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Nonlinear Programming

Lec-10
Separable Programming-II

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Hello friends so welcome to lecture series on nonlinear programming. So in the last lecture we have seen what separable programming problems are, I told you that if the objective function as well as all constraints are separable then we say that the problem is the separable programming problem. Also we have seen in the last class that if function of one variable is given to you how we can find out the linear approximation of that function. Now let us see how to solve those problems okay. So what is the solution processor let us see.

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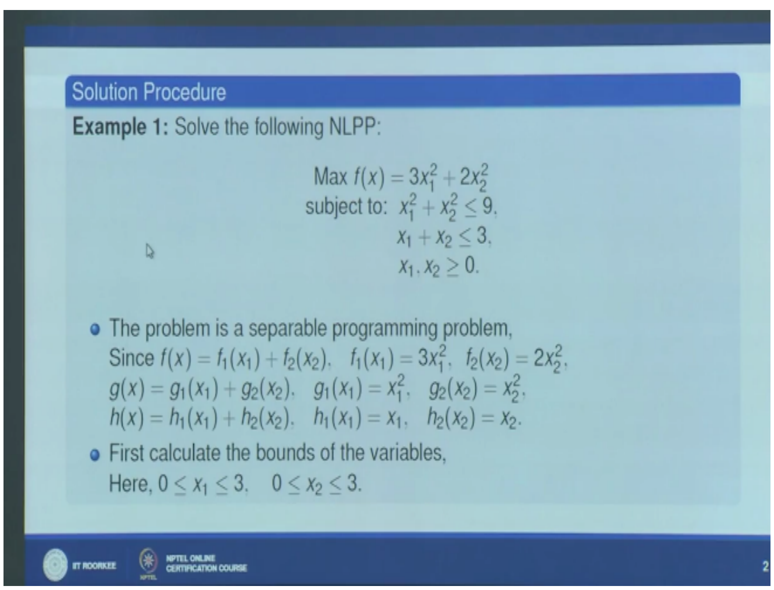

So this problem giving maximizing $f(x)=3x_1^2+2x_2^2$ subject to $x_1^2+x_2^2$ less than equals to 9, x_1+x_2 less than equals to 3 and x_1 x_2 non negative is a separable programming problem, because we can quickly see that the objective function can be separated into two functions and each is the function of one variable only. The first constraint is also separable and second constraint is also separable and non negativity condition is also separable, so it is a separable programming problem okay. Now how to solve this problem, so let us see.

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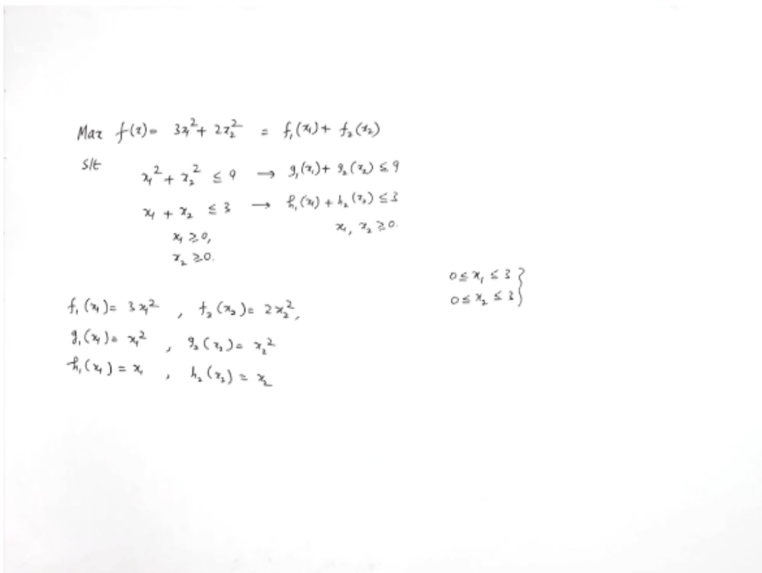

So what is the problem it is maximizing $f(x)=3x_1^2+2x_2^2$ subject to it is $x_1^2+x_2^2$ less than equals to 9, x_1+x_2 less than equals to 3 and x_1 is greater than equals to 0 and x_2 is greater than equal to 0 okay. Now the objective function can be rewritten as f1(x_1)+f_2(x_2) this constraint can be rewritten as g_1(x_1)+g_2(x_2) less than equals to 9, this constraint can be rewritten as h1(x_1)+h_2(x_2) less than equals to 3, and non negativity condition you can remain it as it is, you can leave it as it is.

So what is f_1(x_1), f1(x_1) is $3x_1^2$ as you can simply see function of x_1, and f_2(x_2) is $2x2^2$ if you see g_1(x_1), g_1(x_1) is $x1^2$, g_2(x_2) is x_2^2, h1(x_1) is x_1, and h_2(x_2) is x_2 okay. So this is because the problem is separable programming problem. Now to solve this problem first of all you find the bounds of the variables. How many variables are involved in this problem x_1 and x_2. First find bounds of the problem.

So you can simply see, you can easily see that minimum value of x_1 is 0 okay and from this constraint maximum value of x_1 which it can attained is 3. And if you substitute this over here so it is 3^2 means and 9 is less than equal to 9 that is true. So we can simply say that the bounds of x_1

is 0 to 3. Similarly, if you see the lower bound of x_2 is 0 clear and the upper bound from this consent we can say it is 3, whereas others satisfy this constraint.

So the x_2, the bound for x_2 will be less than equals to 3 less than equals to 0. So first of all find bounds corresponding to the variables involved in the problem, here we have only two variable x_1 x_2, so find bounds of corresponding to all the variables.

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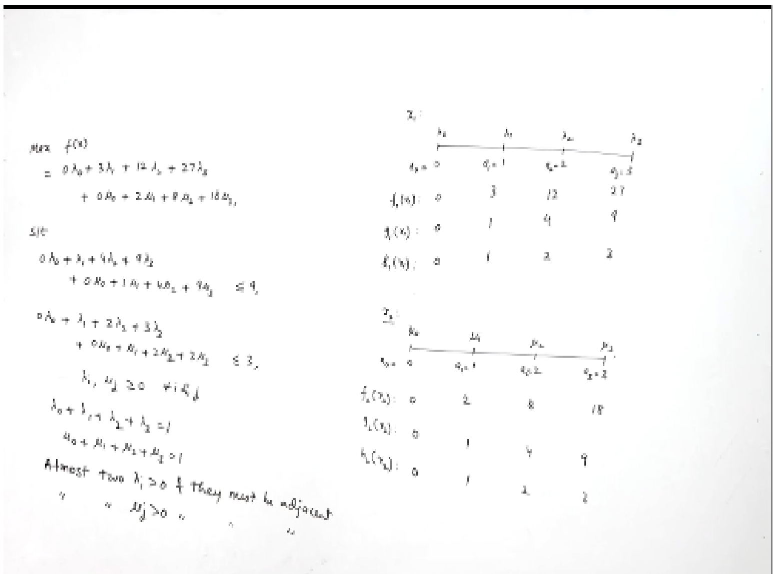

Next is now once finding the bounds we will find now it is the function of x_1, it is the function of x_1, it is function of x_1 okay. Now we will see x_1 is between 0 and 3 wherever x_1 okay is between 0 and 3 wherever x_1 okay is between 0 and 3. So we can mark grid points as 0, 1, 2 where these are the grid points this is a_0¸a_1, a_2, a_3 okay this is a0 this is a_1 this is a_2 this is a3 so what is f1(x1) at these grid points what is f1(x1)? h3(x1) is square, what is f1(x1) when a0 is 0.So you substitute 0 here it is 0 you substitute $_1$ here it is 3 you substitute $_2$ here it is 4 x 3 it is 12 and then it is 27, okay.

Now you focus on g1(x1), g1(x1) is x1$_2$ at a0 = 0 this is 0 so it is 0 when you substitute a_1 = $_1$ basically we are finding the value of f_1 g_1 and x_1 at these grid points when a_1 is $_1$ it is $_1$ when a_2 is $_2$ it is 4 when a3 is 3 it is 9 and h_1 (x1) it is x_1 so it is 0, 1, 2, 3,okay. This is correspondent to label x_1 now for x_2 the bound of x1, x_2 both are same as we have already seen so again we will divide this in between 0 and 3.

We can mark grid point as 0, 1, 2, 3 okay this is only for convenience we have make this as a grid points 0, 1,$_2$, 3 now again this is a0 suppose it is a_1 it is a_2 and it is a3,now it is $f_2(x_2)$ now at a0 = 0,$f_2(x_2)$ is 0 and at a_1 = $_1$ it is 2 x 1 is 2 at a_2 = 2 it is 8 and it is 18, 9 x 2 it is 18 now it is $g_2(x_2)$ is $x1^2$ x_2^2 and 0 it is 0, 1, 4, 9 and at $h_2(x_2)$ which is x_2 so it is 0, 1, $_2$, 3. So we first find the bounds correspond to all the variables involved in the problem.

Then fix up the grip points correspond to the all the variables find the value of f1(x1) g1(x1) h1(x1) and so on okay, similarly for the second variable and so on okay. Now we will write the linear approximation of this problem how to write, now the objective function will be maximizing f(x) is equal to, now suppose correspond to this grid pint we have λ0, λ1, λ_2 and λ3 and corresponding to variable x_2 we are μ0 we are having μ0, μ1, μ_2, μ3,okay.

As we have already seen okay so how to write f(x), f(x) is nothing but $f_1(x_1) + f_2(x_2)$ we first find the linear approximation of f1(x1) okay so linear approximation of f1(x1) is 0 x λ0,3 x λ1, 1_2 x λ_2, $_2$7 x λ3 + $f_2(x_2)$ how to find the approximation of $f_2(x_2)$ that is 0 x μ0, $_2$ x μ1,8 x μ_2,18 x μ3 so what will be f(x), f(x) is nothing but the sum of linear approximation of $f_1(x_1)$ and $f_2(x_2)$ so it will be 0 x λ0 + 3 x λ_1 + 1_2 x λ_2 + 27 x λ3 + 0 x μ0 + 2 x μ_1 + 8 x μ_2 + 18 x μ3 subject to what are the conditions.

That involved the first constraint involved the sum of g1 x1 and g_2 x_2 okay, so what is the linear approximation of g_1 s_1 it is 0 into λ_0 1 into λ_1 4 into λ_2 9 into λ_3 okay that is λ_0 fa0 I mean g1 n_0 + λ_1 g_1 a_1 + λ_2 g_1 a_2 + λ_3 g_1 with so it is $0\lambda_0 + 1\lambda_1 + 4\lambda_2 + 9\lambda_3$ + corresponding to g_2 x_2 it is 0 μ_0 = $1\mu_1$ + $4\mu_2$ and $9\mu_3$ and it is given towards that is < = 9 it is given towards in the problem that $x1_2$ + $x2^2$ is less than < = 9 so it is this is the linear approximation of $x1^2$ and this is a linear approximation of $x2^2$ and sum must be < = i.

Now $x_1 + x_2$ < = 3 the second constraint so what is the linear approximation of x1 it is 0 into λ_0 + λ1 +2 λ2 + 3λ3 + and corresponding to s_2 x_2 it is again 0 into μ0 + 1 x μ_1 + 2 into μ_2 and less than equal to 3 and all variables all λi is in must be > = 0 for all i and j all variables must be non negative okay and of course $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$ must be 1 okay and $\mu_0 + \mu_1 + \mu_2 + \mu_3$ must be 1 okay, basically what we are doing first a fall we are seeing one variable x1 forget about the other variables we are making the linear approximation.

Corresponding to that variable now we are focusing on the second variable forget about the remaining variables, and we are making the liner approximation corresponding to those variables so these λ is and these μj is may not be same so we are making different constraint $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$ should be 1 and $\mu_0 + \mu_1 + \mu_2 + \mu_3$ should be 1 okay. Now at most the next condition at most 2λ is > 0 and they must be adjacent, similarly for λj is for μj they must be adjacent okay the same condition is for μ also, so this will be the linear approximation of this problem this non linear programming problem okay I mean separable programming problem, now the position arises how to solve this problem okay we have find the linear approximation of the given f we are find the need approximation of all the constraints.

But how to solve this problem now so this problem is clearly a LPP a linear programming problem we can easily see it is a linear programming problem now but we have another restriction and the restriction is at most two λj is 2λ is > 0 and they must be adjacent okay and at most to μj are > 0 and they must be adjacent these is that additional condition which we are having okay so how to take care of this condition while solving this LPP by simplest method so let us see. So first it is < = 9 so we add a slack variable here s1 to make it an equation again it is < = type and we will add a slack variable s2.

To make it an equation and s_1 s_2 again must be non negative okay so how to find x_1 x_2 in this problem, so x_1 and x_2 can easily we find out s1 will be nothing but x_1 will be nothing but it is $0\lambda_0+1\lambda_1+2\lambda_2+3\lambda_3$ and x2 is nothing but $0\mu_0+\mu_1+2\mu_2+3\mu_3$. Now let us see how to solve this problem so this is all we have discussed okay so x_1 x_2 can be easily be find out x1 will be nothing but it is $0\lambda_0+1\lambda_1+2\lambda_2+3\lambda_3$ and x2 is nothing but $0\mu_0+\mu_1+2\mu_2+3\mu_3$. Now let us see how to solve this problem so this is all we have discussed okay.

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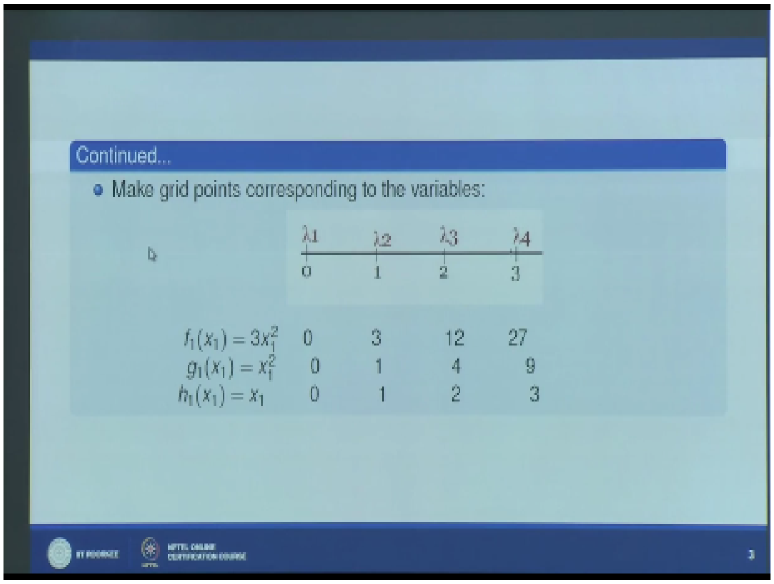

So this simply a notation λ_1, λ_2, λ_3, λ_4 here I have taken λ_0 λ_1 λ_2 λ_3 so it is not a problem these are simply notations okay. Now how to solve this problem let us see.

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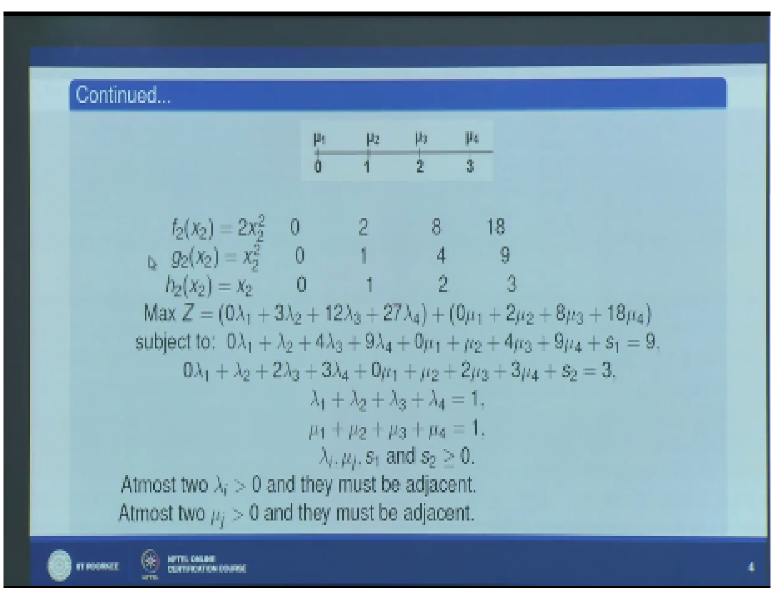

So first we will make the first table of the first simplest table we have seen that this is the LPP okay, what is the here it is λ_1 and here we are having λ_0 okay, so how to solve now S_1 or s_2 then this entering is not possible not allowed because λ is must be adjacent okay so we leave this enter this, this is the pivot element we make 0 here with the help of this – t_3 time this in the entire row, this – 9 time this in the entire row and this plus 27 time is in the entire row get the next table okay. Now which is most negative entry -18 if we enter -18 here so it is μ_4 and μ is already in the basis.

You can see if we enter this variable the minimum ratio here is 0 upon 9 0 upon 3 this is not defined and one upon one is one, so we have to leave either this variable or this variable either s_1 or s_2 and then μ_4 will come in the basis and μ_4 and μ_1 are not adjacent. So it is not possible okay so we cannot enter a μ_4 we can enter μ_4 only when μ is giving the basis but μ is not leaving because this ratio is not minimum these two are minimum. So we cannot enter this next is you can enter the next negative which is μ_3 if you enter this so again 0 upon 4 is 0, 0 upon 2 is 0 and one upon one is one so this is minimum.

So you will leave either s_1 or s_2 okay and if you leave s_1 or s_2 μ_3 is coming in the basis and μ_3 and μ_1 are not adjacent so again we cannot enter μ_3 okay. The next is μ_2 if you enter μ_2 so it is 0 upon one is 0 0 upon 1 is 0 1 upon 1 is one you have to leave one of the variable say you leave s1 if μ_2 is entering so μ is adjacent to μ_2 it is allowed okay because we have an additional condition if two λ to most two λ are greater than 0 and they must be adjacent and similarly for μ gs.

So this condition we have to take care of the entire simplest algorithm if this condition is violated so that means our approximation our linear approximation of this function not possible okay, so this enter this is pivot element we complete the entire table again making the identity corresponding to μ_2.

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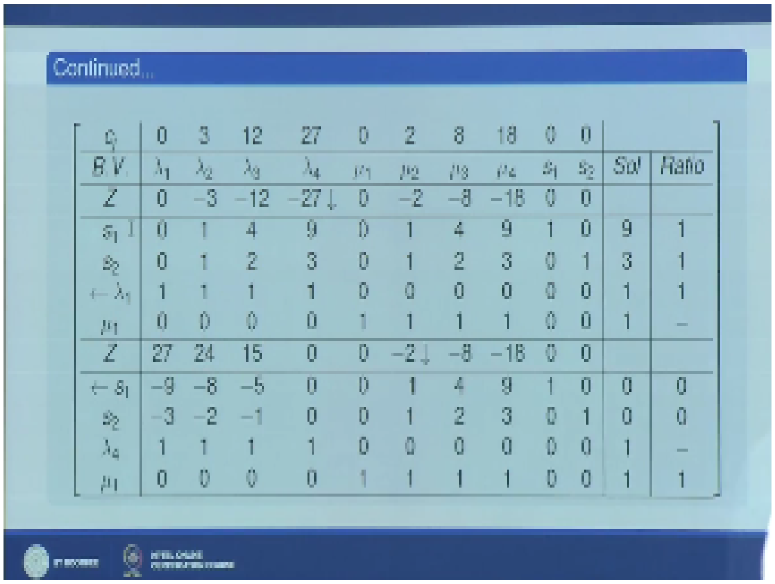
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c_j	0	3	12	27	0	2	8	18	0	0		
B.V.	λ_1	λ_2	λ_3	λ_4	μ_1	μ_2	μ_3	μ_4	s_1	s_2	Sol	Ratio
Z	0	−3	−12	−27↓	0	−2	−8	−18	0	0		
s_1	0	1	4	9	0	1	4	9	1	0	9	1
s_2	0	1	2	3	0	1	2	3	0	1	3	1
← λ_1	1	1	1	1	0	0	0	0	0	0	1	1
μ_1	0	0	0	0	1	1	1	1	0	0	1	–
Z	27	24	15	0	0	−2↓	−8	−18	0	0		
← s_1	−9	−8	−5	0	0	1	4	9	1	0	0	0
s_2	−3	−2	−1	0	0	1	2	3	0	1	0	0
λ_4	1	1	1	1	0	0	0	0	0	0	1	–
μ_1	0	0	0	0	1	1	1	1	0	0	1	1

Now in this table we get all this entries all capital Z, all this entries as positive this means all relative profits are either negative or 0 so this is in optimal table okay and what is in optimal solution optimal solution is $\lambda_1 = 0$ $s_2 = 0$ $\lambda_4 = 1$ and $\mu_1 = 1$, and when you substitute in x_1 and x_2 x_1 is this and x_2 is this when you substitute this values over here so you will get x_1 x_3 and x_2 x_0 so $x_1 = 3$ and $x_2 = 0$ are the optimal solutions and value of the function is 27.

So you can see graphically also if you see here what is the first constraint it is a circle the region inside a circle the center 00 radius 3 okay, so this is the circle.

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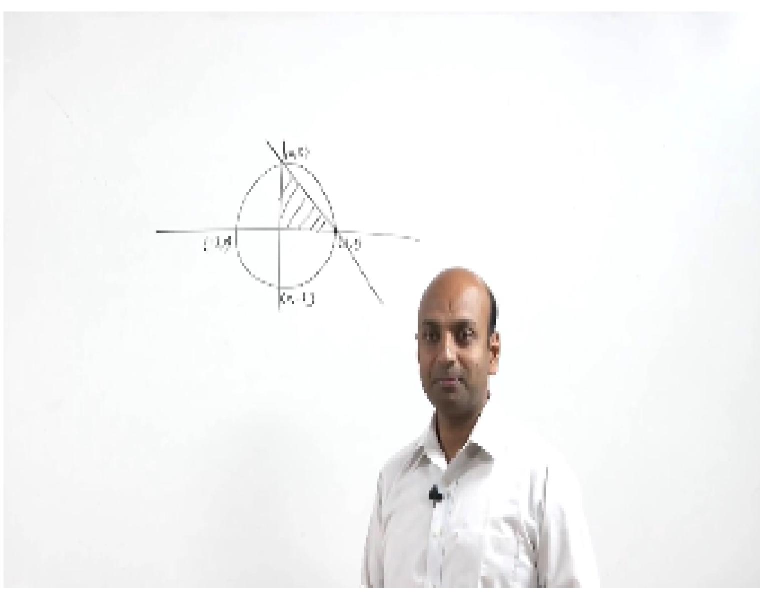

This region and $x_1 + x_2 \leq 3$ so it is passing from this two points and this is the physible issue this region is inside the circle also and $x_1 + x_2 \leq 3$ also, and x_1 x_2 non negative okay so we are obtaining optimal solution at this point 3, 0 for this problem okay x_1 x_2 x_3 is 0, so in this way we can solve such type of problems. Now suppose we have this problem again this problem is separable.

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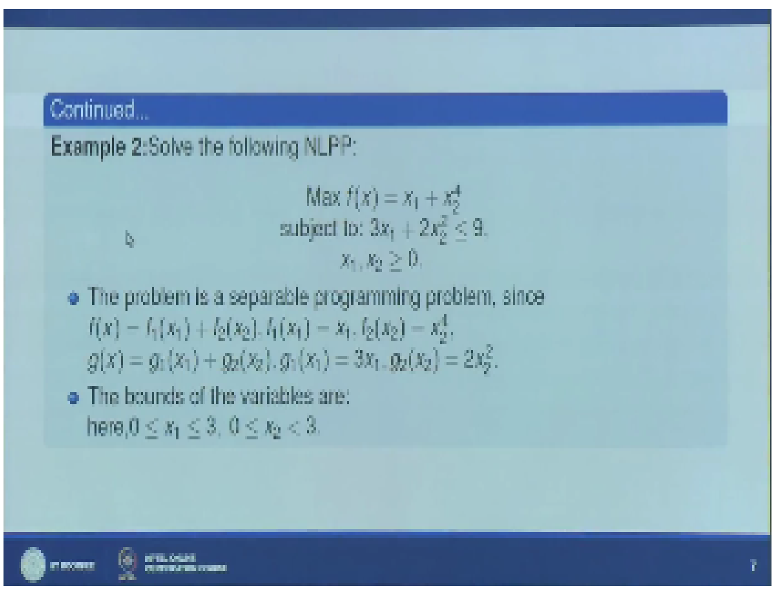

You can simply see because the first function is x_1 f_1 x_1 is x1 f_2 x_2 is x^4 g_i x_1 is $3x_1$ g_2 x_2 is $2x_2^2$ which is less than equal to 9 so this problem is clearly an separable programming problem now the important to note here is in the trail problem x1 is linear.

In the trail problem x_1 is linear so no need to make linear approximation of correspondent to x_1 because in the entire problem the objective function as well as all the constraint the variable x_1 is linear so no need to mark linear approximation of correspondent to x_1 correspondent to x_2 we have to make clear approximation because it is non linear correspondent to x_2 how to make it.

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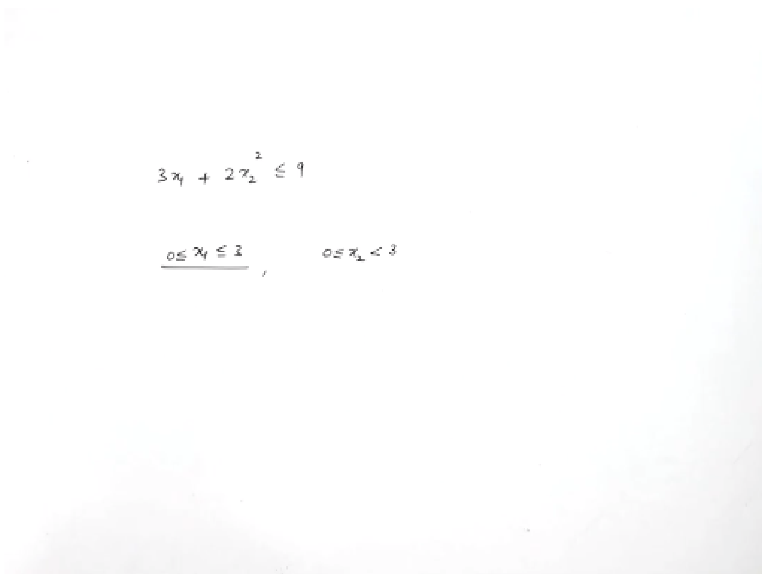

Again first we have to find bounce correspondent to x_1 x_2 now what is the constraint having it is having only one constraint that is a $3x_1^2+2x_2^4<=9$ this is the constraint or $3x_2$ is here now x1 is greater than equal to 0 it is given and the maximum value which as x1 has taken as 3, 3*3=9 which is holder in equality.

And for x_2 , x_2 must be equal to 0 now if you take x_2 as 3 so it is 3^2 that is 9 9*2 is 18 and 18 can't be less than equal to 9 so x_2 can't be 3 now if you make it 2 so 4 *2= 8 so that is the optimal bound for x_2 between 2 and 3 which is in fraction okay 2 point something 2 point something will be a upper bound for x_2 initial for taking that we are making it less than 3 okay.

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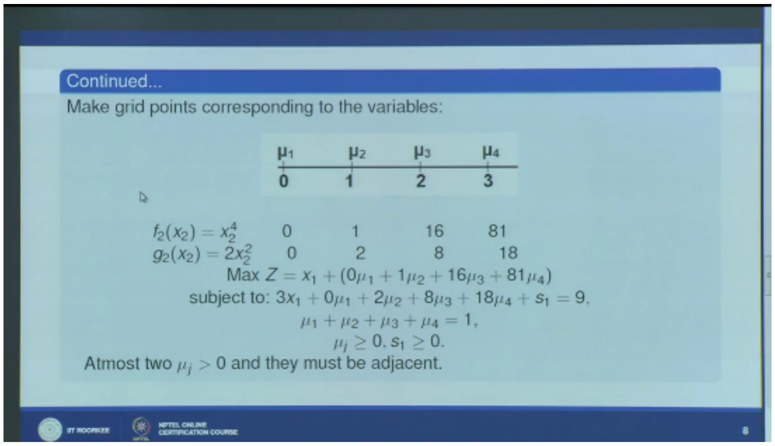

Now corresponding to x_2 only we will make the mark the grade points and make the linear approximation okay now this is the grade points 0, 1, 2,3, okay because it is less than 3 of course 3 is not there in domain of x_2 so the constraints will automatically take care of that thing because when we are putting it less than equal to 9 so constraint will automatically will not make 1,2,3, in such that it will exceed 9 okay.

So constraints will automatically take care of the feasibility of the problem so we can make f_2 x_2 has x_2^4 from the problem here it is x_2^4 and g_2 x_2 is $2x_2^2$ so you make all the functional values and constraint values we minimize the problem in the same way as we did earlier okay at most there will be an adjacent in same way.

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c_j	1	0	1	16	81	0		
B.V.	x_1	μ_1	μ_2	μ_3	μ_4	s_1	Sol	Ratio
Z	−1	0	−1	−16 ↓	−81	0	0	
s_1	3	0	2	8	18	1	9	9/8
← μ_1	0	1	1	**1**	1	0	1	1
Z	−1	16	15	0	−65 ↓	0	16	
← s_1	3	−8	−6	0	**10**	1	1	1/10
μ_3	0	1	1	1	1	0	1	1
Z	37/2	−36	−24	0	0	13/5	22.5	
μ_4	3/10	−4/5	−3/5	0	1	1/10	1/10	
μ_3	−3/10	9/5	8/5	1	0	−1/10	9/10	

We form the simplex table using the same algorithm okay and now here also you can easily see if you enter 4 here it is most negative if you enter this 4 by the minimum ratio it is 9 upon 18 that is 1/8 I mean ½ okay and it is 1/1 that is 1 so ½ which is minimum so s1 will leave not 4 is not an adjacent so this content.

If you enter 3 so by minimum ratio it is 9/8 and it is 1/1 that is 1 minimum is here so this will leave now it can enter because 1 is leaving so now 3 can enter okay so same process we will apply in the entire algorithm through so we got the optimal solution as this now one thing to be noted that it is not necessary that in the optimal solution all these entries for separable programming problems are gather greater than or equal to 0 because there is condition also that at most 2 γ or they must be an adjacent so it may be possible so all this may not be greater than or equal to 0 okay.

So this is the optimum solution that we can obtained, now some problems which appear not to be separable but they can ne separable, they can make separable how let us see. We have the 1st problem it is minimizing of this function; function is you can easily see that the function is separable because you can make f_1 x_1 as in the 1st problem x_1^2 okay f_2 x_2 s.

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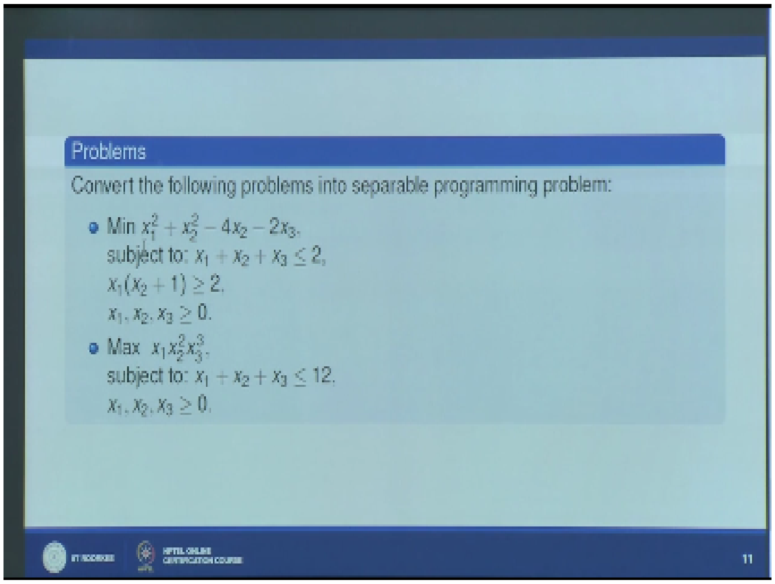

$X_2{}^2 - 4x^2$ and f_3 x^3 as $- 2x^3$, so it will be sum of 3 functions f_1 $x_1 +$ f_2 $x_2 +$ f_3 x_3 the 1st constraint is also separable, you can easily see it is $x_1 + x_2 + x_3 \leq 2$ so g_1 s_1 is x_1, g_2 s_2 is x_2, g_3 s_3 is x_3, the next constraint.

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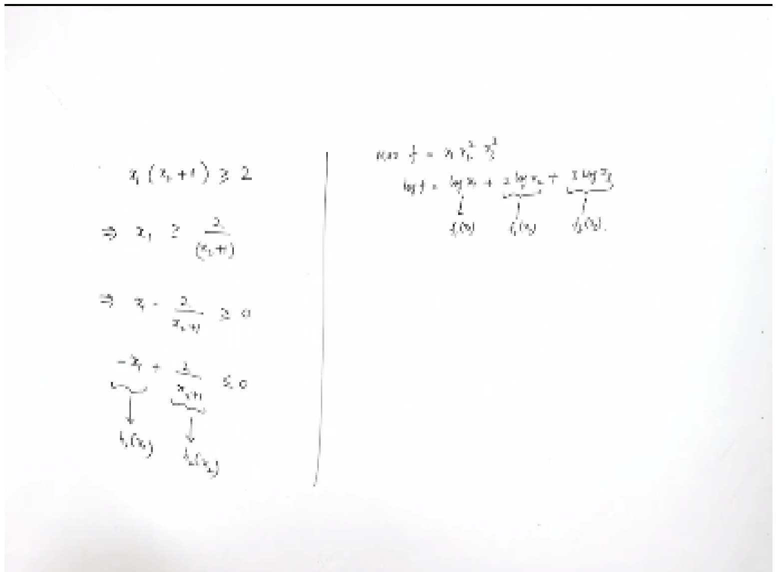

It is not separable because it involve product of x_1 x_2 it is not separable, now it can make separable how, let us see, now this quantity because $x_2 \geq 0$ so $x_2 + \geq 1$ okay so this will be ≥ 2 upon on $x_2 + \geq 1$ now this quantity can never be 0 because it is ≥ 1 x_2 ≥ 0, so we can take it in the denominator and we will not change the sign, so this implies $x_1 – 2/ x_2 +1 \geq 0$ or $– x_1 + 2/ x_2 \leq 0$. So now it becomes separable.

Now it is x1and it is s_2 x_2 now it is separable, which is otherwise not appearing as a separable programming problem. So some problems we have to make it separable we can make it separable and obtain the technique to solve it. Now suppose our next problem, now how can you make it separable, the objective function is maximizing f= x1 x2 $x3^3$. The constraint is obviously separable as a linger constraint it is obviously separable and other is the non negative restriction. The only thing is this is not separable, so how to make separable.

Now you take the log on both the sides' log of x1 + 2log x2 + 3 log x 3, which is now separable, maximizing f as same as maximizing log of f okay, so we can simply maximizing this function okay, with the same of maximizing log of f, so now it is separable this is f1 x1 it is f2 x2 and f3 x3. Now f2 x2 and f3 cannot be 0 because if it is zero, so maximum will be 0 okay, maximum cannot be 0 because x1 + x2 $\leq$ 2 the constraint is there okay.

So maximum is not 0, so that is why log of this is defined okay because variables are 0 as we have already seen in the feasibility condition. So in this way you can solve separable programming problems and we have also seen that there are some problem which is not separable but can be made separable so thank you very much.

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