Integral Equations, Calculus of Variations and their Applications
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Lecture 48
Variational Problems in Parametric Form

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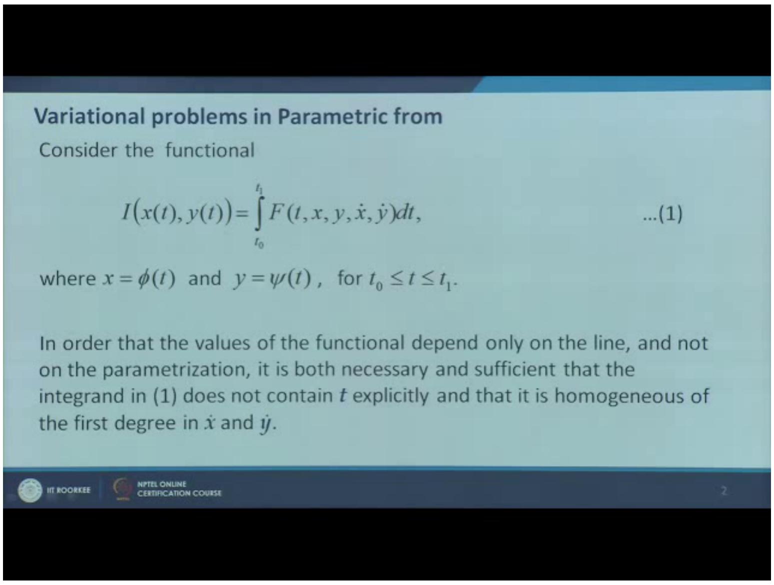

Hello Friends! Welcome to my lecture on Variational Problems in the Parametric form. Sometimes it is convenient advisable to (())(00:27) the variational problem to consider the Variational Problems in the Parametric form. In the Parametric form what we have is we have the functional eye where x i depending on x and y where x n and y are functions of the parameter t and the integral is from t 0 to t 1 f(t) x y x dot y dot dt where x is a function of t and y is a function of t x is given by phi(t) y is given by psi(t) for t 0 less than or equal to t less than or equal to t 1.

Now in order that the values of the functional depend only on the line and not on the parameterization it is necessary and sufficient that the integrant in one does not contain t explicitly and that is that it is homogeneous in the first degree in x dot and y dot. Now what do you mean by the homogeneity in the first degree the first degree in x dot and y dot, here is the definition.

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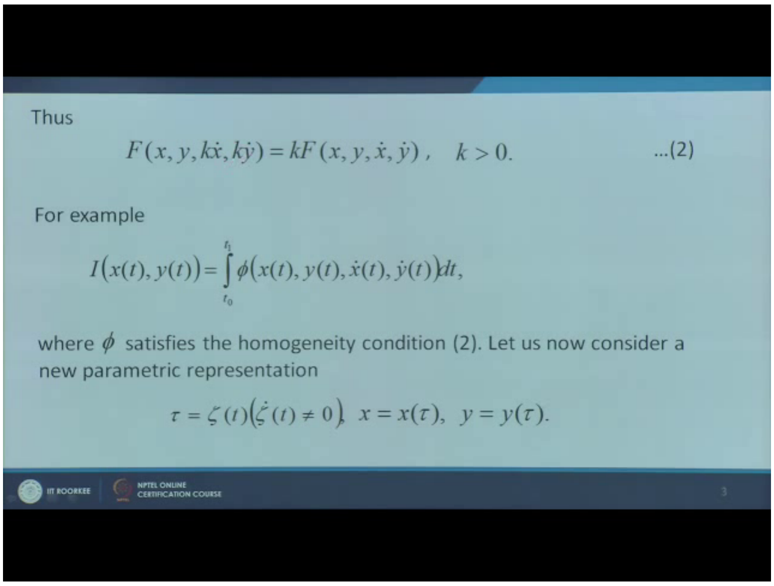

So the function f will be called homogenous of first degree in the variables x dot y dot if f (x) y k (x dot y dot), k (x dot) k (y dot) gives you k times f(x) by x dot y dot. So here you can see in the functional not occur explicitly and then when we replace x dot y k(x dot) y dot by k (y dot) then what we get is k times f(xy) x dot y dot.

So we will call f to be homogenous function of the first degree in x dot and y dot. For example if you consider the functional the variational problem ix t y(t) equal to t 0 to t 1 phi(x t) y(t) x dot t y dot t dt where phi satisfies the homogeneity condition this one

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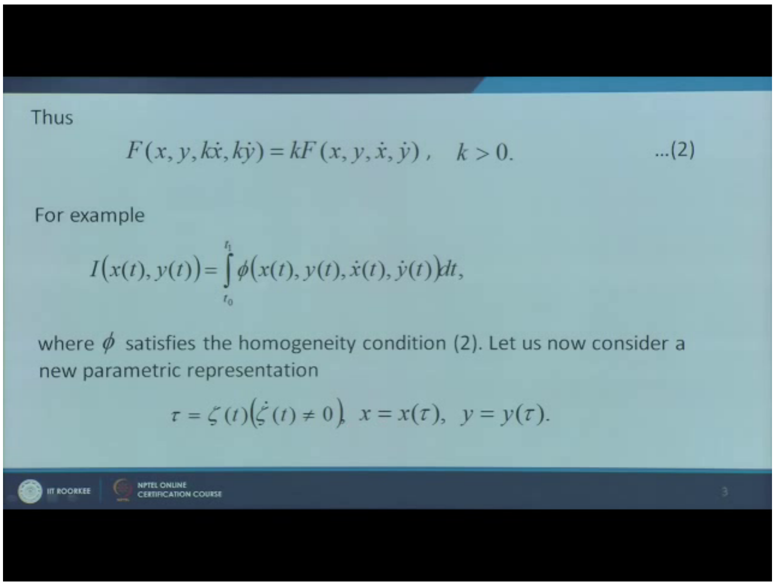

Then the we let us show that it does not depend on the parameterization that means if we do another parameterization of this problem then the it become remains in variants. So if let us consider indo parametric representation say (zeta), Tau is equal to zeta (t) where zeta dot t is not equal to 0 and x is a function of Tau y is a function of Tau, so Tau is another

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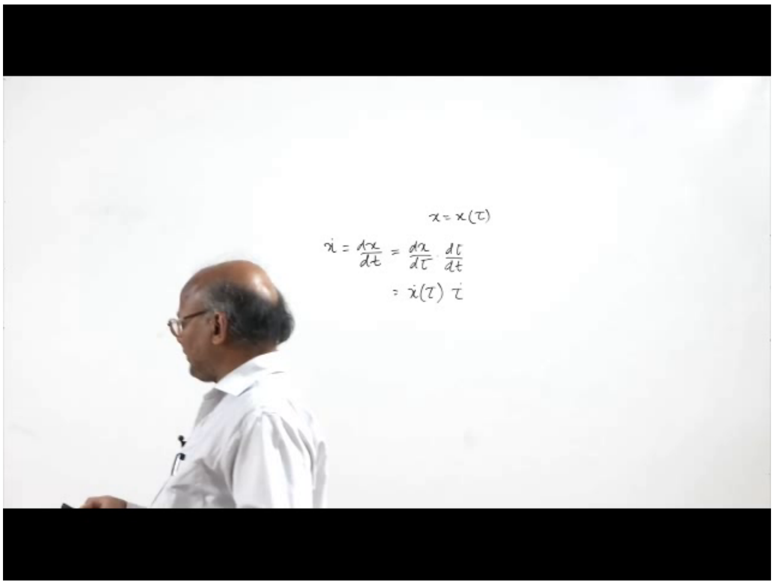

And the what we will see is that the integral t 0 to t 1 (phi x(t), y(t) x dot t y dot t dt) can be written as tau 0 to tau 1 phi x (tau) y (tau) and x dot t will be equal to x dot t is dx by dt, x dot is equal to dx by dt since x is a function of tau x is a function of tau what we have written further x is equal to x (tau). So this is equal to dx over d tau into d tau y dt. And so we can write it as x dot tau into tau dot.

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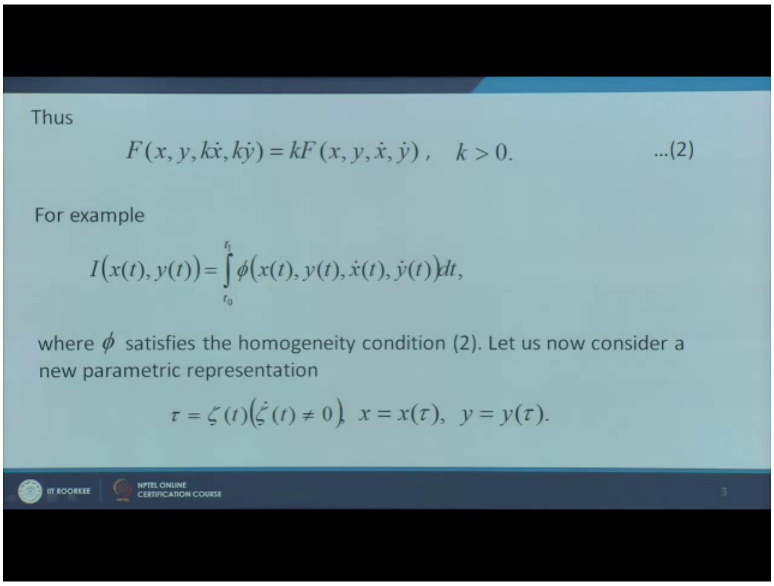

So what we will have here is that x (tau) dot tau x (tau) dot tau x (tau) dot tau into zeta dot t because tau is equal to we have taken tau equal to zeta t tau equal to zeta t.

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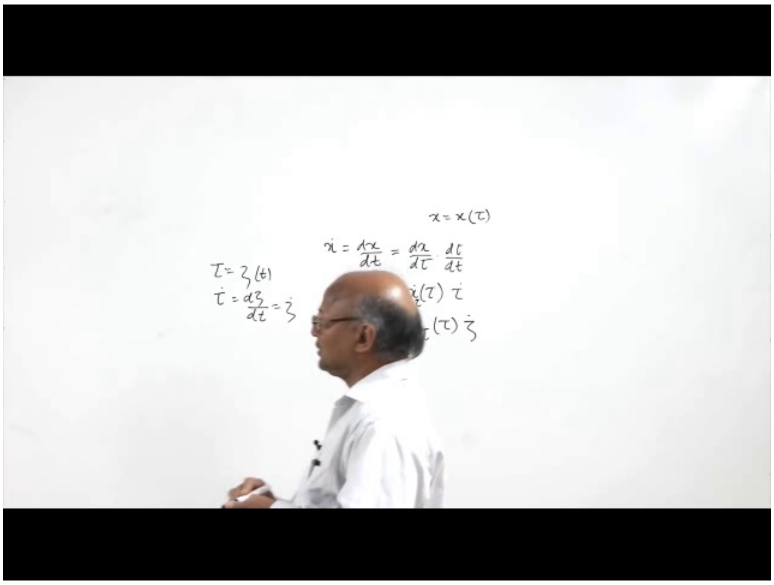

So tau dot will be zeta dot zeta tau over d zeta over dt equal to zeta dot. So this is nothing but tau dot is equal to x tau dot tau into zeta dot.

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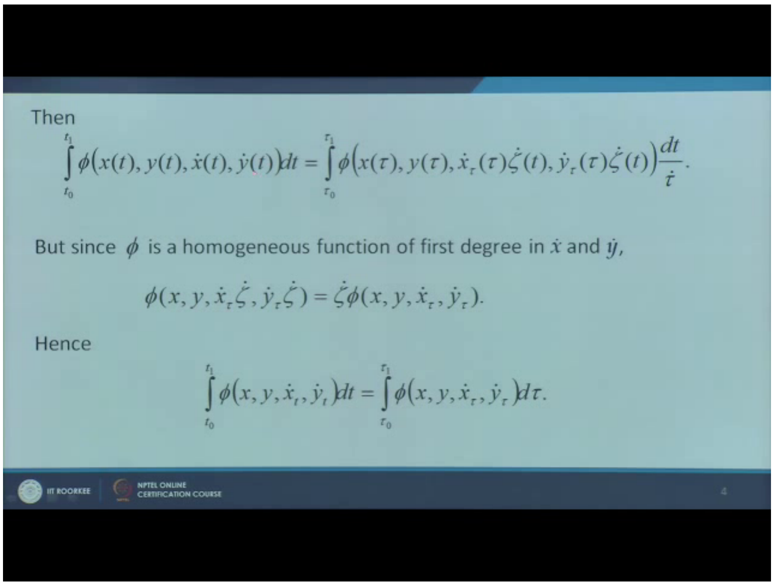

So we will have here x (tau) dot tau into zeta dot t into (())(04:21) y dot t is dy over dt which will be dy over d tau into d tau y dt so will write it as y tau dot tau into zeta dot t and again what we have here you can see tau is equal to zeta t tau equal to zeta t, so d tau equal to d tau dot t into dt d tau equal to d.

So dt will be replaced by d tau over d tau over tau dot. So zeta dot is nothing but tau dot. So we will have here tau dot so now then we will have so this, this is same as this and since phi is a homogeneous function of the first degree in x dot y dot. So phi x by x tau dot tau into zeta dot comma y dot y tau dot tau into zeta dot is equal to this zeta dot times phi x by x tau dot y tau dot because this tau dot this zeta dot x acts as a scalar k.

So this is of the form phi x by k x tau dot and then k y tau dot. So this will be k times phi x by x tau dot y tau dot. So you can replace this value here when we replace this value here then what will happen this zeta dot will cancel with tau dot because tau dot and zeta dot are same. So we shall have integral x 0 to t 0 to t 1 phi x y x tau dot x t dot y(t) dot your this x dot t is same as x t dot and y dot t is we have written as y t dot.

So this is equal to integral tau 0 to tau 1 phi x y x tau dot y tau dot d tau. This tau as a suffix and t as a suffix we have added to show that this is the derivative with respect to t and this is the derivative with respect to tau. Now you can see here that when phi is a homogeneous function of the first degree in in x dot and y dot then the this is of the functionality independent of the parameterization. So the integrant remains unchanged with the change in the parametric representation.

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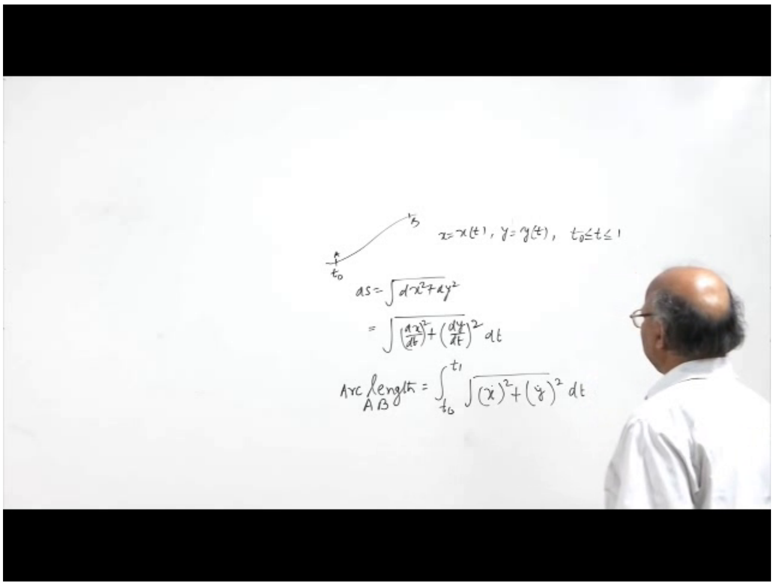

Now for example you can consider let us say we know that the arc line from along the line from t 0 to t 1 if you take the curve any curve say this point corresponding to t 0 the point corresponding to t 1 and the then the arc lines we know is under root dx square plus dy square within the parametric form I can write it as if the curve is given as x equal to x(t) y equal to y(t) where t varies from t 0 to t 1.

Then dx is under root dx square plus dy square which we can write as dx by dt whole square dy by dt whole square into dt. So the length of the curve from t 0 to t 1 ok the arc length equal to arc length ab is equal to t 0 to t 1 ds that is under root x dot square plus y dot square into dt where x dot y dot are derivatives with respect to derivatives of x and y with respect to t.

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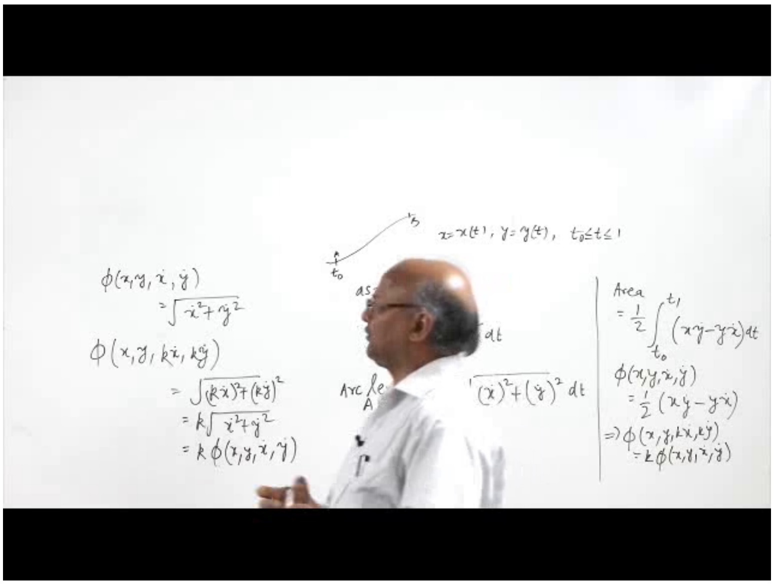

Now here you can see phi (x, y, x dot, y dot) is equal to square root x dot square plus y dot square. So this is homogeneous degree function in x dot y dot because when we replace x by x dot y dot by k x dot k y dot this is k this is also k so then this is equal to under root k x dot square plus k y dot square is equal to k times under root x dot square plus y dot square.

So this is equal to k times phi (x, y, x dot y dot) So phi is a homogeneous function of degree 1in x dot and y dot similarly the area bounded by certain curve if you find the area bounded by certain curve then area is given by 1 by 2 integral from t 0 to t 1 x y dot minus y x dot dt. So this is also here what we have phi x y x dot y dot is equal to 1 by 2 x y dot minus y x dot.

So this implies phi(x, y k x dot k y dot) equal to you can replace x dot by k x dot y dot by k y dot and you see that it is k times phi x by so these are functional where the integrant phi (x y x dot y dot) is a homogeneous function of degree 1 in x dot and y dot. So the examples of the functional we are actually interested in.

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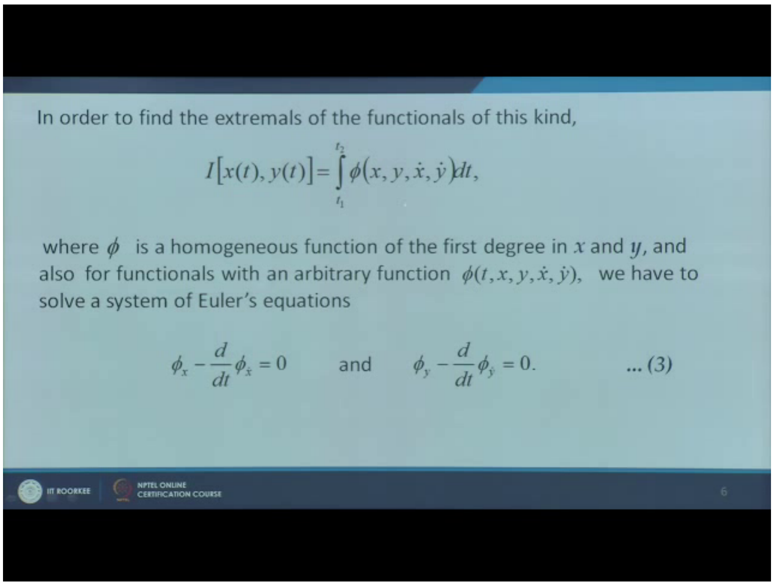

Now say in order to find the extemals of functional of the form where phi does not contain t explicitly and it is a homogenous function of a degree 1 in x dot and y dot. We what we do is we get the system of Euler's equation to find the extemum of this functional or extemals of the functional of this form we have the system of Euler's equation because x and y are the dependant variable t is the independent variable.

So phi (x) minus d over d t phi (x dot) equal to 0 phi (y) minus d over dt phi (y dot) equal to 0. But here what happens is that because t does not occur explicitly and x dot and y dot phi(x, y) x dot y dot is a homogeneous function of degree 1 in x dot and y dot. The two Euler's equations are not independent of each other.

While if while in the case where we have phi t x y x dot y dot and t is independent variable x and y are dependent variables then we have these two equations as independent of each other and we solve them according to the example which we have done earlier where we had discussed the variational problem in case where we have more than one dependent variable.

So (ther) we can solve the problem there, here these two equations are not independent of each other. So to find the extemal we must consider one of the Euler's equation and integrated together with the equation defining with the choice of the parameter.

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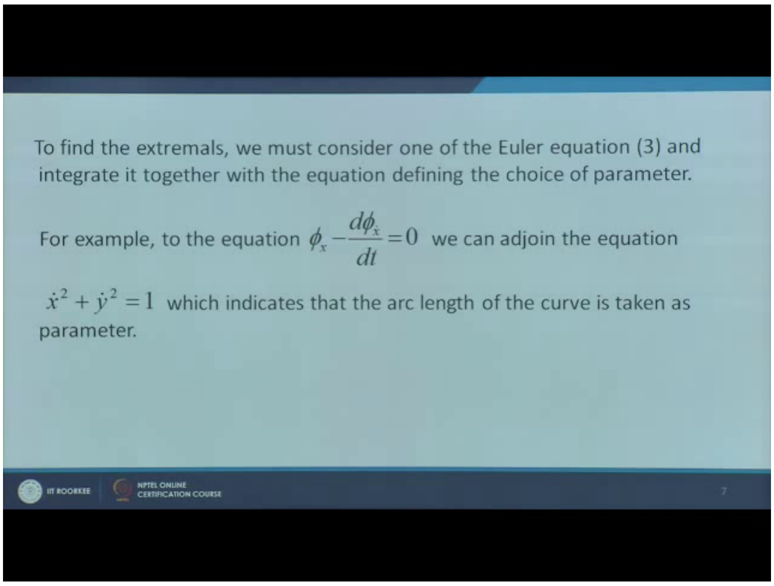

Now for example we have the equation phi(x) minus d over dt phi(x dot) equal to 0 ok. So for example we will solve one of the two Euler's equation say for example phi(x) minus df phi(x dot) over dt equal to 0. And together with the equation x dot square plus y dot square equal to 1 which indicates that the arc lines of the curve is taken as a parameter.

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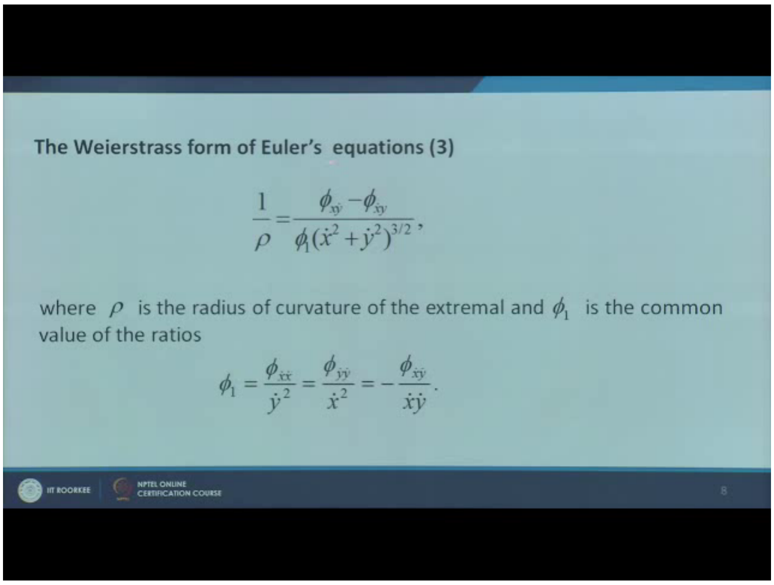

Now to solve this equation we will need the Weierstrass form of Euler's equation which is 1 over Rho equal to phi(xy dot) minus phi x dot y over phi 1 x dot square plus y dot square to the power 3 by 2 where Rho is the radius curvature of the extemal and phi 1 is the common value of the ratios.

So Phi 1 is given by phi x dot x dot over y dot square phi y dot y dot over x dot square minus phi x dot y dot over x dot y dot. This is the Weierstrass form of the Euler's Equation we can solve a Euler's equation by using this Weierstrass form also.

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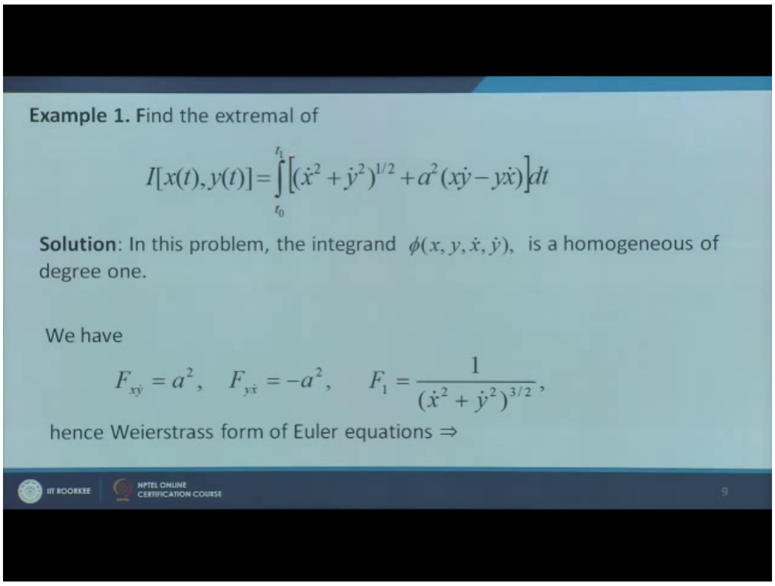

Say for example let us consider this equation; find the extemal of I[x (t) y (t)] t 0 to t 1(x dot square plus y dot square) to the power half plus a square (x y dot minus y x dot)] dt where a is (real) some constant. Now in this problem the integrant phi(x, y, x dot, y dot) is a homogeneous function of degree 1 in x dot y dot.

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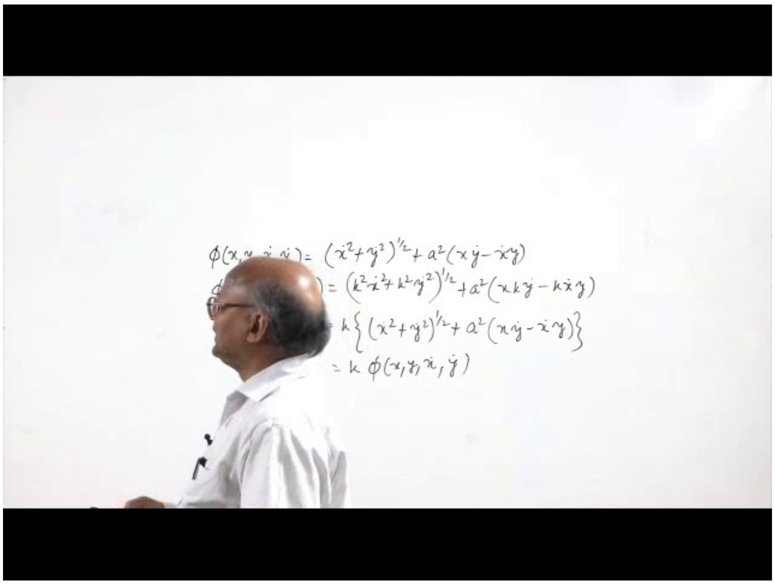

So let us see you can replace phi(x, y, x dot, y dot) is equal to (x dot square plus y dot square) to the power half plus a square times (x y dot minus y x dot). Now you can see when we replace x dot by k x dot and y dot by k y dot then what we get is k square (x dot square k square y dot square) raise to the power half plus a square (x k y dot minus k x dot y) which is equal to k times phi k times (x dot square plus y dot square) raise to the power half plus a square times(x y dot minus x dot y) which is equal to k times (phi x y x dot y dot).

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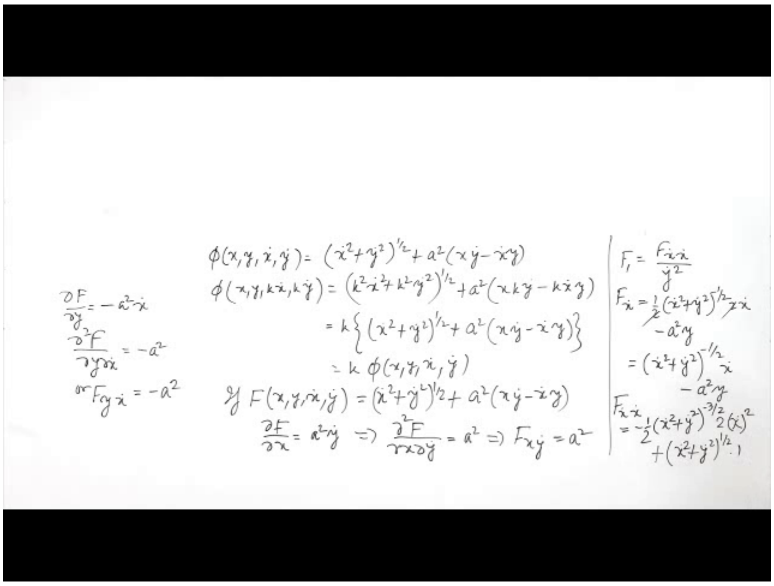

So it is a homogeneous function of degree 1 in x dot y dot. Now you can see here that if you instead of phi you write it denote it by f, ok.

So in this solution we have actually denoted phi y f. So if you write f x y (x dot y dot s) instead of phi you write it as f x do square plus y dot square to the power half plus a square x y dot minus x dot y then you can see that if you replace if you write it as f(x) y x dot y dot then you differentiate with respect to x partially with respect to x what you get is a square (y dot) and when you differentiate with respect to y dot again you get delta square f by delta x delta y dot which is equal to a square.

So this is this implies f (x y dot) equal to a square. Similarly if you differentiate f with respect to y what we will get is with respect to y minus a square x dot and when we differentiate it with respect to x dot further we will get minus a square so we can say that this is f by x dot. Now in order to find f 1 f 1 means phi 1, so phi 1 is phi (x dot) x dot over y dot square.

So f 1 is equal to f x dot x dot divided by y dot square. So let us find f x dot x dot, so first we find f x dot f x dot is equal to 1 by 2 x dot square plus y dot square to the power minus half into 2 x dot into 2 x dot and then we have minus a square y when we differentiate with respect to x dot. So this 2 will cancel and we get x dot square plus y dot square raise to the power minus half into x dot minus a square y.

Now let us differentiate this further with respect to x dot again. So f x dot x dot here we have a product of two functions each containing x dot so we have to use that formula for product of derivative of the product so minus 1 by 2 x dot square plus y dot square raise to the power minus 3 by 2 into 2 x dot into x dot. So 2 times x dot square x dot square and then this is minus a square sorry this is further we have x dot square plus y dot square raise to the power half into derivative of x dot is 1.And derivative of minus a square y with respect to x dot is 0. So this f(x dot x dot).

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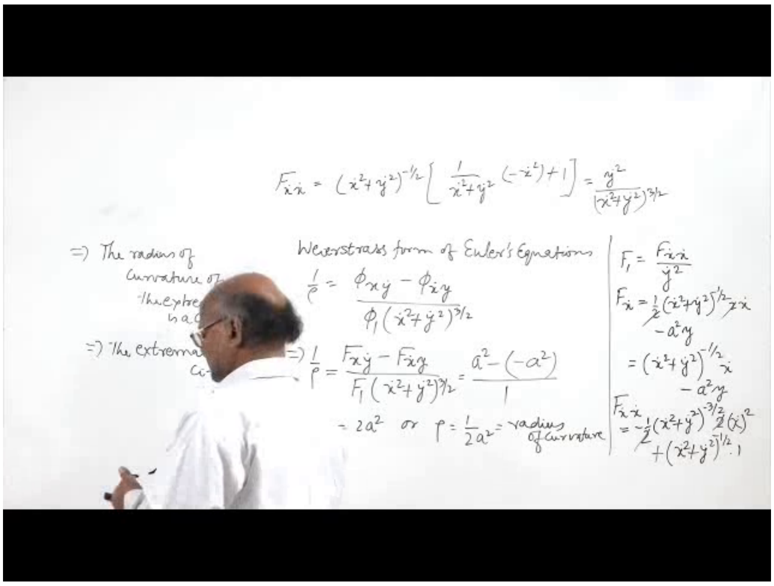

We can simplify it we can write it as here it is minus half. So we can write it as x dot square plus y dot square raise to the power minus half and here we will have minus x square plus y dash ok 1 upon x dot square plus y dot square minus x dot square plus 1.

So this is x dot square plus y dot square minus x dot square so we get y dot square divided by x dot square plus y dot square divided by to the power 3 by 2. So f (x dot is this and for phi 1 ok therefore (phi 1) therefore hence f 1, f 1 is equal to y dot x square upon (x dot) square plus (y dot) square to the power 3 by 2 f (x dot x dot) divided by (y dot) square. So 1 upon so this will cancel.

And we will get f 1 ok f 1 is 1 upon (x dot) square plus (y dot) square to the power 3 by 2. Now let us use Weierstrass form of Euler's equation. So we go to the Weierstrass form of Euler's equation so 1 by Rho, 1 by Rho equal to phi(x) y dot minus phi (x dot y) divided by phi dot 1 (x dot) square plus (y dot) square to the power 3 by 2. Now in place of phi we are writing f here.

So we have 1 by Rho f(x y dot) minus f(x dot y) divided by f 1 x dot square plus y dot square to the power 3 by 2. And this is equal to, this is equal to f(x y dot) f x ok f(x y dot) we have find f x y dot came out to be we had found f x y dot, f x y dot is equal to a square. So we have a square minus and this is (minus a square).

And then we have f 1 into x dot square plus y dot square to the power 3 by 2 equal to 1. So we get 1 by Rho is equal to 2a square or Rho equal to 1 by 2 a square. Now here Rho is the radius of curvature, so radius of curvature is a constant quantity and therefore the extremal are circles. So this Rho this Rho is the radius of curvature,.

So we have the radius of curvature of the extremal, the radius of curvature of the extremal is a constant. And therefore the extremal is a (circ) so extremal because the radius of curvature of circle is a constant quantity. Now it is the radius now what we have is . Let us now so the extremals are circles now find the extremal of this problem.

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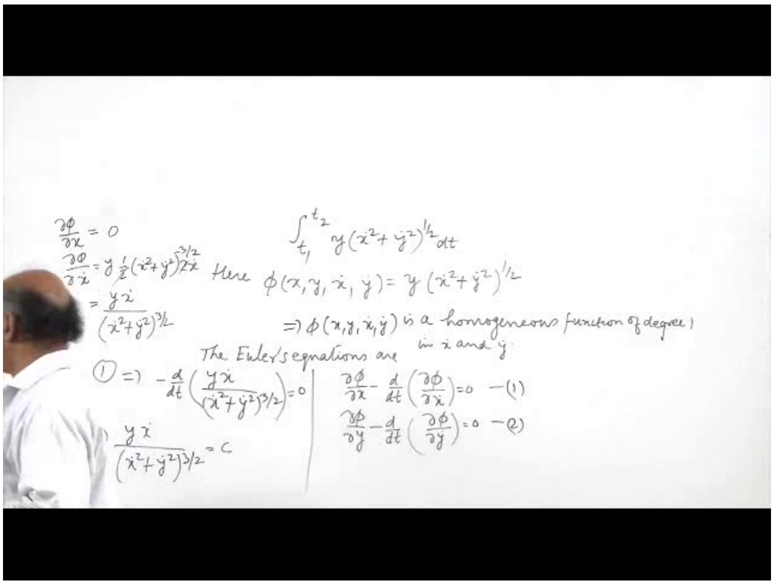

Let us see this problem. So we have integral over t 0 to t 1, t 1 to t 2 y times (x dot square plus y dot square) to the power half dt. So here phi(x, y, x dot, y dot) is equal to y times (x dot square plus y dot square) raise to the power half.

And therefore we can (ans) moreover we notice that when we replace x dot y k x dot y dot y k y dot then what we get is y times (k x dot) whole square to the power half. So this will give you k y into x dot square plus y dot square to the power half. So phi is a homogeneous function of degree 1 in x dot and y dot ok.

Now so the euler's equations are delta phi by delta x minus delta phi by delta we have here x and y are dependant variables (t is) delta phi by delta x minus d over dt of delta phi over delta x dot equal to 0, and delta phi over delta y minus d over dt of delta phi over delta y dot equal to 0. Now we will pick up one equation from the here and find the extremals of the given problem.

And then we show that the second equation is not independant on the first equation we solve second equation there also we will get the same extremals. So let us see how we will pick the first equation. So let us take the first equation here delta phi over delta x if we find delta phi over delta x what we will get 0 here. And if we find delta phi over delta x dot then what we will get y times 1 by 2 x dot square plus y dot square raise to the power minus 3 by 2 into 2x dot.

So this 2 will cancel and will get y x dot over x dot square plus y dot square to the power 3 by 2. So this will give you 0 ok the first Euler's equation. The first Euler's equation gives us minus d over dt delta phi by delta x dot, so y x dot divided by x dot square plus y dot square raise to the power 3 by 2 equal to 0, ok because this is 0.

So now integrating with respect to t what we get y(x dot) divided by x dot square plus y dot square to the power 3 by 2 equal to some constant. Let us say c ok. Now there arise two possibilities. The possibilities are the constant c is 0 and the constant c is not 0. So let us discuss the two cases separately.

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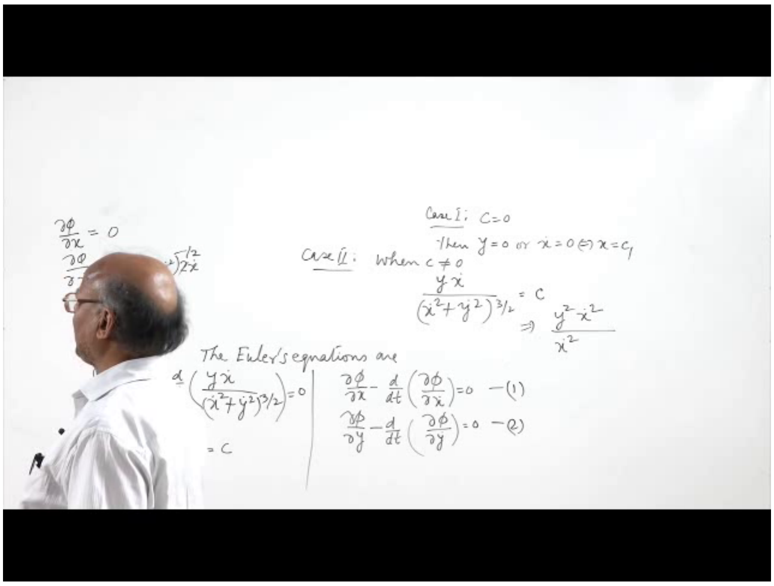

So we will let us consider the case 1 c is equal to 0. The constant c is equal to 0 will give you y equal to 0 or x dot equal to 0. So when c is equal to 0 either y is 0 or x dot is 0. So x dot equal to 0 gives you x equal to some constant. You can say c 1, ok so when c is equal to 0 y equal to 0 could be extremal or x equal to a constant could be an extremal.

Now let us see the case 2 when c is not 0, when c is not 0 then we will get y x dot divided by x dot square plus y dot square raise to the power 3 by 2 equal to c. So we will get here y x dot ok y x dot upon x dot square plus y dot square to the power 3 by 2 we will get we had to solve it so we will get y square, ok it is to square it y square x dot square upon x dot square we have x dot square to the power del phi over del x dot it should be minus half not 3 by 2.It should be minus half here because?

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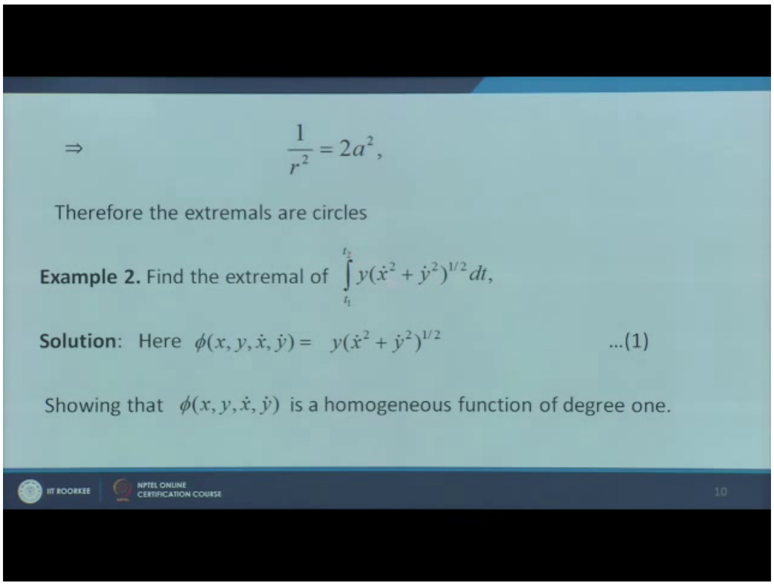

We are differentiating phi with respect to x dot we are differentiating this is phi, phi y phi equal to y times x dot square plus y dash square to the power half when we differentiate with respect to x dot we get y times 1 by 2 x dot square plus y dot square to the power minus half into 2x dot. So x dot y upon x dot square plus y do square to the power half. So this is half here.

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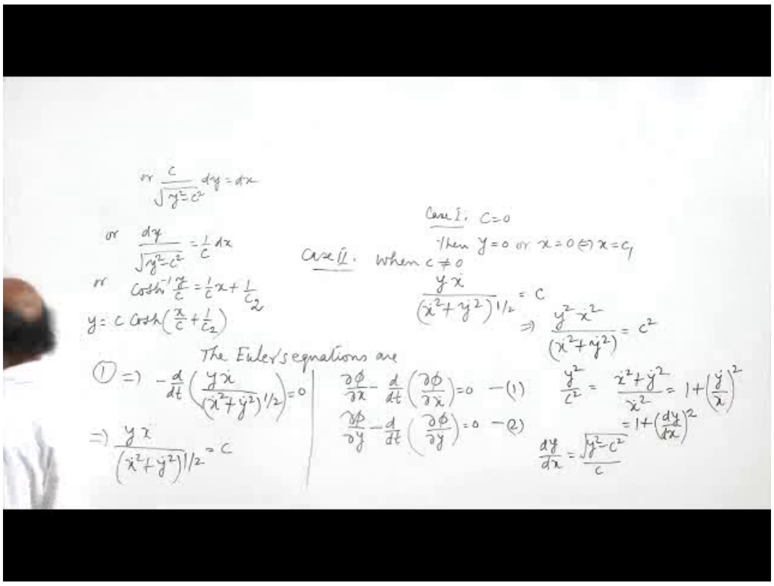

So we get y x dot upon x dot square plus y dot square to the power half and this will give you when we square this y square x dot square upon x dot square plus y dot square is equal to c square. So r we can say y square by c square equal to x dot square plus y dot square divided by x dot square which is 1 plus y dot upon x dot whole square y dot is dy by dt x dot is dx by dt. So y dot over x dot is 1 plus dy by dx whole square.

So 1 plus dy by dx whole square. So what we will get dy by dx equal to y square minus c square under root divided by c or we can say c upon under root y square minus c square dy is equal to dx. So this will give you when we integrate this so we can say dy upon under root y square minus c square is equal to 1 by c t dx ok.

So this gives you when we integrate we get cos hyperbolic inverse y over c equal to 1 over c into x plus 1 over d some constant c 3 we can write or we have already taken c 1 we have taken so let us write c 2 here. So then what we will get y equal to c times cos hyperbolic y (sorry) cos hyperbolic x y c plus 1 by c 2. So one solution we had as y equal to 0 and over solution was x equal to c 1 and this is the third solution y equal to c times cos hyperbolic x by c plus 1 by c 2.

So there are 3 extremals of the given problem which we have found by solving the first Euler's equation. Let us show that the second Euler's equation when we solve it also gives us the same same extremals. So let us what we will do is when we will simplify this second Euler's equation we shall see that we get this equation y x dot upon x dot square plus y dot square to the power half is equal to c which when we solve by taking the k c equal to 0 and c 0 equal to 0, we are arrive at the 3 extremals y equal to 0 x equal to c 1 and y equal to c cos hyperbolic x by c plus 1 by c 2.

So we shall simply prove that the second Euler's equation when simplified gives rise to the same equation this one and therefore when solved this equation gives us the same extremals. So let us solve the second Euler's equation.

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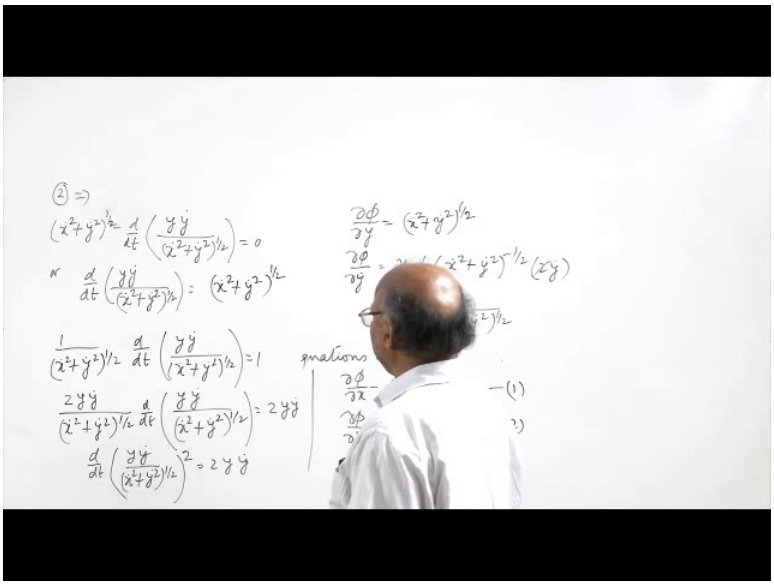

So second Euler's equation we need to find phi delta phi over delta y delta phi over delta y if we find it is x dot square plus y dot square raise to the power half. And when we find partial derivative of phi with respect to y dot what we will get y times 1 by 2 x dot square plus y dot square raise to the power minus half into 2 y dot. So this will cancel and we will get y y dot divided by x dot square plus y dot square to the power half.

And therefore second Euler's equation gives us partial derivative with respect to y so 1 upon x dot square plus y dot square minus d over dt (y y dot divided by x dot square plus y dot square to the power half equal to 0. We can write it in this form or d over dt y y dot upon x dot square plus y dot square to the power half is equal to 1 by x dot square plus y dot square, no wait wait.

We have this (x dot square plus y dot square to the power half), (x dot square plus y dot square) to the power half so we get here (x dot square plus y dot square) to the power half . Now what we will do is this gives you 1 upon x dot square plus y dot square to the power half into d over dt y y dot upon x dot square plus y dot square raise to the power half equal to 1, ok.

Let us divide by x dot square plus y dot square to the power half then what we do is we multiply both sides by 2 by y dot. So 2 by y dot upon x dot square plus y dot square raise to the power half d over dt of(y y dot over x dot square plus y dot square) raise to the power half equal to 2 y y dot.

Now we can see that the left hand side is the differential of , when we differentiate y y dot upon x dot square plus y dot square whole to the power half raise to the power half, what we get? 2 times y y dot upon x dot square plus y dot square to the power half into the derivative of y y dot upon x dot square plus y dot square to the power half.

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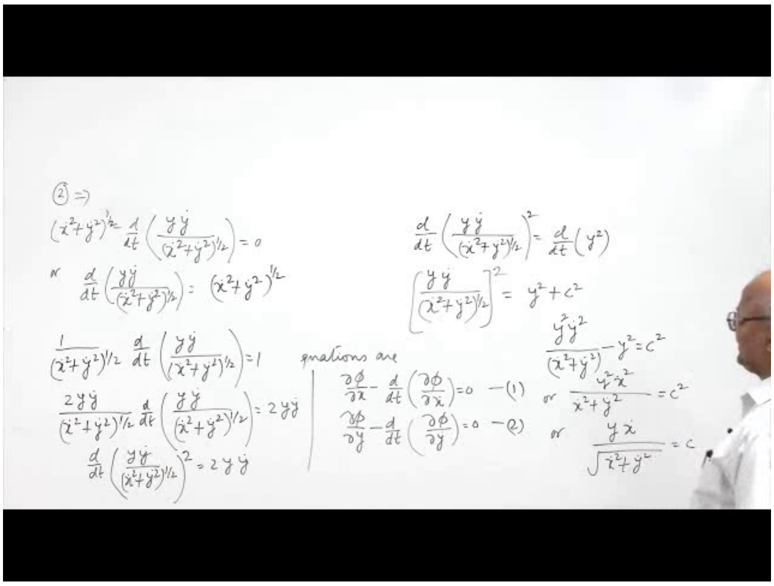

So what we get is this also we can write in the following form, y y dot d over dt (y y dot over x dot square plus y dot square) to the power half equal to d over dt (y square), ok. I will integrate both sides so y y dot upon x dot square plus y dot square this to the power 2. So this square is equal to y square that is right plus c square constant we can take as c square.

So what we will get y y dot upon x dot square plus y dot square ok this is to the power half here. Let me write it again x dot square plus y dot square raise to the power half whole to the power 2. So what we will get y square y dot square upon x dot square plus y dot square minus y square equal to c square.

So when we take LCM what we get y square y dot square will cancel and we will get x dot square y square x dot square equal to c square or we will take the square root and we will have y x dot upon square root x dot square plus y dot square equal to c. And this is what we wanted y x dot upon x dot square plus y dot square to the power half is equal to c.

So second equation also leads us to the same equation which when we solve we get the extremals as we have got the extremals from the by solving the first equation. So the two equations are not independent we can solve any one equation and use the parametric form and the parametric relation and get the extremals.

So this is the second example on this variational form in our next lecture we shall consider the variational problems with moving boundaries. So far we have considered the variational problems where both the end points x 1 y 1 x 2 y 2 were fixed now we shall take the variational problems where either one or both the end points are not fixed they are moving.

So that is, that will be discussed in the next lectures ok, thank you very much for your attention.