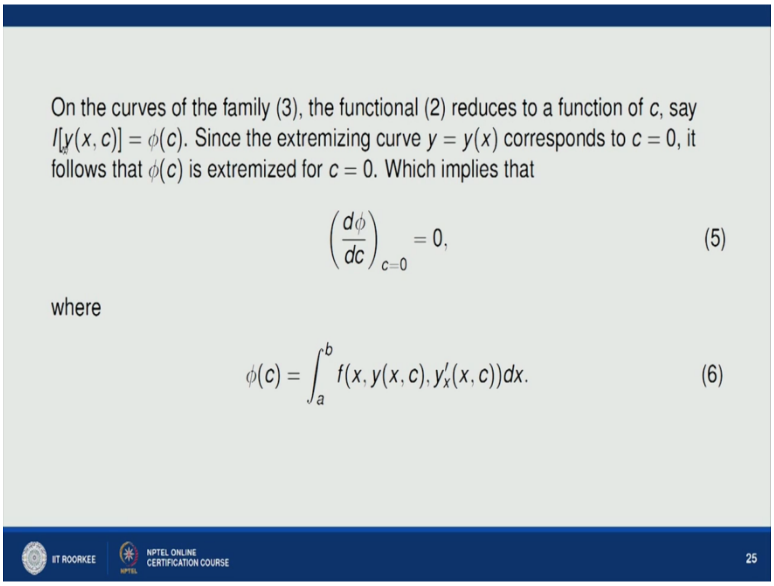


So we can say that this functional at c equal to 0 will be a maximum value for this I of y x, c. So we can say that phi of c is extremized for c equal to 0 and we already know the theorem for function that if phi achieves the extremum at c equal to 0 than its derivative with respect to c, at c equals to 0 has to be 0.

So evaluating the derivative here we have phi of c equal to a to b, f of x, y x,c, y x dash x, c d of x. So we need to find out say the derivative of this phi of c with respect to c and it is coming out to be that phi dash c is equal to, since f is a function of x, y and y x, so here and c is contained only in these 2 things .

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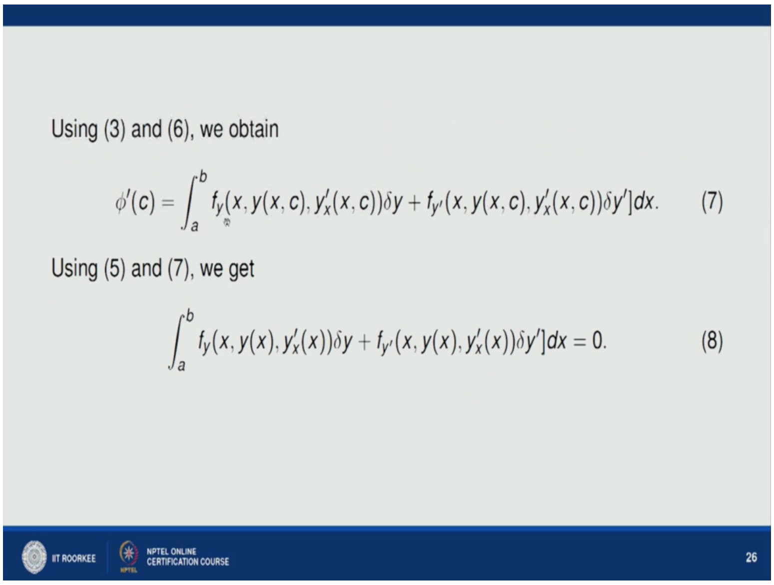


So we find out some derivative we use the definition of composite function and we can write it phi dash c as a to b derivative of f with respect to y and yx, c and derivative of yx, c with respect to c will be what? If you look at here you have y of x, c, right?

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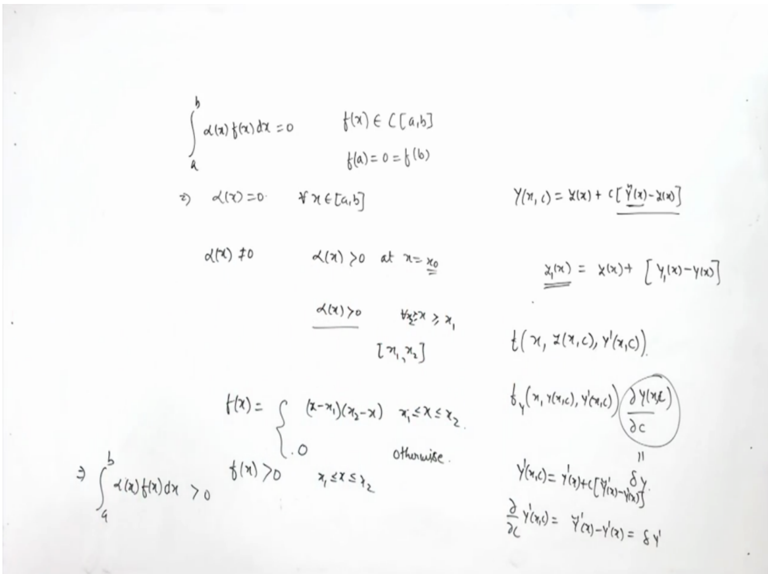

And so here we have fx, yx, c and y dash x, c, right? And if you want to find out say derivative of this, so derivative will be f of y x, yx, c, y dash x, c and then derivative of y x, c with respect to dabba c. Now what is yx, c? Is this, if you find out say derivative of yx, c with respect to c, you will get only delta y. So this we have represented as delta y, so here the first term is written as fy, x, y, x, c, y dash x, c into this is nothing but your delta y.

In a similar way if we differentiate delta differentiate y dash x, c with respect to c you will see you will get Delta y dash, so here let me write it here, y dash x, c will be what? It is nothing but y dash x plus c times y tilde dash x minus y dash x this. Now if you find out dabba by dabba c of y dash x, c it is nothing but your y tilde dash x minus y dash x and this we can represent as delta y dash.

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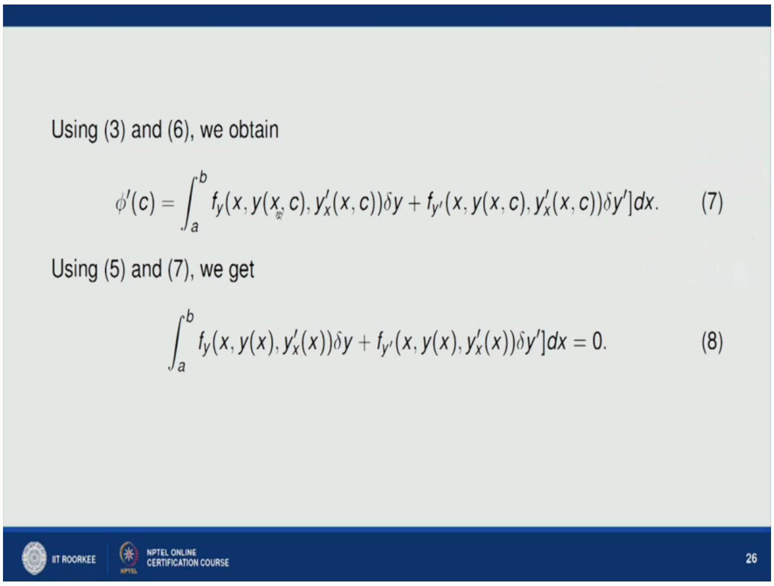


So using this we can write Phi dash c as a to b, fy x, yx, c yx dash, x, c delta y, so this is nothing but dabba y x, c y del c. Similarly we can say that the second term will be fy dash delta y dash, now here since y dash c equal to 0 then this is equal to 0. Now here we can simplify the second term in a way such that you can use integration by part and shift the derivative of Delta y dash on fy dash. So here we differentiate, on integrating the second term by parts subjecting to the boundary conditions delta y a equal to 0 and delta y b equal to 0 we may obtain like this.

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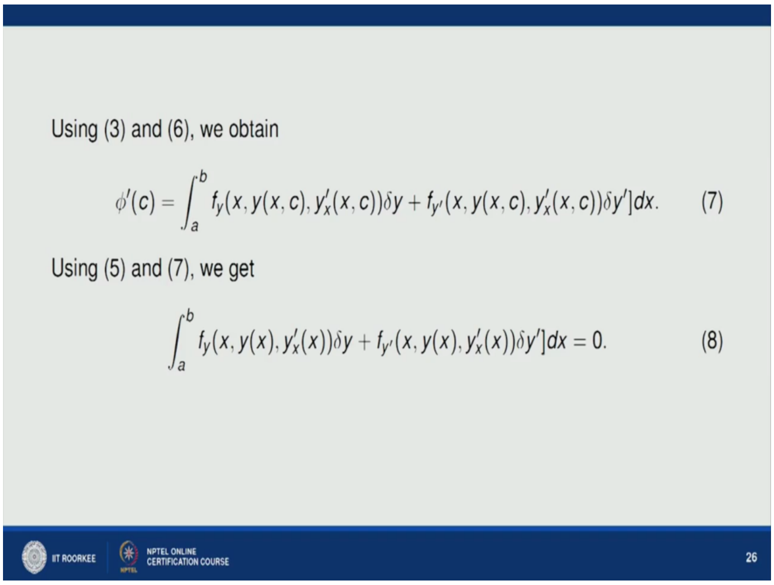


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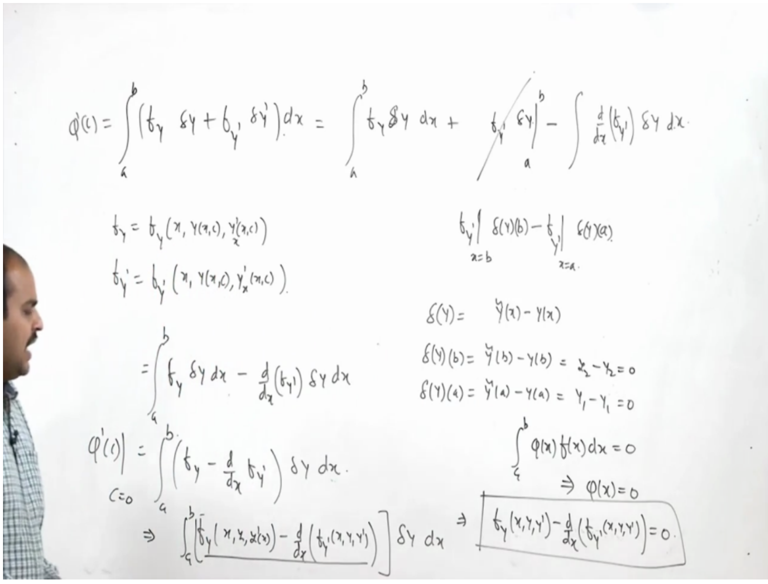

Let me write it here, so here we can write a to b fy delta y plus, f of y dash and this is with respect to d of x. So here I'm using a short notation for f of y, in fact f of y is written as f of y, x, y, x, c, y dash x, c and this derivative with respect to x, so we are denoting it like this similarly we can write it f of y dash. Now here we can say that first term is as it is a to b, f o fy delta y and d of x plus we're just looking at this thing and we are using integration by part.

So here we can write it that f of y dash delta y and at a and the minus write it here d by dx of f of y dash and Delta y and Delta x, dx sorry, this is d of x. So here if you look at, what is the value here? So here it is f of y dash evaluated at x equal to say b into delta y of b minus f y dash evaluated at x equal to a and Delta y evaluated at a. Now here if you remember delta of y is basically y tilde x minus y of x.

And if you calculate delta y of b than it is nothing but y tilde b minus y of b and here remember that both y and y tilde b which achieves the same value that is satisfying the boundary conditions so we can say that this is nothing but y2 minus y2 it is equal to 0. Similarly you can calculate that delta y evaluated at a is equal to y tilde a minus y of a that is nothing but y1 minus y1 is equal to 0.

So it means that this boundary term will simply vanish and this can be reduced to a to b, say fy, delta y d of x minus d by dx f of y dash delta y d of x. So we can simplify and you can write it this that a to b this is the expression for phi dash c, right? So phi dash c is equal to this thing, so here we can write it f of y minus d of dx f of y dash delta y d of x. Now please remember here I'm just using the short iterations, so it means that here this fy and all these things is written like this, so here f of y dash is equal to f of y dash your x, y x, c y dash x x, c and we already know that this function y dash c evaluated at c equal to 0, so here we are evaluating at c equal to 0.

So if evaluated at c equal to 0 then this is nothing but a to b, f of y, let me write, now the x of y, y dash x minus d by dx of f of y dash, x, y, y dash into delta y and d of x. Now please remember here I hope you remember that by resumption we have assumed that f is an thrice differentiable function in terms of its argument, so we can say that fy and this whole thing is continuous function respect to x.

Similarly your delta y is a continuous function but here delta y is nothing but y tilde x minus y of x and if you remember this y tilde x is an arbitrary function, so we can say that this delta y is an arbitrary continuous function in the argument, so delta y is an arbitrary continuous function satisfying the condition that delta y b is equal to 0 and delta y a is equal to 0.

So if you recall the lemma here, so lemma is this that a to b and you can say that phi of x, f of x, d of x equal to 0, so if we say that f of x satisfying is an arbitrary continuous function satisfying the condition that f of a equal 0 and f of b equal to 0 then this implies that your phi of x has to be 0. So using this lemma we can say that the integrand here is equal to 0, so this will give you that f of y, xy y dash minus d of d of x f of y dash that is xy dash is equal to 0 and this form this equation is known as Euler's equation.

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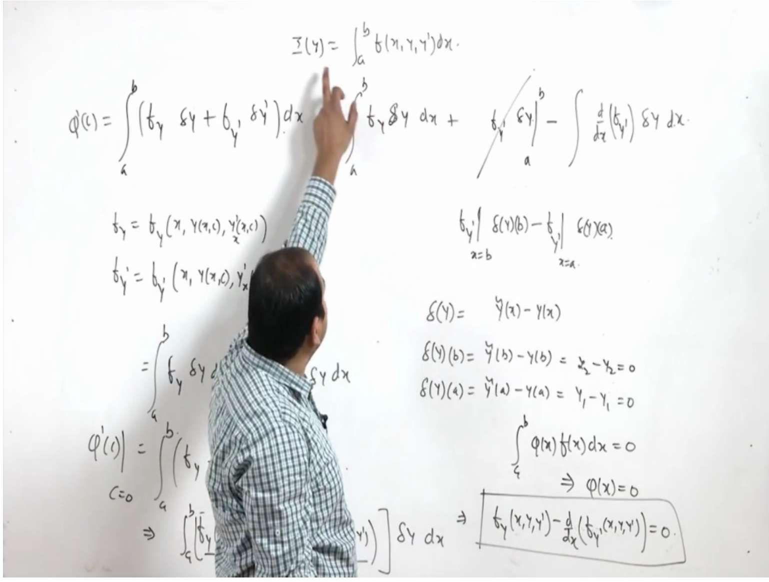

So it says that if y is the curve which extremize the functional I'm just writing this function I of y is equal to a to b, f of x, y, y dash d of x, so if I say that if y is the curve which extremizes then this y must satisfy this equation, this is a differential equation if you simplify this, this will be a second-order differential equation, let me write it here.

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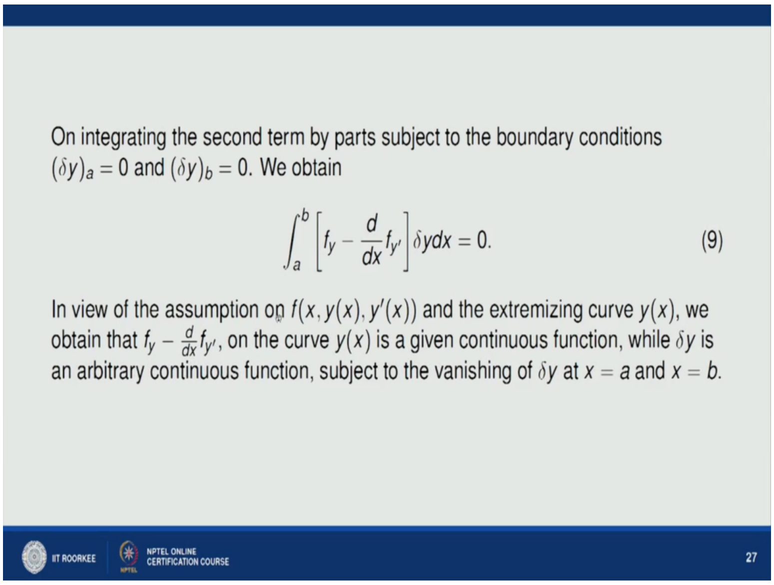


So this we have just achieved that in view of the assumption on f of x, y of x, y dash x and the extremizing curve y of x, we obtain that f of y minus d by dx f y dash on the curve y of x is given continuous function, while delta y is an arbitrary continuous function, subject to the vanishing of Delta y at x equal to a and x equal to b.

So we can apply this lemma and we can say that fy minus d by dx fy dash is equal to 0 on the extremizing curve y equal to y of x. So it means that if y equal to yx is an extremizing it must satisfy this equation which is known as Euler equation and if you simplify this f of y is this and if you simplify d by dx of fy dash then f of y dash maybe function of x, y and y dash, so I write it minus f of xy dash minus f of yy dash y dash minus f of y dash y dash y double dash is equal to 0.

So this Euler's equation is reduced to this second-order differential equation in terms of y double dash x. So here y is extremizing curve then it must be a solution of this second-order differential equation in y of x. So in so this here we stop here this is a necessary condition for an extremizing curve that if it is an extremizing curve it has to satisfy Euler function Euler equation.

So in next class we will discuss, in next lecture we will discuss that how to find out this extremizing curve and what are the other sub cases for function f of xy, y dash that in which this Euler's equation reduce to a simpler form? And we will try to solve certain other problems, so thank you for listening us and thanks.