Integral Equations, Calculus of Variations and their Applications
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Lecture 50
Variational Derivative and Invariance of Euler's Equation

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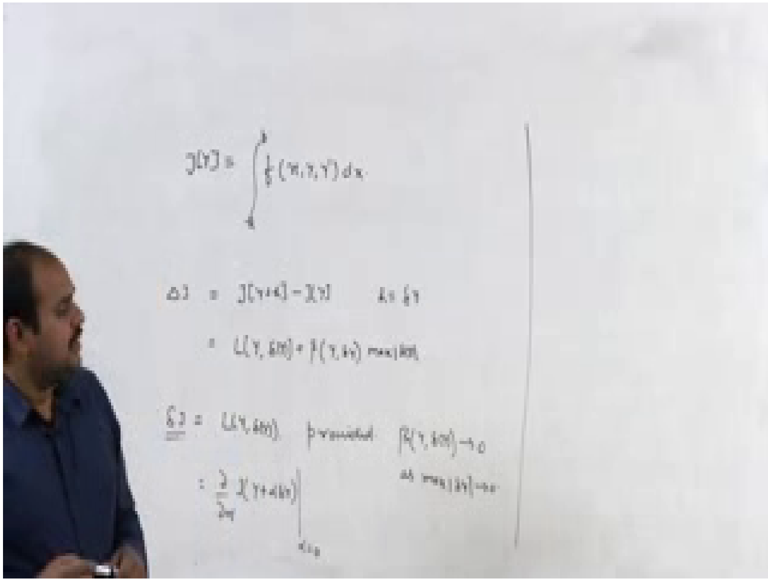

Hello Friends! Welcome to the today’s lecture in this lecture will discuss the concept of variational derivative and some application of variational derivative. So if we recall we have already define variation and differential of functions. For example: If we have function define by a to b f (of x, y, y dash) d(x), then we have to define the variation in functional J as J of say (y plus h) minus j of y and this we have a defined as this l (y) here h is your delta of (y).

So here we can say that this we can write it l of (y) delta of (y) and plus beta times you can say b y and delta y maximum of modulus delta y. So this we have defined differential of this functional j y as l of (y) delta y as the differential variation of this functional j delta j is defined by this awaited that this beta y, delta y is tending to zero as maximum of modulus of delta y into zero.

And here we have defined delta j as this and we may also define it like this dev by dev alpha j of (y plus alpha delta y) at evaluated at alpha equal to zero. So here this is a definition which we have discussed and with the help of this variational or we can say (function) variation of this functional j we have derived the Euler equation which is a necessary condition for an extremal to satisfy this thing.

So if y is the extremal of this function j then that must satisfy the Euler's equation . Now with the help of this we can also define the concept of variational derivative which is known as functional derivative also. So we want to see how we can define variational derivative of functional derivative. So if you recall we can discuss the maxima or minima or you can say extrema of a function of several variable with the help of partial derivative.

So the necessary condition for finding the extremum of a function of n variable is that the partial derivative is equal to 0. So here we want to define the single kind of quality as partial derivative for the functional. So that quantity we call this as functional derivative or variational derivative.

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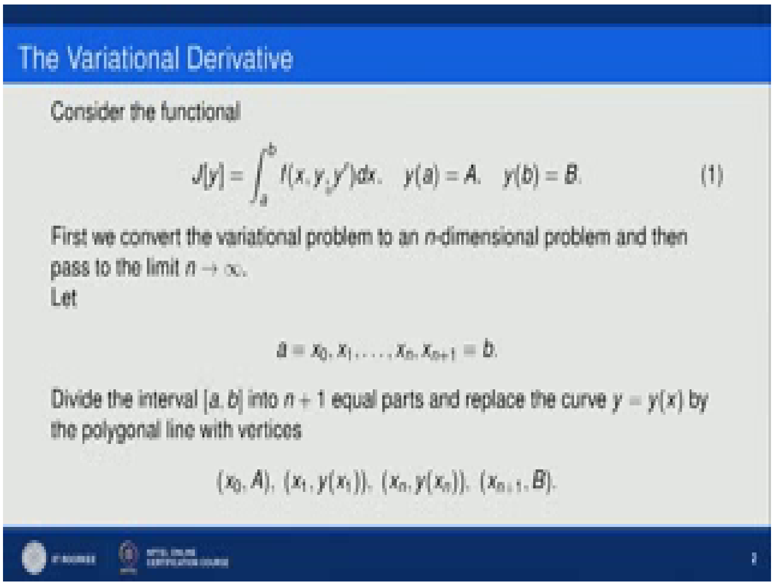

So for this let us define the functionality of y given as a to b f(x) y dash dx with the condition that y(a) is equal to capital A and y(b) is equal to capital B.So to find out say variational derivative we recall that we have discussed that the functional can be understand as function of infinite many variable.

And for that we have utilized the concept of Euler's finite difference method. So here also we use the method of Euler's finite difference and convert this functional into a function of infinite many variable and try to relate the variational derivative with the partial derivative of function of several variables.

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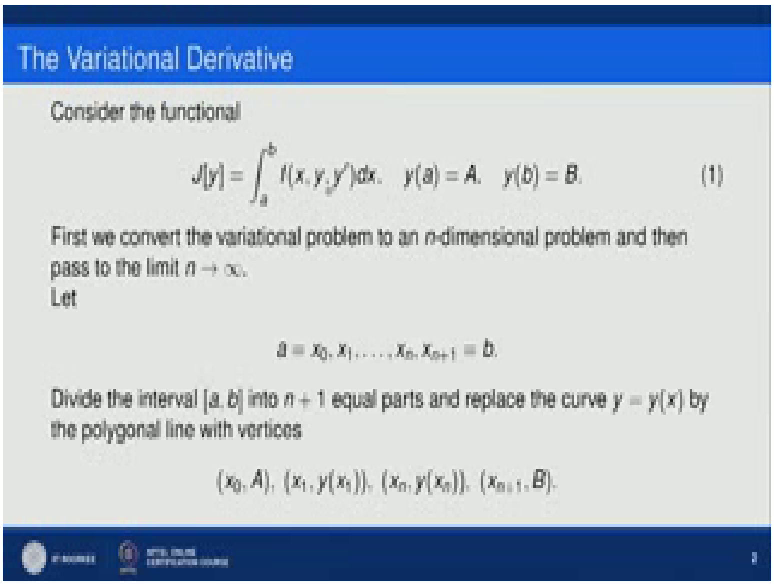

So for that you we first truncate this problem this functional into a problem of n dimensional problem on of a function of an independent variable. For that and then we try to pass the limit as n tending to infinity and try to see what relation we will get. So for that we divide the interval a to b into plus 1 equal part so we say that x 0 is a x n plus 1 is equal to b and all in between element are divided into n plus 1 equal parts.

And we can say that we can replace the curve y equal to y(x) by the polygonal line with vertices. This we can see it like here.

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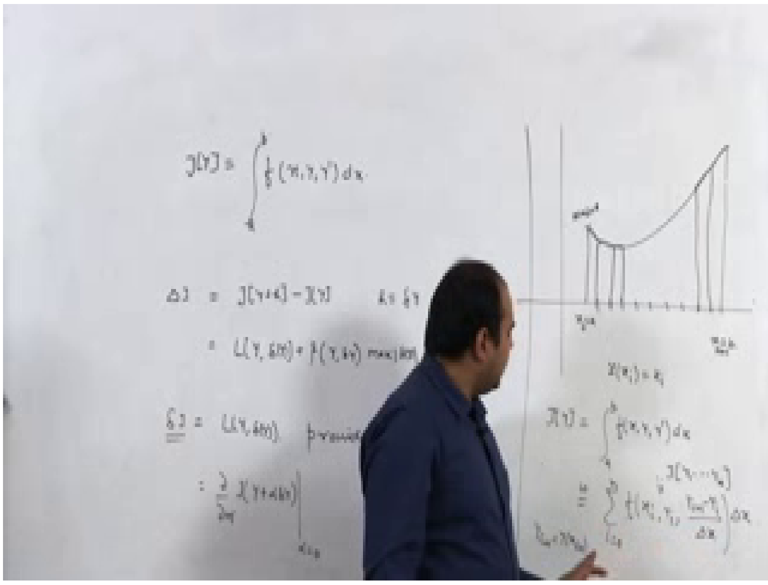

So here we have a curve say y(x) like this and this is your x equal to a and this is your x equal to b. So here what we try to do we divide this into n plus 1 equal part. So that your x 0 is a and x n plus 1 is equal to b and we simply say that this curve is approximated by these lines, ok. So it is like this. So here it is this and this.

So here basically here we are considering that this curve is approximated by these polygonal lines, ok. So and here the value of this y(x 0) is equal to a. And we define y(x i) as y(i). So here if you look at the function j(y) it is given as a to b, f(x, y, y dash) d(x) this can be approximated by the summation i equal to 0 to n f(x i y i) and y(i plus 1) minus y i upon delta x and delta (x).

So here we are denoting y (i plus 1) as y(x i plus 1) and we denote this quantity we define this quantity as say we can define this quantity as say j y 1 to y n. So the difference is that here this functional is approximated by this function of n independent variable y 1 to y n here. Please recall here that your y 0, y 0 which is given as y (x 0) it is already given as a and similarly y (x n plus 1) that is y (n plus 1) that is also known that is b. So it is basically function of n independent variable that is y 1 to y n.

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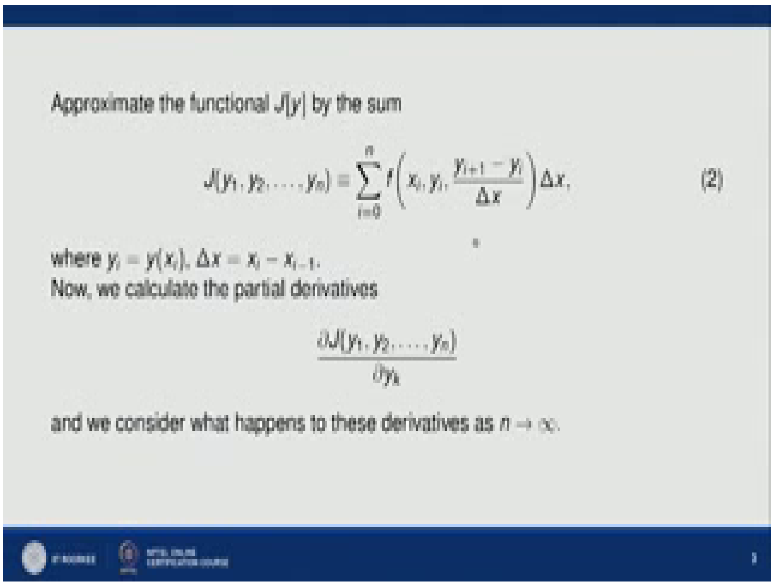

So here so it means that what we have done is that we have approximated the functional j y by this sum which is depending on the variable y 1 to y n and it is given as this summation I equal to 0 to n f(x i y i) y I plus 1minus y I divided by delta x delta x. So here y dash is approximated by this sum y i plus 1 minus y i upon delta x.

So here this notation we are using that y i equal to y(x i) and delta x represents the length of the interval that is x i minus x i minus 1.

So now (we) since it is a function of n independent variable, now let us try to find out same derivative of this j which is given by this sum with respect to say one of the independent variable let us say this is y k and try to find out the partial derivative this dev j (y 1 to y n) with respect to y k. So dev j upon dev y k we are trying to find out.

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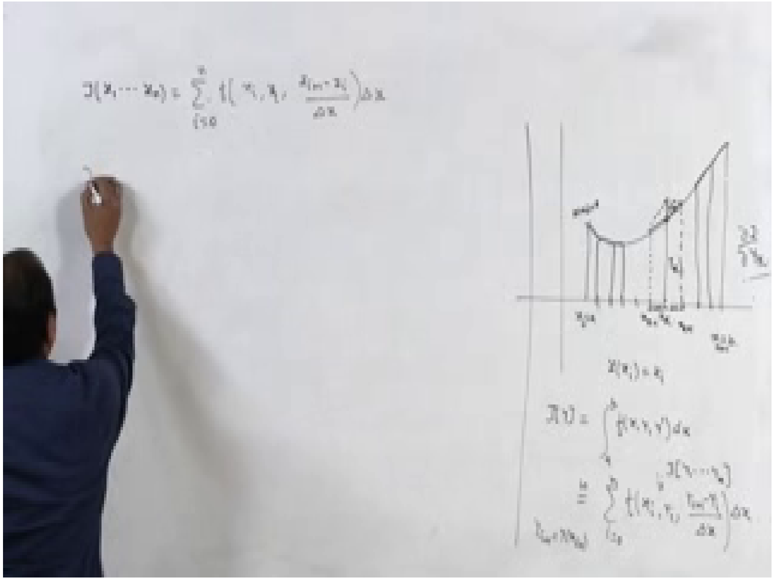

So basically it is what let us consider here somewhere say this corresponding to your say x k so this is your x k and here this represents your y (k) and this is your x k minus 1 and here we have x k plus 1. So here we have this thing, is that ok. So what we try to find out here we have the these straight lines here ok we try to find out dev j by dev y k it means that if we do small change in the values of y k if we perturbed y k little bit and then what is the corresponding change in this function j will occur.

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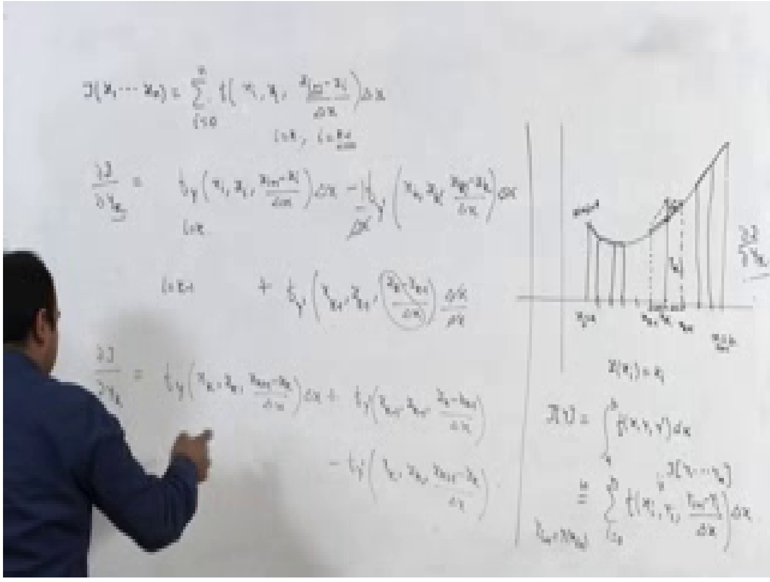

So that we wanted to check, so to find out dev j upon dev y k let us consider the function given in this particular form and then you find out dev j by dev so dev j by dev y k. So dev j upon dev y k and if you remember here this (dev y k) this y k will appear in two terms first when i is equal to k. So when i equal to k then we have y k here and corresponding to i equal to k minus 1 there y k will appear here.

So when you find out say derivative with respect to y k then we will get f(y) and this I am considering for i equal to k term so that is f(y x i y i) and y (i plus 1) minus y i upon delta x delta x and then you consider this particular term. So minus f (y dash) this position is corresponding to y dash and here we have x i y i and y i plus 1 minus y i upon delta x and here we have delta x. And then we have this is i equal to k so this is k this is k plus 1 this is k here.

And then we when you differentiate this minus y k upon delta x then you will get minus 1 upon delta x. So this delta x delta x will be cancelled and now consider the term corresponding to i equal to k minus 1 which gives plus and for corresponding i equal to k minus 1 here we have f(y dash) and we have x (k minus 1) y(k minus 1) and y of this k minus y (k minus 1) divided by delta x and then when you delta x divided by because of this term you will have one more 1 upon delta x coming into picture, so we will cancel this.

And we can see that dev j found over y k is given by f(y) x i (y) sorry this is i equal to k. So here we can get i as k. So I can write it here f(x k y k) y k plus 1 minus y k upon delta x delta x plus we write it in a different form we can write it here f(y dash) and x(k minus 1) y(k minus 1) y k minus y k minus 1 upon delta x minus this term that is f(y dash) here and this is x k y k and comma y k plus 1 minus y (k) upon delta x here.

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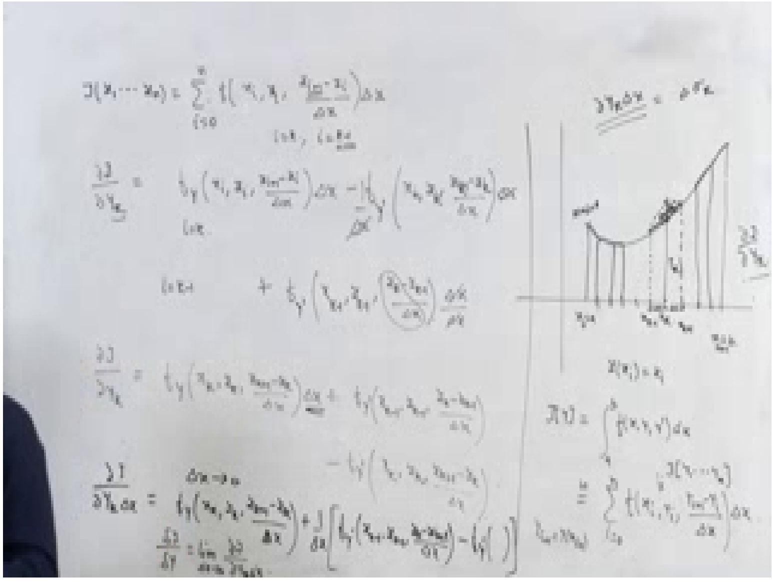

So when you write it like this and then if you take delta x tending to 0. So if I take the limit as delta x tending to 0 means as your this length of the interval is tending to 0, it means that the number of partitions is tending to infinity. So here if we are doing this then just look at the limit here then this term is tending to 0, because we have a delta x here and if you look at these two terms, these two term will tend to the same limit that is f(y dash) x y and y dash.

So here these two term will also cancel each other, so it means that we try to find out say limit of this say in dev j upon delta y k into delta x. And then if you calculate this then this quantity is basically what this quantity will f(y) x (k) y (k) and y(k plus 1) minus y k upon delta x plus 1 upon delta x will be here. And here we will write this thing f(y dash) and x(k minus 1) y k minus 1, y(k) minus y k minus 1 divided by delta x minus this quantity f(y dash) here.

Now now we try to take the limit both the sides as delta x tending to 0. Then please recall here the term in denominator here delta dev y k delta x this basically represents what, dev y k represents this term this deviation and delta x represents this thing. So it means that this dev y k into delta x represent the area between this shaded line and this this shaded line and this dark line.

So it means that this represent a kind of area you can call this as say delta say y k, you can write it like this or delta some sigma k you can write it here. So it means that you can write it like this. So it means that if you say that this delta x tending 0 means the area between this and this dotted line and this dark line is also tending to 0.

And then we want to find out say the change in the functional j, so as you take that delta x tending to 0 then this will tend to what limit. So we say that if delta x tending to 0 this limit exists we call this limit as delta j for delta y. So this is the limit of delta x tending to 0 dev j upon dev y k and delta x here. So the whatever limit will be that is denoted by delta j by delta y here.

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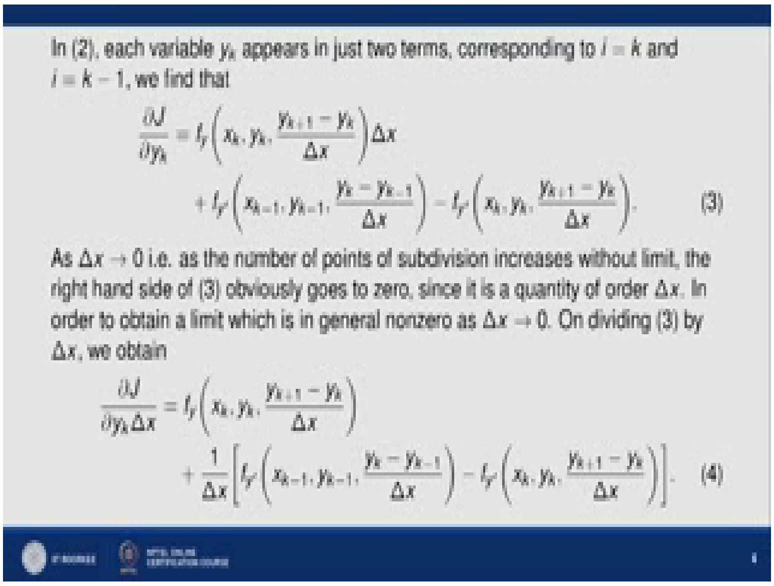

So this we are doing it here so as we have seen that this three represent the partial derivative of j with respect to y k which we have just calculated here and here we divide this 3 by delta x then we can calculate dev j by dev y k delta x equal to this quantity this is just we have written here. Now as you take the limit delta x tending to 0 then if this last term will tend to some limit as delta j by delta y which we which is given by f(y) x(y y dash) minus d by dx of f (y dash) x y y dash.

That is clear here that here as delta x tending to 0 this will tend to what this will tend to let me to find out delta j by delta y we take the limit when delta x tending to 0 then this x k will say tend to x and y k will be y,

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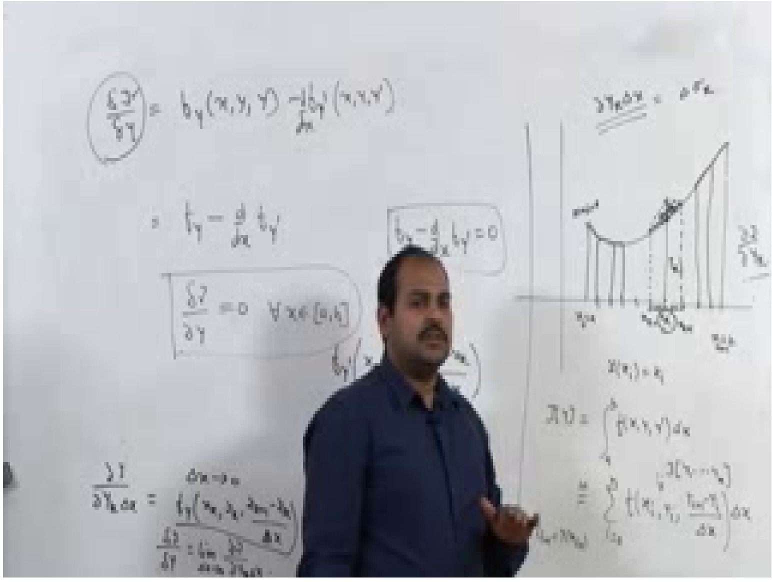

So you can write it delta j upon delta y which is given as this f(y) x y y dash minus now here this will be what here if you look at this term it is what it is x(k) y(k).And let me write it here in a clear manner this term is your f(y dash) and (x k, y k, y k plus 1 minus y k) upon delta x.

So if you tick the limit of this minus this as delta x tending to 0 then this will converge to minus (f(y dash) d by dx of f(y dash) x y y dash here . So we can say that your variational derivative of functional j with respect to y is given by this quality which is known as f(y) minus d by dx of f(y dash). Now if you recall the Euler's equation then it is nothing but the (right hand side) left hand side of the Euler's equation?

In Euler's equation the necessary condition for a curve to be an extremum of a function is at f(y) minus d by dx of f(y dash) is equal to 0. So here if we want to define the same Euler's equation in terms of variational derivative then we can say that Euler's equation is nothing but that the delta j upon delta y is equal to 0 for every x belonging to this interval a to b.

So Euler's equation I can write it like this. Now here we may discuss one more interpretation of this delta j by delta y. Here if we look at what is this basically here, here we are finding say limit of this quantity. Now what is the limit of this quantity? Here this represent that here we have perturbed only the value of y only in the neighbourhood of say x k.

So here we try to find out say partial derivative of this function j from y 1 to y n with respect to y k it means that in the neighbourhood of x k we perturbed the value of y k that this we perturbed the value of y k and try to see what is the corresponding change in the functional.

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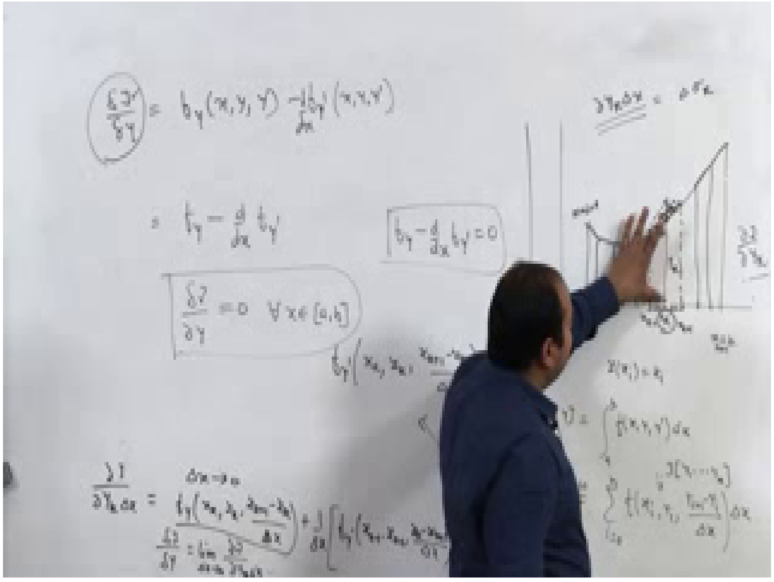

So it means that here this delta y here we have considered the change only in the neighbour hood of x k and we denote it as the like this. So in general we can generalize into a new concept new thing we can say it like this.

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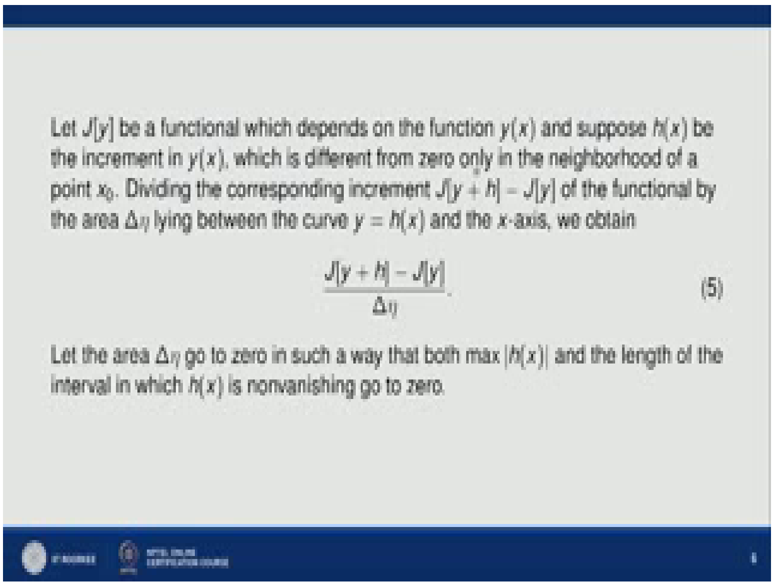

That if let j y be a functional which depends on the function y(x) and suppose h x be the increment in the y(x) which is different from 0 only in the neighbourhood of a point x 0. So here if you recall we have done in the neighbourhood of x k. Now let us say that x 0 be a point in which we find out say (diff) increment in y(x) which is denoted by y(x).

So we say that dividing the corresponding increment j of y plus h minus j (y) of the function by the area delta eta. What is the delta eta? Delta eta is area lying between the curve y equal to h(x) and the x axis. In fact you can also consider that this delta eta is the area between the curve y plus h and y. So here if you look at this quantity this quantity is j(y plus h) minus j(y) divided by delta eta. So here if you look at if you find out say limit of this as delta eta goes to 0 is the same thing which we have considered here.

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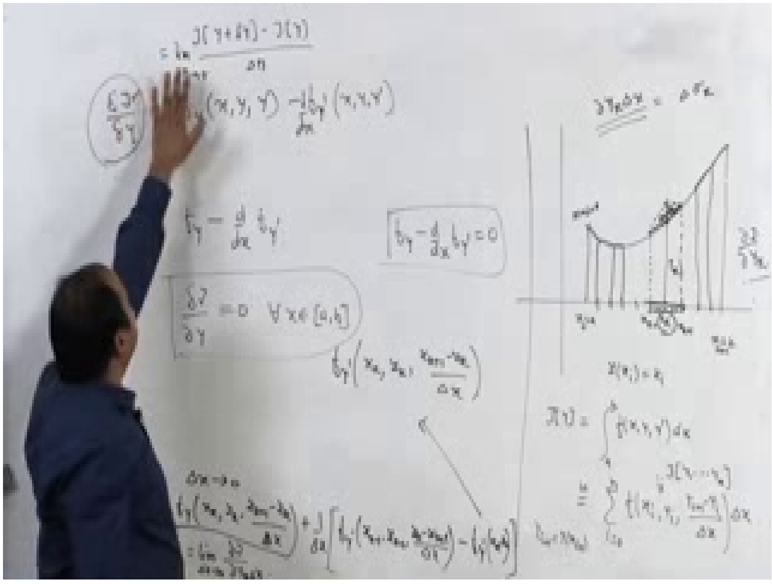

So here we can interpret this quantity delta j by delta y as you can simply say that it is nothing but you have j(y) plus delta y here minus you can consider this as j(y) divided by the area which delta y covered that is you can consider that as delta eta. So here we can say that what is a limit of this, so this I can define as limit delta eta is tending to 0. So we can also define the variational derivative in the sense of functional like this.

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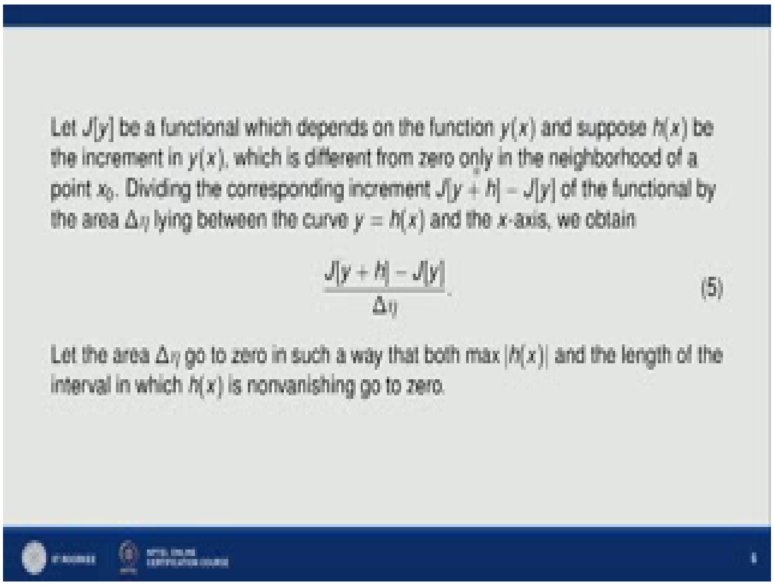

So here we say that if the area delta eta goes to 0 in a such way that both maximum of modulus of h x and the length of the interval in which h x is non vanishing go to 0.

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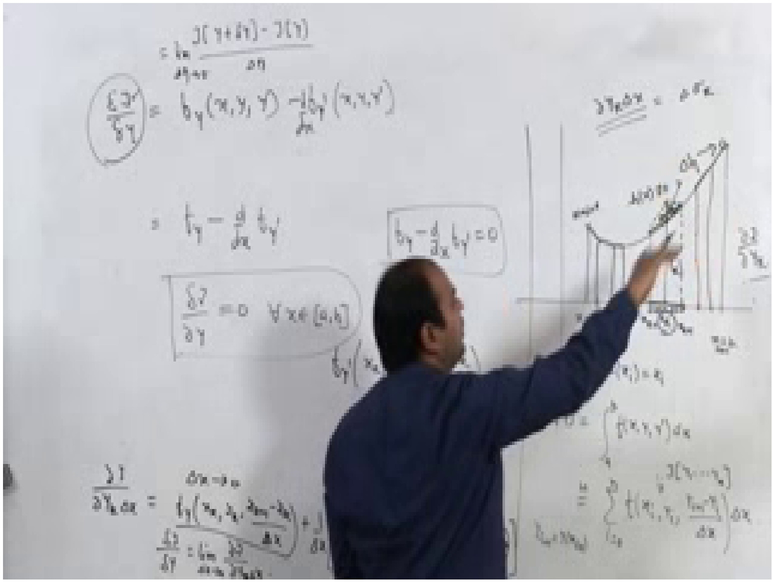

So here delta eta is basically what delta eta is the area of this certain line and this dark line, so here we say that this delta x tending to 0 and delta y is also tending to 0. So it means that we are saying that this is denoted at delta eta, so we can say that delta eta tending to 0 means the height is tending to 0 as well as the width of the area where it is non zero h a 6 h s h(x) is non zero that is also tending to 0.

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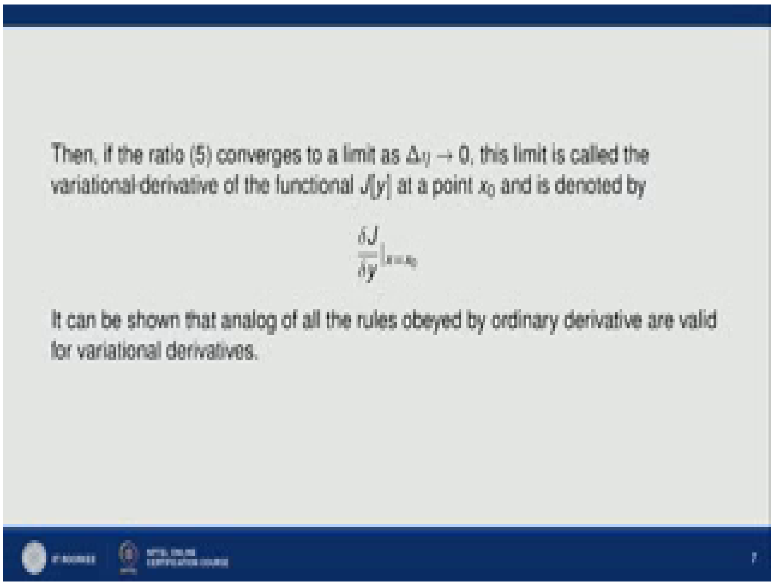

So here we can say that then if the ratio 5 converges, ration 5 means this thing, so if the ratio j(y plus h) minus j(y) divided by delta eta converges to some limit as delta eta tending to 0 we call this limit as delta j by delta y in the neighborhood of x equal to x 0, and we denoted this as this and we call this as variational derivative of the function j y at a point x 0. And as we know that there are certain rules which ordinate derivative satisfy this similar kind of rules we can also prove for this variational derivative.

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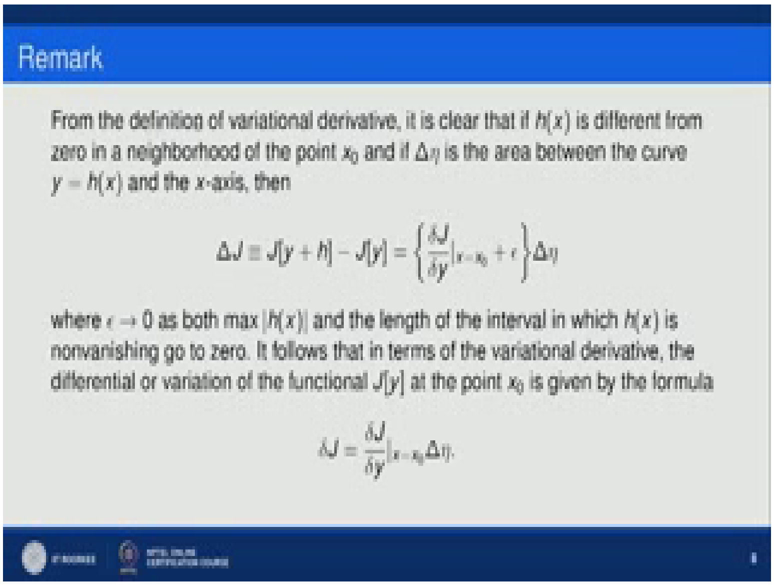

Now moving on say next thing we can find out say relation between variation and the variational derivative. So here we can say that from the definition of variational derivative it is clear that if h x is different from 0 in the neighbourhood of the point x 0 and if delta eta is the area between the curve y equal to h(x) and the x axis then j(y plus h) minus j y is equal to dev j by delta delta j by delta y defined at x equal to x 0 plus epsilon into delta eta.

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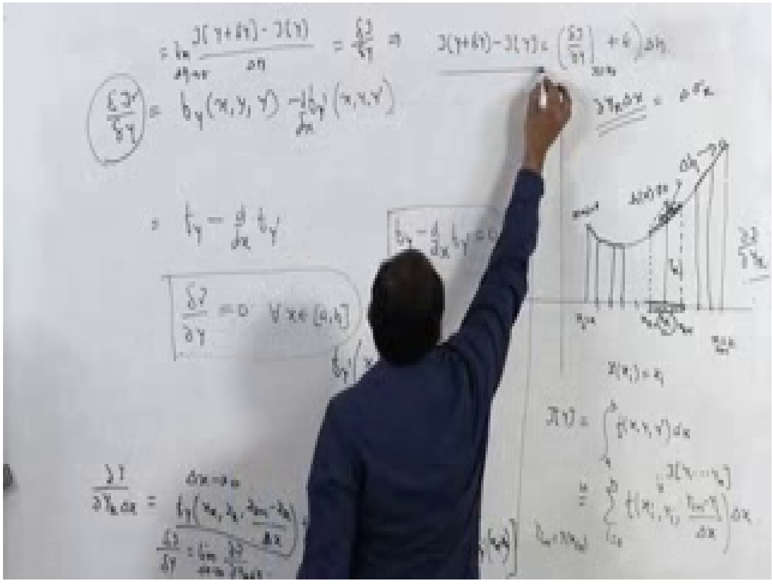

Infact you look at here, so this is what this is we have defined at delta j upon delta y and if you remove the limit then we can write it here this as j(y) plus as delta y minus j(y) equal to we can say that it is delta j on delta y.

Now since we are considering delta y only in the neighbour hood of point x equal to x 0. So we can write it at x equal to x 0 plus here you can take any function say epsilon into delta eta.

So as epsilon tending to 0 this delta eta is tending to 0, so here we can define this is nothing but differential or variation of the functional. So here we can say that we can link variational derivative and variation of the function in this way, is that ok.

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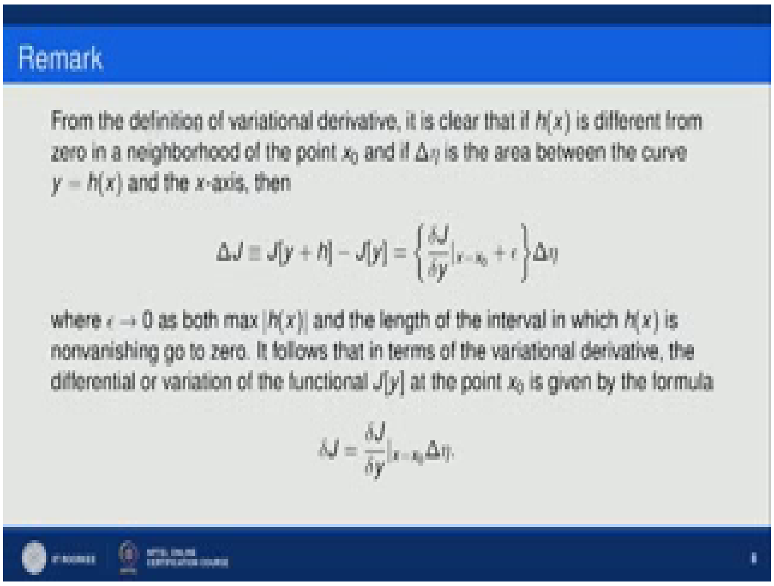

So here we say that delta j is given as j(y plus h) minus j(y) delta j by delta y given at x equal to x 0 plus epsilon into delta eta. So where epsilon tending to 0 as both maximum of modulus of h x and the length of the interval in which h x is non vanishing go to zero. It follows that in terms of the variational derivatives the differential of variational of the functional j y at a point x 0 is given by this formula.

So here you can say that the limiting case as epsilon tending to 0 we can say delta j is equal to delta j by delta y given at x equal to x 0 into delta eta. So this is the relation between variation and variational derivative of a functional j. So as an application of variational derivative we are going to discuss the problem of this property which is known as invariance of Euler's equation.

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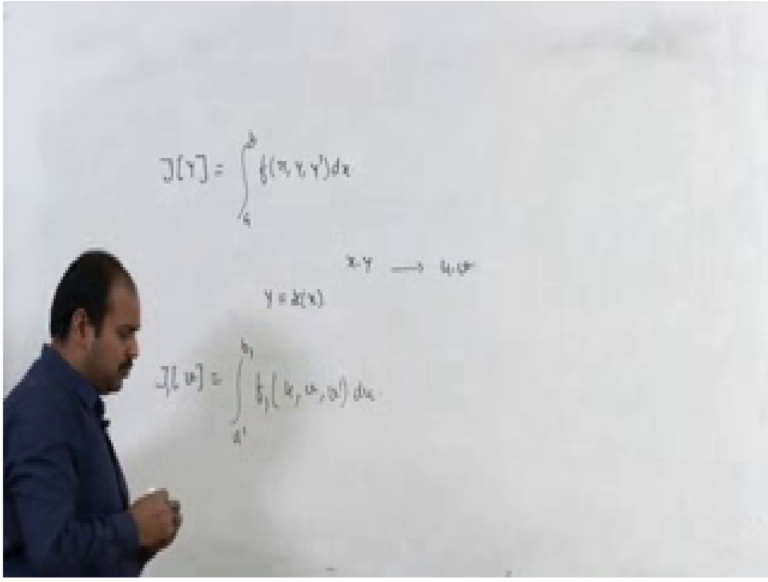

So what is this property if you recall we have a j(y) which is given as a to b f(x, y dash) d of x. So here we are using the rectangular coordinate system x y and here we are considering that y is a function of y(x) here. So now if we want to consider the same problem in some other coordinate system then it means that if we write it in some other coordinate system say for example here we are using xy plane but now let us suppose that we are considering the new coordinate system uv plane and we can rewrite this problem as j(y 1) say j 1(v) as some thing like a 1 to b 1 say f 1 of u v and v dash and du.

Then what is the relation between Euler's equation given in this case and the Euler's equation given in this case. Whether can we do this kind of change of variable here also, because this thing we can do it in a multiple integral when when when we solve some kind of a multiple integral when we are stuck in some kind of problem, then we always try to find out a new co ordinate system in which your problem is simplified and you can easily solve. So here also we try to find out a similar kind of thing here.

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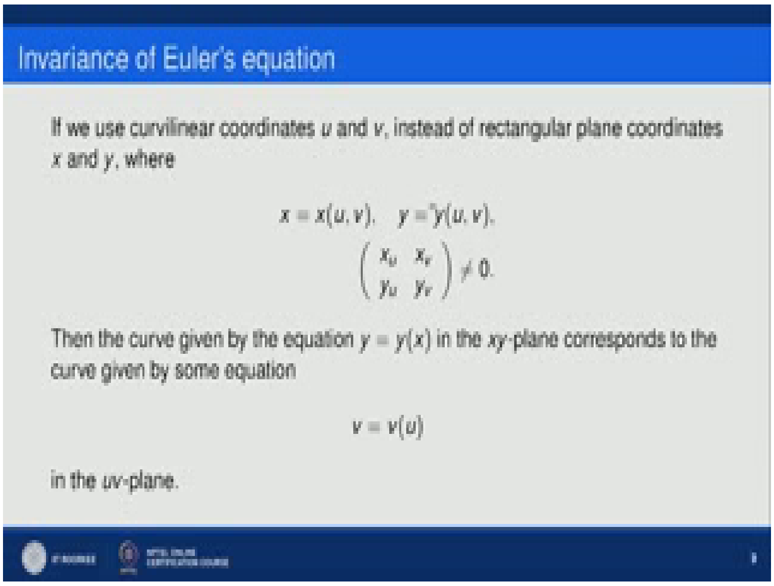

So here let us consider this that if, if we use curvilinear coordinates u and v instead of rectangular planar coordinate x and y where x and y and u and v are related by this, that x can be written as function of u and v and y is a given function of u and v and y is a given function of y(u and v). Now since u and v and x and y are simply representing the coordinate system so here we are assuming that this Jacobean dev x y with respect to u v is non zero.

It means that x u this determinant x u , x v y u y v is non zero. Then we can say that then curve given by the equation y equal to y (x) in the xy plane correspond to the curve given by the sum equation v equal to v(u) in the uv plane. So if we do this then we can change the functional into a new function which is j 1(v).

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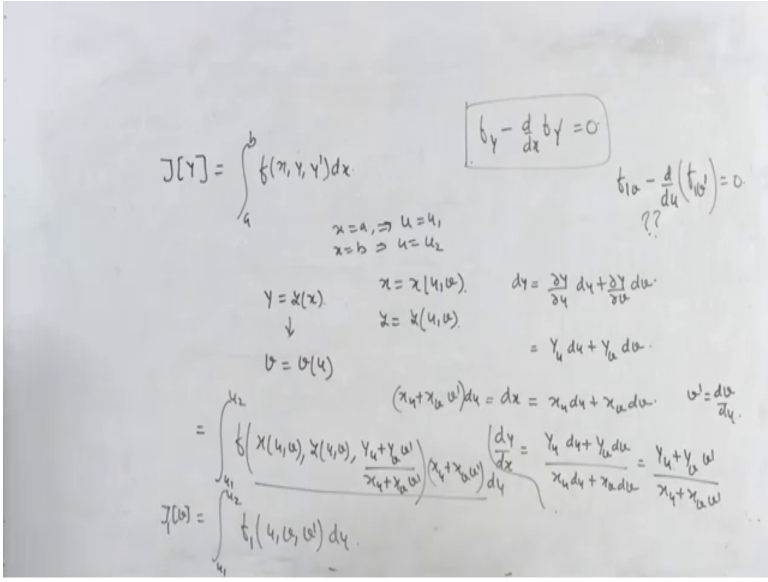

So here how we can do this change we simply say that this here your y equal to y(x) is given by the corresponding curve is given by v (u). So here x is given as x (u and v) and y is y(u and v). So here we simply replace this as the corresponding value of x when x equal to a then we can say that this correspond to some value of u equal to say u 1.

Similarly when x equal to b you can say that u is equal to some u 2 or the corresponding values are there. And here we can write it as x(u, v) y(u, v) and to find out dy by dx we will use this formula that dy is given as a dev y by dev u du plus dev y by dev vdv, so this we can find out as y(u) du plus y(v) dv similarly we can find out d(x) that this x u d u plus x v d v and we can find out dy by dx as you can write it here y u d u plus y v d v divided by x(u) d u plus x (v) dv.

And if we assume that v is a function of v is a function u then we can write this as y(u) plus y(v) and here we can write it v dash, v dash represent the derivative of v with respect to u. So here v dash represent dv by du and here we have x (u) plus x(v) and v dash.

So it means that here dx we can write it here as x(u) plus x (v) v dash, is that ok. And d (u) here. So dx can be written as this so it means that in place of dy by dx we can write it this quantity that is y(u) plus y(v) v dash divided by x(u) x(v) v dash and in place of d (x) we are writing this x(u) plus x(v) v dash and d(u). So this we call this as so corresponding value you can call it u 1 and u 2. So this you can say that u 1 to u 2 this we call as a some function which is given in terms of u and v.

And we call this as f 1 u v v dash and du and call this functional as j 1 which is given in terms of v. So here, here your extremal (card) is y and here your variable is v, is that ok. So now we say that if y is the extremal curve of this it means that y satisfy the condition f(y) minus d by dx of f y dash is equal to 0 then we try to find out that whether v will also satisfy the similar kind of relation or not. It means that we want to find out the relation that 1 that f 1 v minus d by d (u) f(v dash) is true or not f 1 v dash is equal to 0 or not. So this we want to check here.

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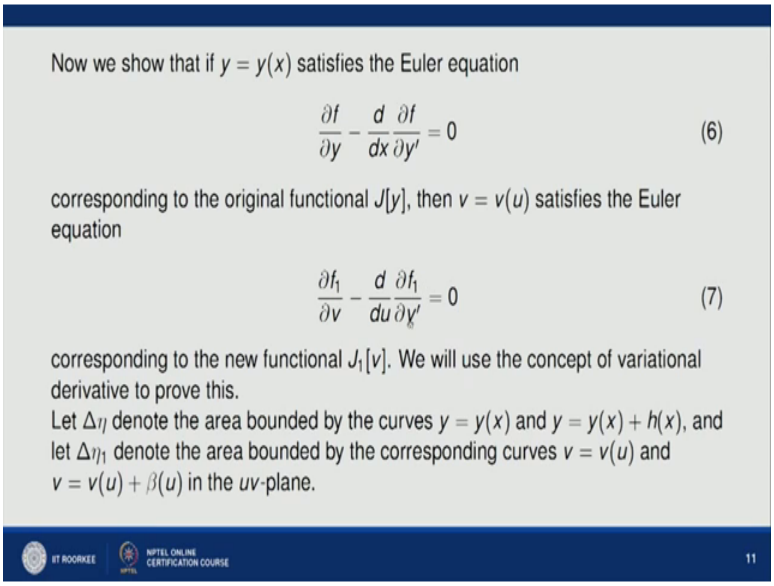

So our claim is if this is true then this will also be true, so this we want to prove here, so here we show that if y equal to y(x) satisfy the Euler equation dev f by dev y minus d by dx of dev f by dev y dash is equal to 0, then the corresponding to the original function this is corresponding to original function j(y) then v(u) satisfy the Euler's equation corresponding to the new functional that is dev f 1 dev v minus d by d u dev f 1 dev v dash is equal to 0 corresponding to the new functional j 1 v.

So here to prove this that if 6 is true then 7 will also true we use the concept of variational derivative. So let us say that let delta eta denote the area bounded by the curves y equal to y(x) and y equal to y(x) plus h(x). So here delta eta is the area bounded by these 2 curves and delta eta 1 denote the area bounded by the corresponding curves, so corresponding to y equal to y(x) we have v equal to v(u) and corresponding to this y equal to y(x) plus h(x) we have new curve that is v equal to v(u) plus beta of u in the u, v plane.

So if delta eta represent the area between these two curves then delta eta 1 represents the area between v equal to v(u) and v (v(u)) plus eta of u in the u, v plane. And we already know that the area is related by this formula,

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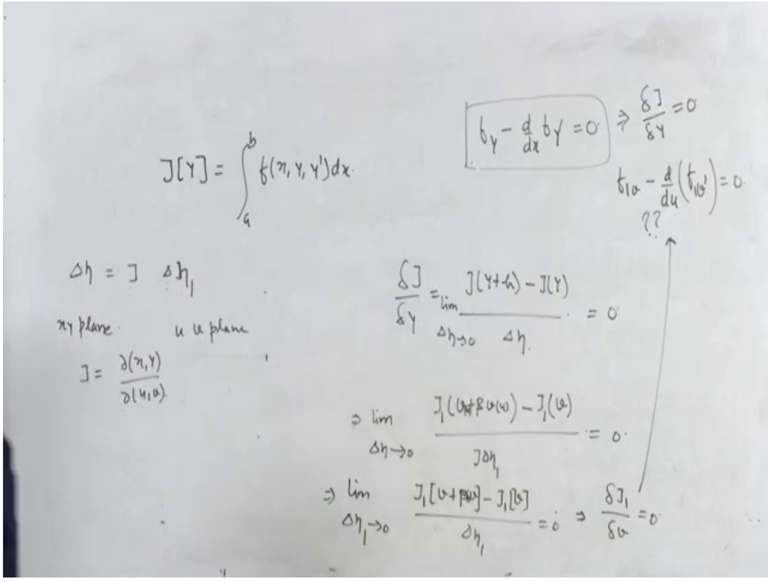

So here we can simply say that if we have area delta eta and in x y plane and delta eta 1 sorry delta eta 1 in the u, v plane then it is related by this that Jacobean delta eta is equal to Jacobean of delta eta 1 where Jacobean is divided by (dev) x, y with respect to u and v. Which is given by this formula (x u x v, y u y v) which is non zero by hypothesis. So this we have already assumed that this is non zero.

Now if you look at the equation number 6 represent what this represent what this represent that delta y delta j with respect to delta y is equal to 0. So here and this is how we are denoting this we are denoting as, so the Euler equation for this function is given by f(y) minus d by dx of f y dash equal to 0 which we have seen that this is nothing but the variational derivative of j with respect to y is equal to 0.

And we also define this delta j by delta y as j(y plus h) minus j (y) divided by say delta eta and limit delta eta tending to 0. So delta j by delta y is defined like this so it means Euler 's equation means that this quantity is equal to 0. So here we want to see whether this quantity is 0 in the new case also or not.

So here if you look at limit now delta eta tending to 0 now j(y plus h) is given by j 1 of your v plus beta v(u), v(u) beta v plus u minus j (y) is given by j 1(v) divided by delta eta I am writing as Jacobean and delta eta 1 right? And we want to show that this is equal to this limit is equal to 0 as delta eta tending to 0. Now we already know that delta eta is given by Jacobean into delta eta 1.

So if delta eta is tending to 0 then delta eta (is) 1 is also tending to 0. So we can write this as that limit delta eta 1 tending to 0 this I am writing as j 1 v plus beta v delta v minus j 1 of v divided by delta eta 1 and that is equal to 0. So this simply shows that this is nothing but in variational derivative says delta j 1 with respect to delta v is equal to 0.

And which is equal to this thing which represent that the Euler's equation in new coordinate system v is also true it means that f 1 v minus d by du f (1 v dash) is equal to 0. So it this simply says that if we have a function given in say x y coordinate system then n y is y is the extremal of this function it means that this Euler's equation is this true it means this is true.

Then in a new coordinate system that is u v plane and the corresponding functional so it means that we have seen that, that in new functional is also satisfies the corresponding Euler's equation.

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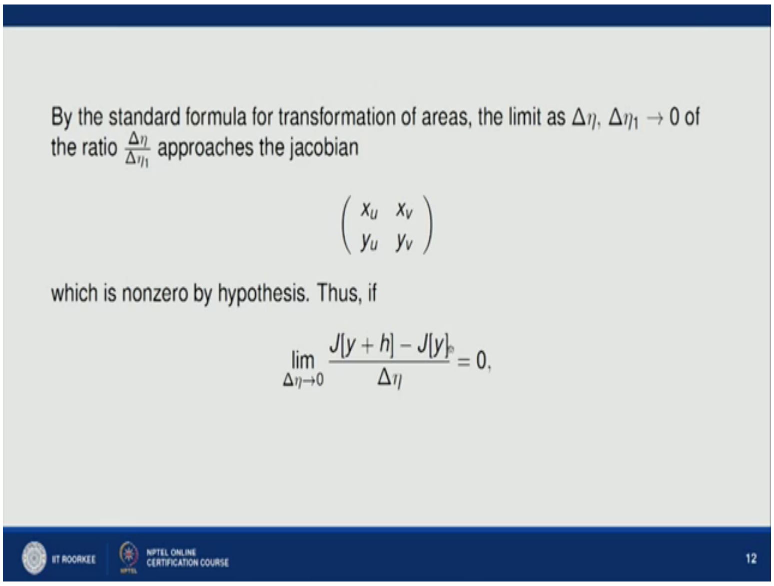

It means that, that if this limit delta eta tending to 0 j(y) plus h minus j(y) divided by delta eta is equal to 0.

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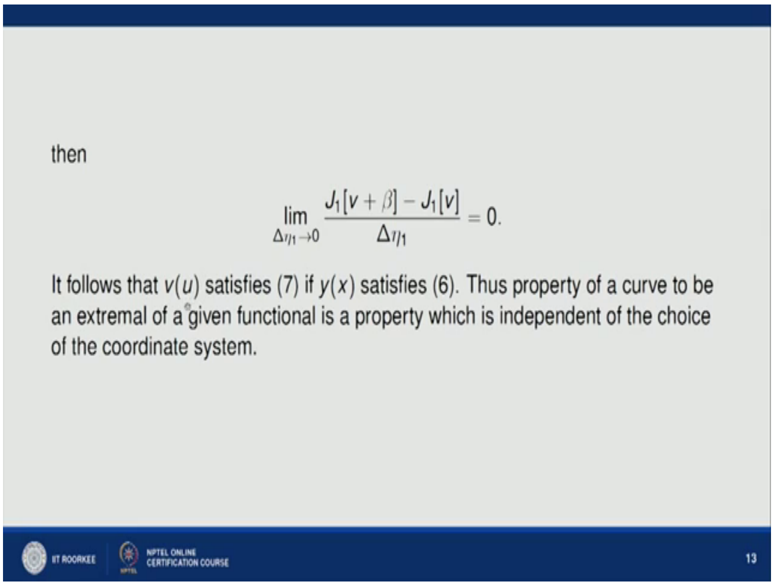

Then this limit limit delta eta 1 tending to 0 j 1 v plus beta minus j 1 v divided by delta eta 1 is equal to 0. Which simply says that the curve v satisfy the Euler's equation n u v coordinate system, So it means that, that property of a curve to be an extremal of the given functional is a property which is independent of the choice of the coordinate system. So it means that whether we have a say coordinate system which is x y coordinate system or say u v coordinate system

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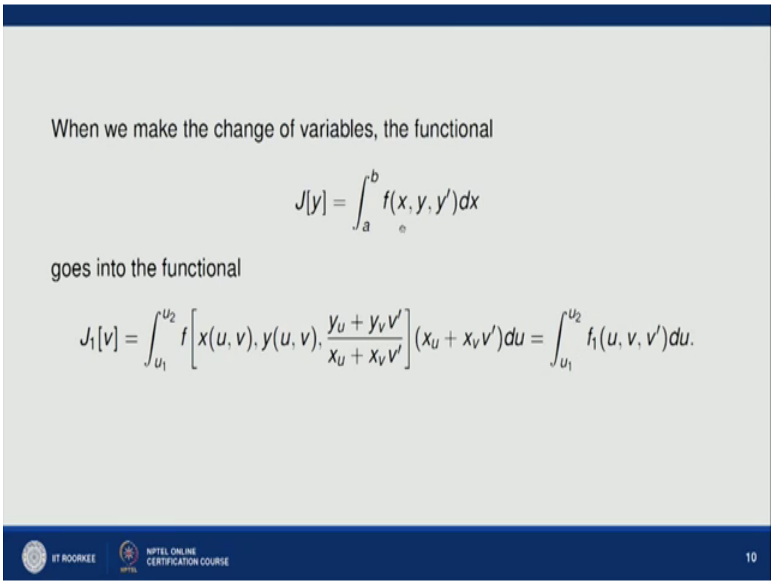

But the Euler's equation in this equation and Euler equation in this system is independent of the coordinate system. So it means that this simply says that Euler's equation are invariant under the change of coordinate system. So here we have discussed the impedance of Euler's equation with respect to coordinate system and next lecture we will discuss the example based on this property of impedance of Euler's equation.

So thank you for listening us we will meet in next lecture