Integral Equations, Calculus of Variations and their Applications
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Lecture 05
Integro-differential equations

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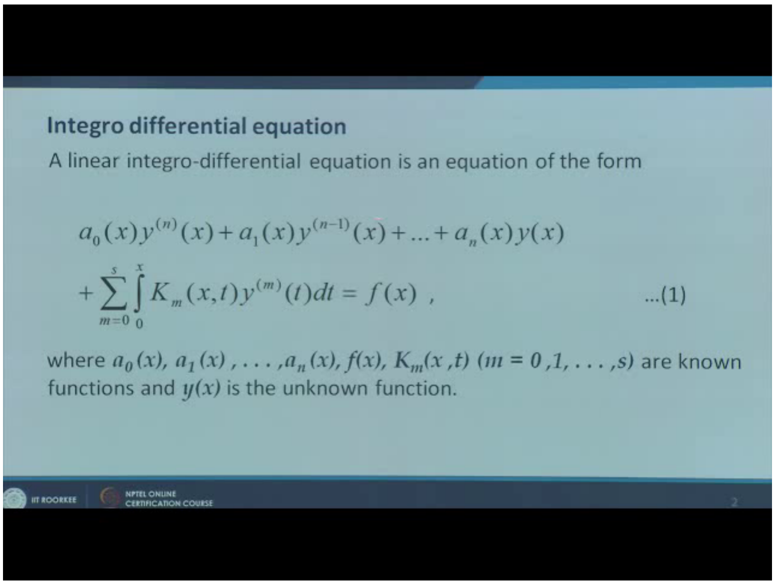

Hello friends! I welcome you to be lecture on Integro differential equations A linear integral differential equation is an equation of the form a 0(x) into y (n) (x), y n (x) is the n th derivative of y with respect to x plus a 1(x) y (n minus1) (x) and so on a n (x) into y (x) plus sigma s is equal to n equal to 0 to s 0 to x k m (x, t) y (m) (t) dt is equal to f (x) where a 0, a m, a n (x), and f x k m(x, t) a m varying from 0 to x are known functions and y(x) is the unknown function.

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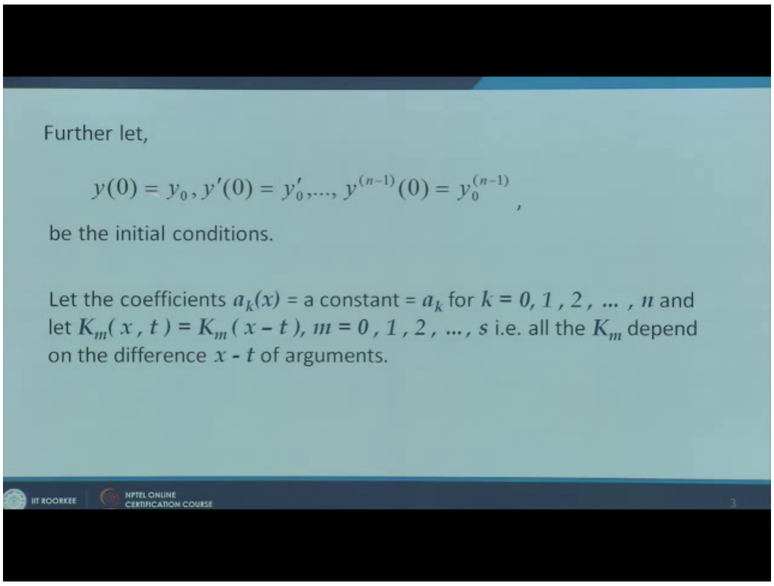

Further we are given the conditions on y, y(0) equal to y 0 y dash (0) equal to y 0 dash and so on y (n minus 1) (0) equal to y 0 (n minus 1) these are the initial conditions.

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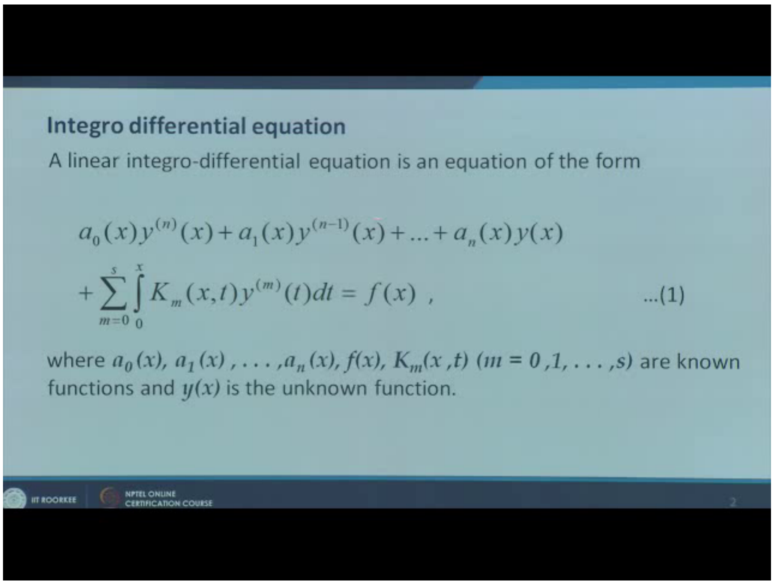

Let us assume that the coefficients a k x of the integro differential equation these coefficients a k x are constants let us denote the coefficients a k x y the constant a k for k equal to 1, 2 and so on upto n. And let us assume that the kernel k m(x,t) kernel k m(x,t) is equal to k m (x minus t) for value all values of m that is 0,1,2 and so on upto s.

That is all the k ms depends on the difference (x minus t) of arguments, without any loss of generality we can assume a 0 equal to 1 because if it is not 1 we can divide the equation by a 0.

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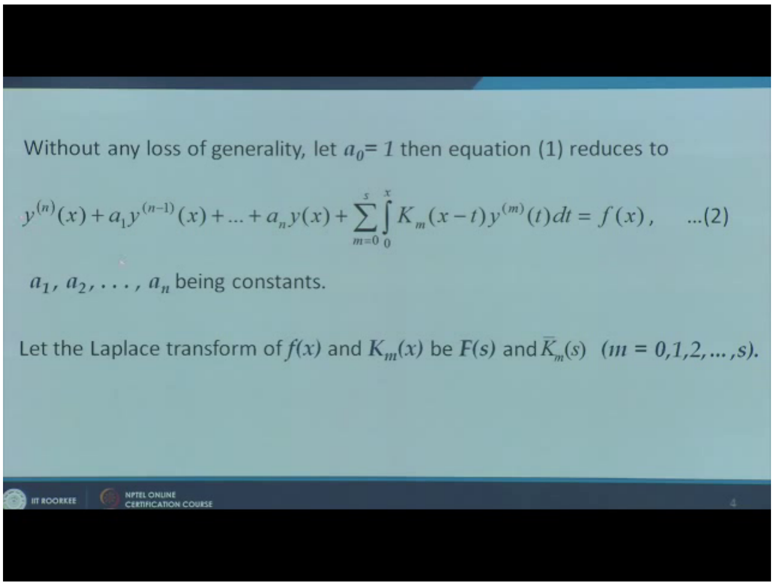

So when we assume a 0 equal to 1 then the equation 1 reduces to this form. So we have y n(x) n th derivative of y with respect to x plus a 1 (n minus 1) th derivative of y with respect to x and so on a n y(x) and then sigma a m equal to 0 to s this integral equal to f (x) a 1, a 2, a n are constants.

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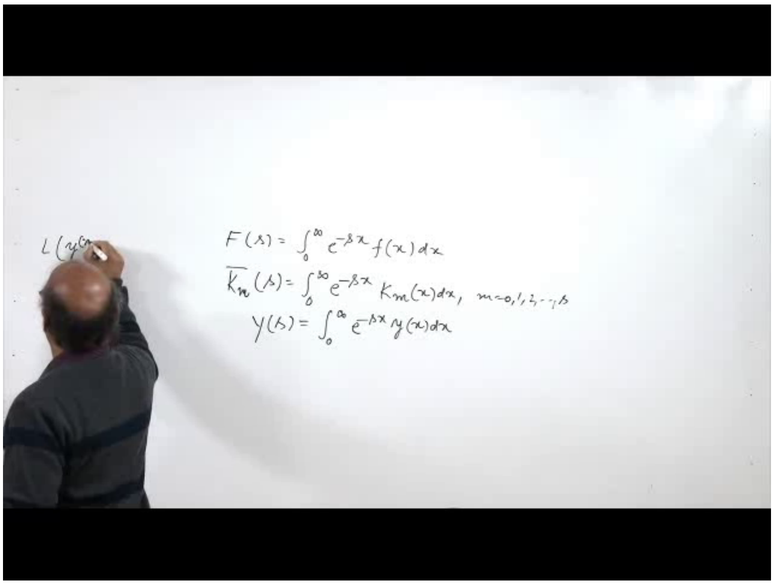

Now let us assume that the Laplace transform of f(x) is f s. So by definition of Laplace transform f s is equal to integral 0 to infinity e to the power minus f(x) into f (x) dx and Laplace transform a m x we assume as k m bar (s).

So k m bar (s) is equal to integral 0 to infinity e to the power minus f x into k m (x) where m is equal to 0 , 1, 2 and so on upto x. Further let us assume that the Laplace transform of the unknown function y(x) be y (s).So we assume y(s) equal to integral 0 to infinity e to the power minus s x into y(x) dx.

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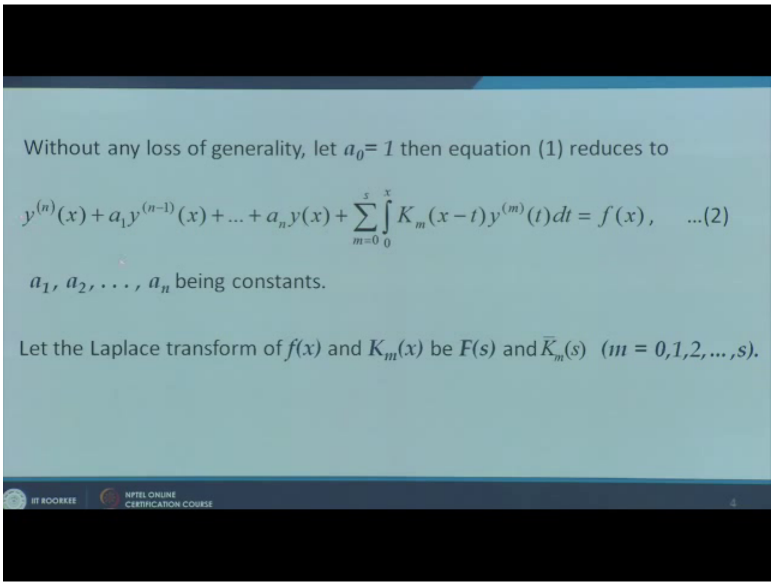

Now what we do is , we take the Laplace transform of both sides of the equation 2 and they recall the formula for the Laplace transform of the n th derivative of y(x).

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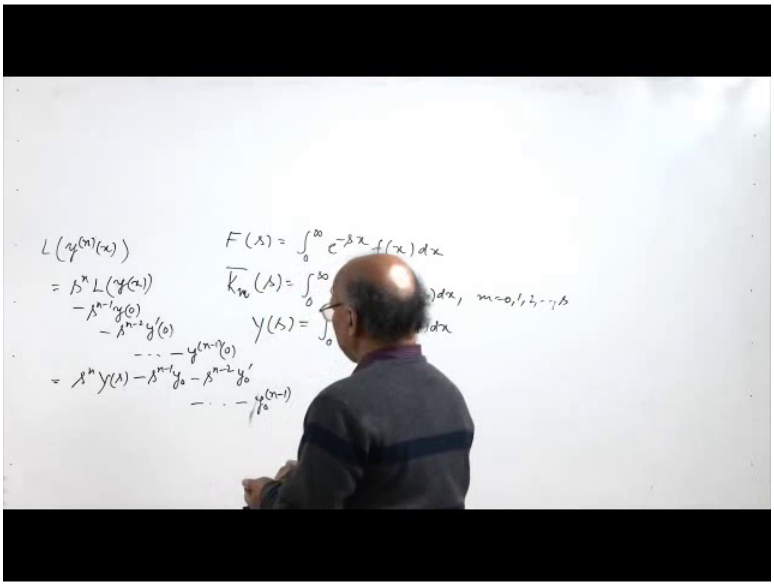

So Laplace transform of n th derivative of y(x) is given by s to the power n Laplace transform of y(x) minus s n minus 1y at (0) minus s n minus 2 y dash at (0) and so on minus y (n minus 1) at (0).

Now according to our hypothesis assumption this is s to the power n Laplace transform of y(x) is y(s) so we have y(s) minus s n minus 1 y(0) is y 0 minus s n minus 2 y 0 dash and so on minus y 0 (n minus 1).

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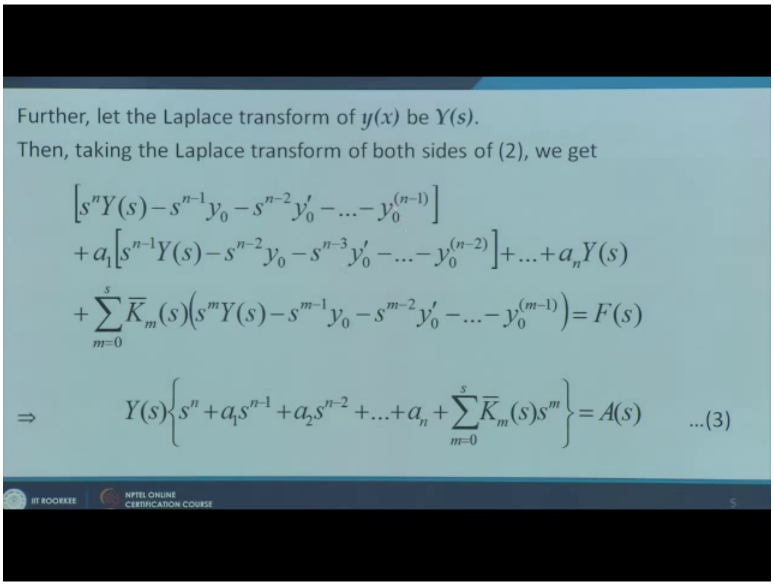

So let us substitute this value when we take the Laplace transform of y n s we get s to the power n y (s) minus s n minus 1 y 0 minus s n minus 2 y 0 dash and so on minus y 0 n minus 1 and then a 1 times Laplace transform of (n minus 1) th derivative of y with respect to x.

So we have s to the power n minus 1 y (s), minus s to the power (n minus 2) y (0) and so on y 0 (n minus 2) and so on we have a m Laplace transform of y (x) which is a n which is y (s) so a n into y (s).And then we have sigma n equal to 0 to s Laplace transform of this integral.

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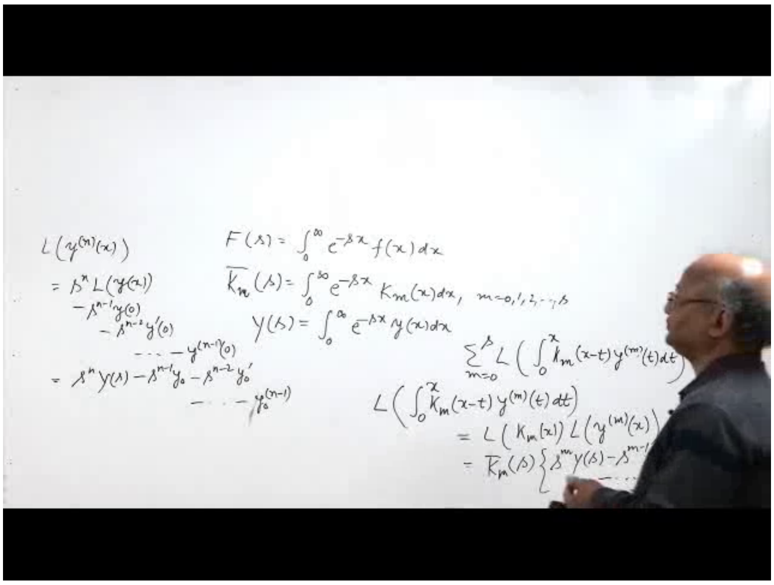

So we have sigma m equal to 0 to s Laplace transform of integral 0 to x we had 0 to x k m(x minus t) into y (m) (t) dt.

Now let us write the Laplace transform of so Laplace transform of integral 0 to x k m(x,t) x minus t) into y (m) (t) dt we applied convolution theorem here Laplace transform of k m (x) into Laplace transform of y m(x). So by applying convolution theorem Laplace transform of integral 0 to x k m(x minus t) y (m) (t) dt is Laplace transform of k m (x) into Laplace transform y(x).

Now this is k m bar (s) and this is again we apply the formula so s to the power m y (s) minus s to the power s (m minus 1) and then we have y (0) which is y 0 s to the power m minus 2 y 0 dash and so on minus s to the power m minus 1.

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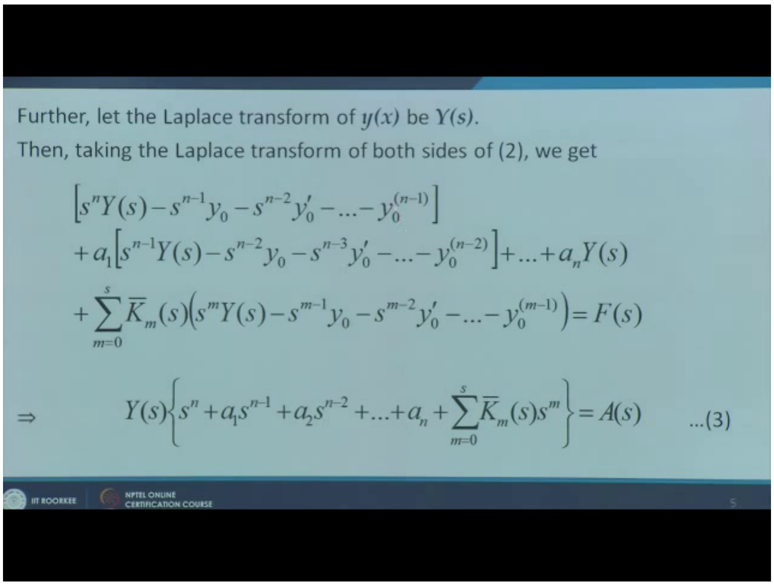

So we will get sigma m equal to 0 to s k m bar (s) into [s m (bar) y(s) minus s m y(s) y 0 and so on y 0 m minus 1] and Laplace transform of the right side is f x, Laplace transform of f (x) we have assumed as F(s). So we get F(s) here .

Now what we do is we collect the quotient of y(s) so when we collect the quotient of y(s) what we have s to the power n then a 1 times s to the power n minus 1 and then and so on a 0 is and so on a n we have and then sigma m is equal to 0 to s k m bar (s) into s to the power m. The remaining thing is a function of s we can write it as a s.

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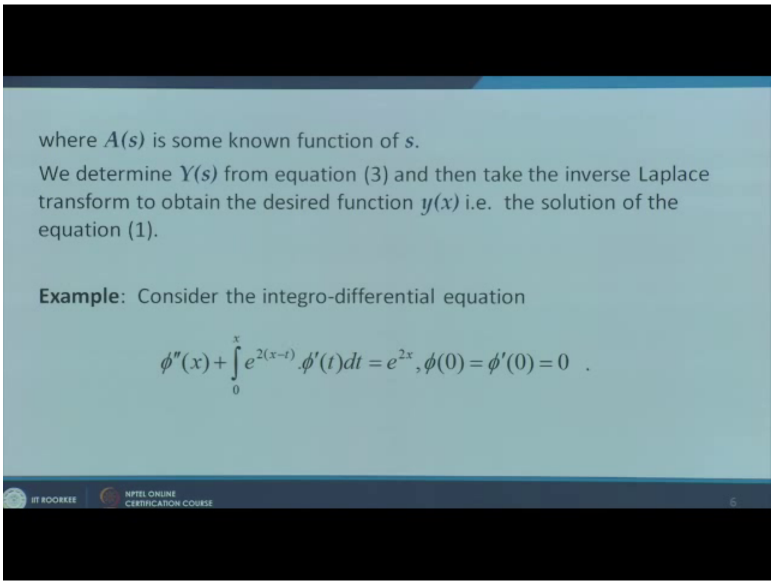

So we get the value of y(s) from here A(s) is a known function of s. So we determine y(s) from this equation 3 and then take the inverse Laplace transforms.

Inverse Laplace transform of y(s) when we do we obtain the desired function y(s) which is the solution of the equation 1 for example let us consider the integro differential equation phi double dash(x) plus integral 0 to x e to the power two times (x minus t) phi dash (t) dt equal to e to the power 2 x. Phi 0 is equal to phi dash(0) equal to 0. So it is an integro differential equation.

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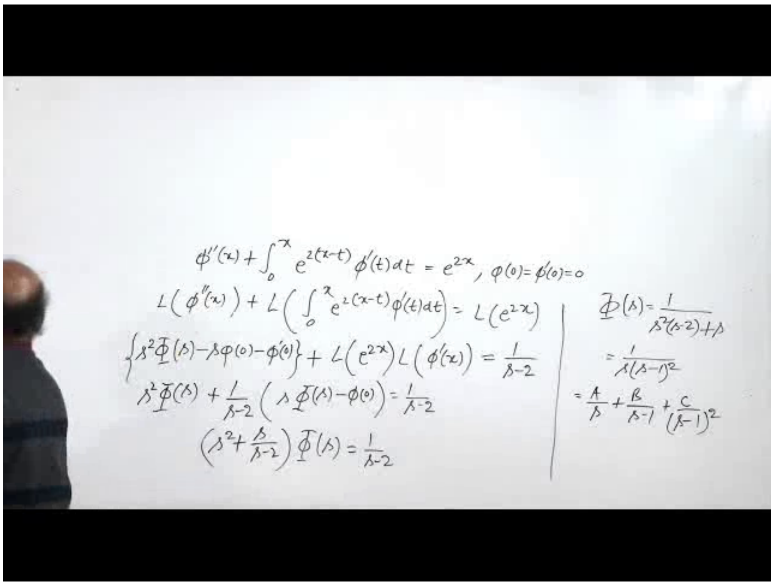

We are given the initial conditions phi(0) is equal to 0 phi dash(0) equal to 0 and phi is the unknown function. So let us take the Laplace transform of this equation. So Laplace transform of phi double dash(x) plus Laplace transform of integral 0 to x e to the power two times (x minus t) into phi dash(t) dt equal to Laplace transform of e to the power 2x.

Now Laplace transform of phi double dash (x) is s to the power 2 phi (s) minus s times phi (0) minus phi dash(0) plus by convolution theorem Laplace transform of e integral 0 to x e to the power two times (x minus t) phi dash(t) dt is Laplace transform of e to the power 2x into Laplace transform of phi dash(x) and Laplace transform of e to the power 2x is 1 over s minus 2.

So we have s square phi (s) phi(0) is 0 phi dash(0) is 0 and Laplace transform of e to the power 2x is 1 over s minus 2, Laplace transform of phi dash(x) is s phi (s) minus phi(0) equal to 1 over s minus 2, phi(0) is 0 so we get s square plus s over (s minus 2) phi(s) equal to 1 over s minus 2, and what we get is we can simplify this.

So s square into (s minus 2) so phi(s) equal to 1 over s square into s minus 2 plus s and what we get is 1 over s into s minus 1 whole square, because this is s cube minus 2 s square plus s so we can take as common and this is 1 over s into (s minus 1) whole square we can break it into partial fractions a over s , b over(s minus 1), c over (s minus 1) square. And we can determine the values of a,b,c very easily from here.

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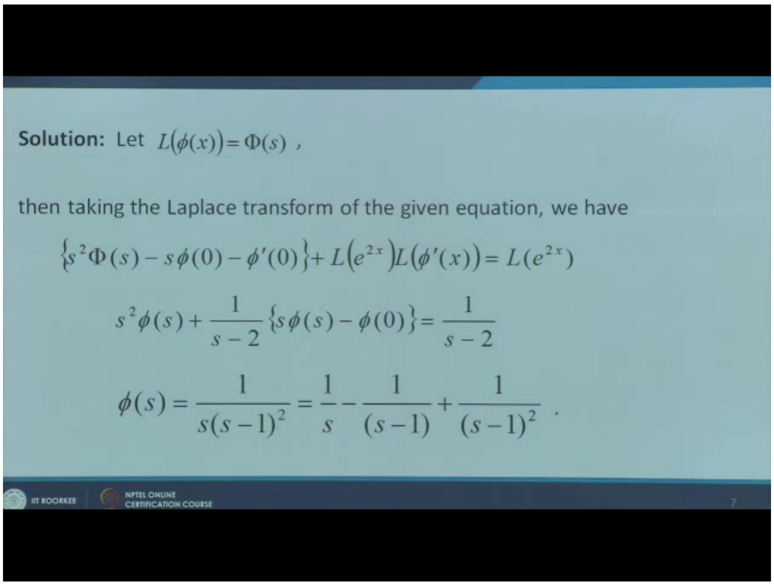

When we determine the values of a, b, c we notice that this is phi(s) phi (s) is 1 over s into (s minus 1) whole square and this is 1 over s minus 1 over (s minus 1) plus 1 over (s minus 1) whole square.

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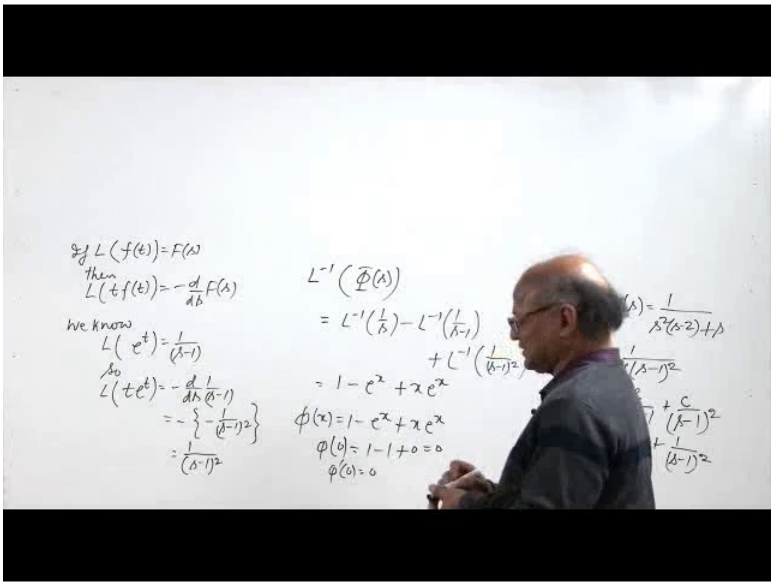

So a comes out to be 1 and b is minus 1 c is 1 we can easily find those values. So this 1 over s plus 1 over (s minus 1) and 1 over (s minus 1) whole square. Now let us take the inverse of Laplace transform.

So inverse Laplace transform of phi(s) equal to inverse Laplace transform of 1 over s minus inverse Laplace transform of 1 over (s minus 1) plus inverse Laplace transform of 1 over (s minus 1) whole square. Now inverse Laplace transform of 1 over s is 1 and inverse Laplace transform of 1 over s minus 1is e to the power x we have to find the inverse Laplace transform of 1 over (s minus 1) whole square.

So for that let us recall the formula if Laplace transform of f(t) is equal to f(s) then Laplace transform of t f(t) is minus d over ds of F(s).So we know that Laplace transform of e to the power t is 1 over (s minus 1). So Laplace transform of t times e to the power t is minus d over ds of 1 over (s minus 1). So this is minus 1 over (s minus 1) whole square so we get 1 over s minus 1) whole square.

So inverse Laplace transform of 1 over (s minus 1) whole square is e to the power t, here we will have x e to the power x. So now inverse Laplace transform of phi(s) is phi(x) so we get phi(x) as 1 minus e to the power x plus x e to the power x which is the solution of the integro differential equation. We can check that phi 0 is the 0, phi(0) is equal to 0 because it is 1 minus 1 plus 0. And we can also see that phi dash(0) is 0.

So it is the solution of the given integro differential equation 1 minus e to the power x plus x e to the power x.

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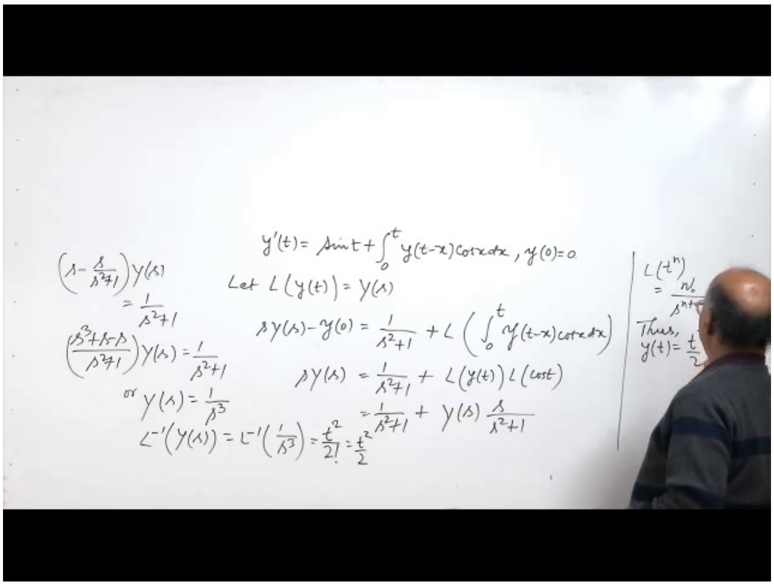

Another example we can take so y dash(t) equal to sin(x) y dash(t) equal to sin(t) plus integral 0 to t y(t) minus x cos x dx and y(0) is equal to 0. So here again let us assume that Laplace transform y(t) is y(s) and then let us take the Laplace transform of both sides of the given equation.

So Laplace transform of y dash(t) will be s y(s) minus y(0) and Laplace transform of sin(t) is 1 over s square plus 1 plus Laplace transform of 0 to t y(t minus x) into cos x dx. Now y(0) is given as 0 so s y(s) 1 over s square plus 1, here we apply the convolution theorem so Laplace transform of y(t) into Laplace transform of cos t. So this is 1 over s square plus 1 plus Laplace transform of y(t) is y(s) into s over s square plus 1.

We then simplify and get the value of y(s) from here so s minus 1 s upon square plus 1into y(s) is equal to 1 over s square plus 1. So we will get s cube plus s we will get this r y (s) equal to 1 over s cube. Let us take inverse Laplace transform, now recall the formula that Laplace transform of t to the power n, if n is a positive integer it is n factorial divided by s to the power n plus 1.

So here we have a (Laplace transform of) inverse Laplace transform of 1 over s ccube will be t square divided by 2 factorial or t square by 2. So y(t) thus y(t) is equal to t square by 2 and it is the solution of the given integro differential equation. So this is what we have here y(t) equal to t square by 2, ok.

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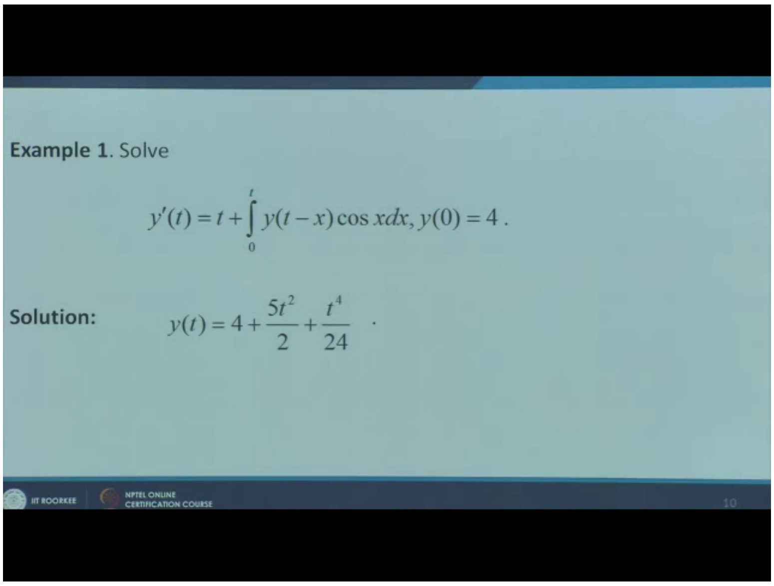

Let us take one more example of an integro differential equation y dash(t) equal to t plus integral 0 to t y(t minus x) cos x dx where y(t) equal to 0 this equal to 4.

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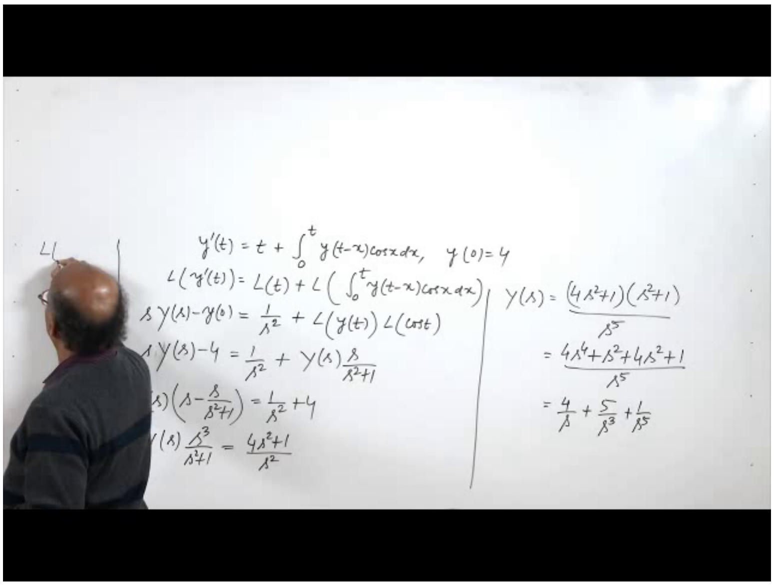

So let us see y dash(t) is equal to t plus integral 0 to t y(t) minus x into cos x dx and we are give y(0) equal to 4. So let us take Laplace transform of both sides of this equation, so Laplace transform of y dash(t) equal to Laplace transform of t plus Laplace transform of integral 0 to t y(t minus x) into cos x dx.

Let us recall the formula for the Laplace transform of y dash(t) so Laplace transform of y dash(t) is s y(s) minus y(0) y(s) is Laplace transform of y(t) and then Laplace transform of t is 1 over s square plus by convolution theorem Laplace transform of integral 0 to t y(t) minus x cos x dx is Laplace transform of y(x) into Laplace transform of or rather I say y(t) and then Laplace transform of cos t.

So this is s y(s) minus 4 because y(0) equal to 4 1 by s square then we have y(s) then we have s over s square plus 1. So let us collect the quotient of y(s), y(s) times s minus s upon s square plus 1 equal to 1 upon s square plus 1 this is y(s) times s cube plus s minus s, so s cube upon s square plus 1 equal to 4 s square plus 1 divided by s square.

And so we have y(s) equal to 4 s square plus 1 into s square plus 1 divided by s to the power phi. Let us multiply in the numerator we have 4 s 4 plus s square plus 4 s square plus 1 divided by s to the power 5 and which is nothing but 4 by s 5 s square by s to the power 5 so 5 by s cube and then 1 by s to the power 5. So this is y(s).

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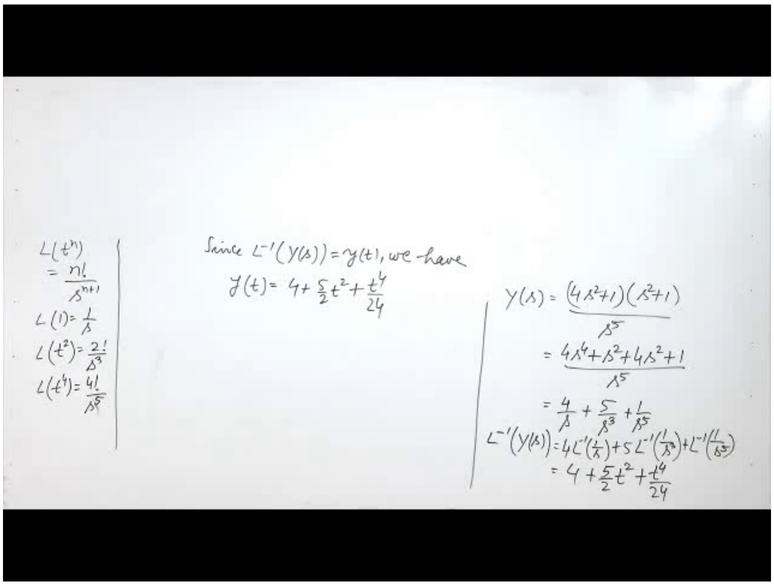

Now let us recall again the formula Laplace transform of t to the power n is n factorial divided by s to the power n plus 1. So by that Laplace transform of 1 is equal to 1 over s Laplace transform of t square is 2 upon 2 factorial upon s cube Laplace transform of t to the power 4 equal to 4 factorial divided by s to the power 5. So applying these formulas inverse Laplace transform of y(s) will be 4 times inverse Laplace transform of 1 y(s) plus 5 times inverse Laplace transform of 1 y(s) cube plus inverse Laplace transform of 1 y(s) to the power of 5.

And this will be equal to 4 plus 1 inverse Laplace transform of 1 y s cube will be t square by 2. So 5 by 2 t square inverse Laplace transform of 1 by s 5 will be t 4 by 4 factorial so t 4 by 24. And inverse Laplace transform of y(s) is y(t) so we get the solution as y(t) equal to 4 plus 5 by 2 t square plus t 4 by 24 which is the required solution of the given integro differential equation

This is what I had to say in this lecture thank you very much for your attention.