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Nonlinear Programming - 1

Lec – 17
Dynamic Programming - IV

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Hello friends so welcome to a lecture series of non linear programming, we were discussing dynamic programming what dynamic programming are what are their applications, in the last 1 to lectures we have seen that how can we solve a problem of shortest path using dynamic programming how can we solve allocation problem using dynamic programming and few more examples based on it.

Now dynamic programming is also useful to solve some non linear programming problems, how can we solve a non linear programming problem let us see this in this lecture okay.

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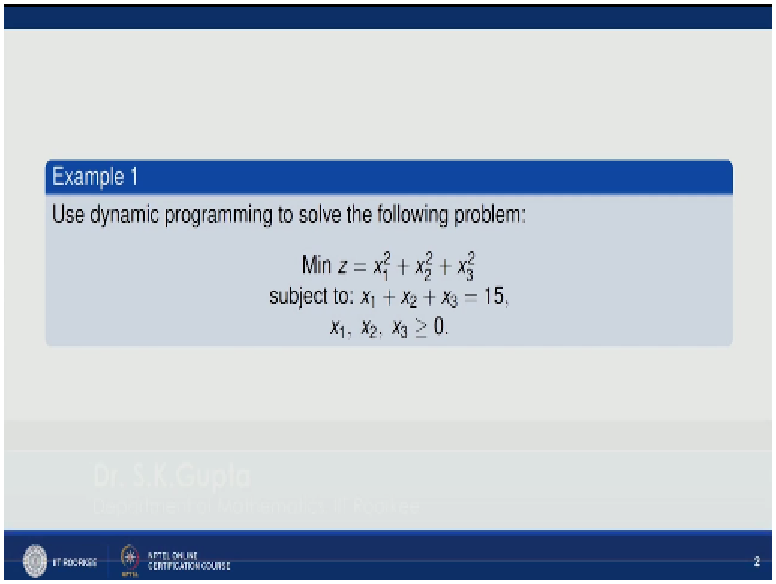

Now take of first example use dynamic programming to solve the following problem it is a very simple problem you seemed objective function is quadratic that is minimum of $x_1^2 + x_2^2 + x_3^2$ subject to $x_1 + x_2 + x_3 = 15$ and x_1, x_2, x_3 all are non negative, so how can we solve the simple problem the simple quadratic problem in fact using dynamic approach using dynamic programming approach, so let us see let us start to solve this problem.

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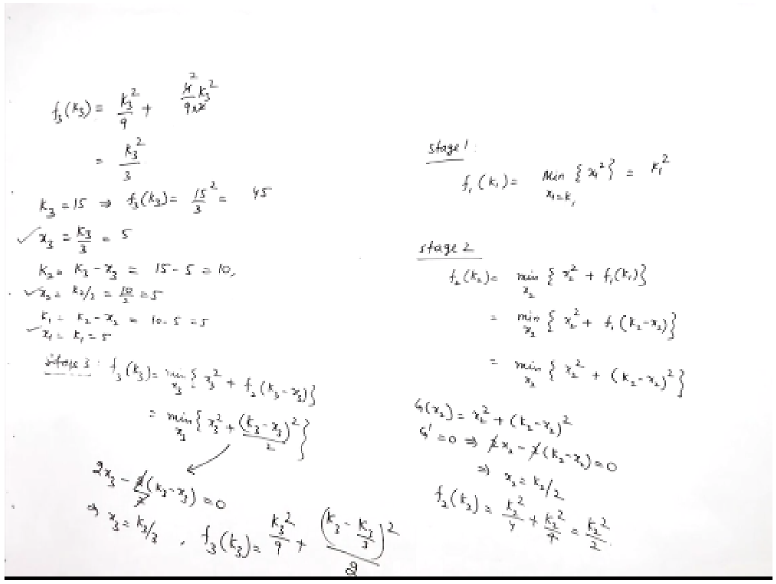

So it is a simple problem minimizing $z = x_1^2 + x_2^2 + x_3^2$ subject to $x_1 + x_2 + x_3 = 15$ and $x_1 + x_2 + x_3$ all are non negative okay so we first mark out some state variables okay how many variables we are having here 3 variables we are having so we will mark 3 state variables so let us suppose k1, k2 and k3 are the state variable, so state variables we mark the state variables say k_3, k_3 we take as $x_1 + x_2 + x_3$ which is equals to 15 k_2 take as $x_1 + x_2$ which = to from here from this expression it is = x_3 – k3 - x_3.

And then k1 which is x_1 which is = from here it is k2 – x_2 so these are the state variables which we let okay basically what we do we first we first take this objective function my x_1^2 okay subject to $k_1 = x_1$ then we taken in the stage we take these two objective functions I mean x_2^2 + whatever we obtain in the first previous stage that is the dynamic approach okay and the last stage that is the stage 3 we take x_3^2 + whatever we obtain in the previous stage okay.

So we will start form stage say stage1 now stage1 is f_1 (k_1) is the net output from stage 1 okay net maximum value which we obtain from the stage 1 that is since the maximization problem so it is minimization of 18 walls only one objective x_1^2 so it is x_1^2 subject to x, so minimum of x_1^2 when x_1 = k1 is k1^2 only so it is k1^2 okay that is first a stage okay in this approach we are moving in a forward direction, we first take this stage then this stage and with the combined stages stage 3 okay it is over choice whether we take backward approach backward recursion or forward direction here I am taking forward direction now stage 2 in stage 2 we take f_2 k_2 what is f_2 k_2 it is minimum of $x2^2$, $x2^2$ is this objective function plus whatever we obtained the previous stage plus f_1 k_1.

Where minimum more x_2 and that is equals to minimum of x_2 of $x2^2$ + f1 and what is k1, k1 is k_2 – x_2 so it is k_2 – x_2 okay k1 is k_2 – x_2 and it is minimum of again minimum of $x2^2$ + now f1 k1 is simply $k1^2$ so f1 k_2 – x_2 will be k_2 – x_2 whole square we simply replace k1 / k2 – x2 so if we replace k_1 / k_2 – x_2 so it will be k_2 – x_2 whole square now we have to minimize this function this function okay and it is only one variable x_2, so we can use the derivative the concept of derivative were simply differentiate this function.

Okay and put it equal to 0 so derivative of this function say it is gx^2, gx^2 is $x2^2$ + k_2 – x_2 whole square so derivative of this function put it equal to 0, so this implies $2x^2$ – 2 k2 – x2 = 0 and this implies to cancels out those this implies x_2 will be k_2 / 2, okay and what will be f_2 k_2 so f_2 will be = we simply substitute x_2 = k_2 / 2 here so it is $k2^2$ / 4 + $k2^2$ / 4 which is $k2^2$ / 4 so this up to this stage we have find out the minimum value of this function, now stage 3 which is a last stage the stage3.

The stage 3 will be f1 f_3 k_3 which is minimum of minimum over x_3 oka , and $x3^2$ = whatever we obtained the previous stage that is f_2 k_2 and k_2 is nothing but k_3 – x_3 so it is plus f_2 of k_3 – x_3 because this objective function I means this function x32 = whatever we obtained the previous stage and stage is f_2 k_2 and k_2 is k_3 – x3 so f_2 of k_3 – x_3 and this further equals to minimum of $x3^2$ + now f2(k2) is $k2^2$ so that means so f2(k3 – x3) simply replace k2/k3 – x3.

So this will be equal to $(k3 – x3)^2$ /2,now again excess it is a one variable problem so we simply differentiate it with respect to x3 and find out the minimum value of this function, so we simply differentiate this with respect to x3 so it is2x3 + -2(k3 – x3)/2 = 0,so 2, 2 cancels out and this implies x3 will be equals to k3/3.

And what will be f3(k3)? For f3(k3) you simply substitute x3 = k3/3 over here so what we obtained, we obtained $k3^2$/9 + when you substitute x3 = k3/3 so it is $(k3 – k3/3)^2$ /2 so that we can easily solve it is f3(k3) will be equals to $k3^2$/9 + it is 4/9 $k3^2$/2 that is it is $k3^2$/3 okay, now what is k3,so this is the maximum value of the entire objective function is it, minimum value of this objective functions set, okay. Now what is k3?

K3 is 15, okay so this implies f3(k3) will be $15^2/3$ which is 45 so this is a maximum value of the objective function, sorry minimum value of the objective function now at which points for what are the values of x1 and x2 and x3 so x3 is k3/3 and k3 is 15 so it is 5 okay, now x3 is 5 now k2 is k3 – x3 okay, k2 is k3 – x3 okay so k3 is 15 – 5 that is 10 so what will be x2, x2 is k2/2 okay so it is k2/2, 10/2 that is 5.

So x3 is 5, x2 is 5 now k1 is k2 – x2 that is 10 – 5 = 5 and it is at x1, x1 = k1 so x1 =k1 so that means equals to 5, so that means minimum is obtained when all x1,x2,x3 are equal and equal to 5 so that is how this simple illustration that how can we solve some non linear optimization problems using dynamic programming approach we simply first defines some state variables state variables k1, k2,k3 depending on the number of variables in all in the problem, okay.

And then we find out the stages, stage 1 stage 2 and so on depending on again depending on the number of variables in all in the problem okay, and then we find out the stages stage 1, stage 2 and so on depending again depending on the number of variables okay, suppose there are n variables involved in the problem so there will be n stages and the final stage will give the optimal solution of the problem okay, so this PPT shows solution also.

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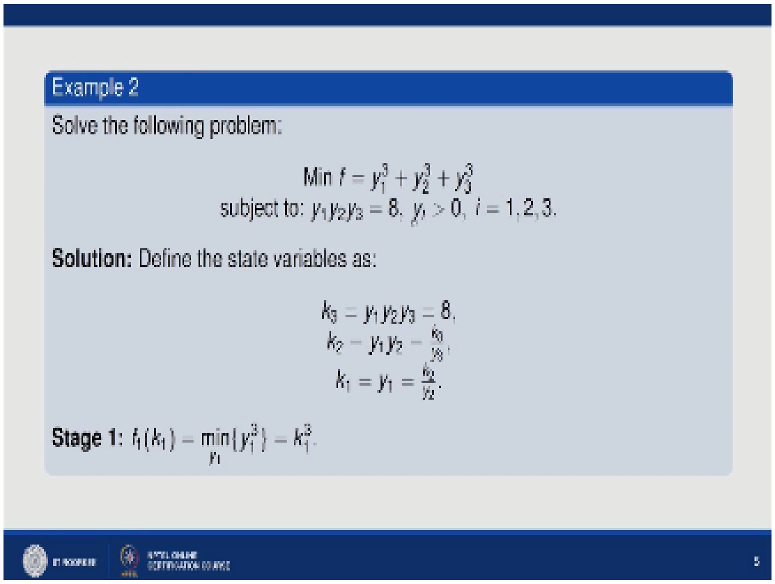

Now the second example again a simple example it is minimum of $f=y1^3+y2^3+y3^3$ subject to y1.y2.y3=8 and y2 >0 strictly greater than 0 for all i 1,2,3 of course y1 cannot be 0 because if y1=0 so the product cannot be equal to 8, if any one of the y either y1,y2 or y3 become 0 so the product cannot be equal to 8 and we need the product equal to 8 so yi cannot be 0, so we are assuming y is strictly greater than 0 over here.

Now again it is a 3 variable problem so we define state variables k1, k2 and k3, so first we define k3, k3 is simply y1 here instead of addition constrained we are having constrained in a multiplicative form y1, y2 and y3 multiplication involved, so we take as first state variable k3 as y1.y2.y3 which is equals to 8, k2 we take as y1.y2 only these two variables okay, which is equal to from the first condition y1.y2 is k3/y3 this value okay.

And k1 is only this y1 which is equals to from this condition again from the second condition y1 is k2 up on y2, so in this way we have defined 3 state variables k2,k2 and k1 okay. Now the stage 1, again the stage 1 we take minimum of the first component of the objective function that is $y1^3$, in the second stage we take $y2^3$+ whatever we obtain the previous stage that is stage 1 okay, it is a addition here.
If it is the objective function also multiplication is involved then we multiply okay, in the stages. So in the first stage it is f1k1 which is equals to minimum of y1, $y1^3$ is here and y1 is equals to k1 so minimum is obtained only at k1 so it is $k1^3$.

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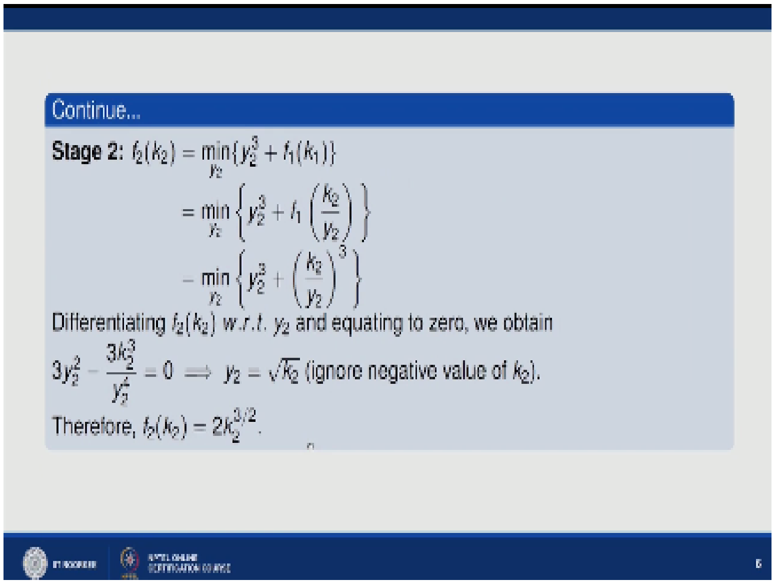

Now in the second stage here second component is $y2^3$ okay, and whatever obtained from the first stage that is f1k1 and k1 is from here k1=k1/y2 so in the next stage okay, on the next stage it is $y2^3$+f1 of k1 is k2/y2 okay, and f1/k1 yeah from here $f1k1=k1^3$ so here k2/y2 $f1(k2/y2)=(k2/y2)^3$. Now it is again a single variable function of y2 you differentiate it with respect to y2 put it equal to 2 then you can get the stationery point from which this value is minimum okay so you different respect to xy_2 put it equal to 0 we can get y_2 under root of k_2 ignore the negative value because y_1 and y_2 is 0 oaky so f_2 k_2 from here we obtain f2 k2 is equal to when you substitute y_2 over here in this expression we obtained f_2 2 equals to 2 times k_2.

So now in next stage again we take y_3^3+ whatever we obtain in previous expression that is f_2 k_2 and f_2 k_2 if you see here f_2 k_2 if you see here k_2 is equals to k_3 upon y3 so simply replace k_2/k_3 upon y_3 and as f_2 k_2 is 2 times 3/2 so you simply replace $2*k_3$ upon y3 again it is a single variable function of k3 of y_3 it differentiated put it equal to 0 find out the stationery point which is equal to minimum value of this objective function.

So in this way we get $y_3=k_3$ 1/3 and minimum value of this function now we note that k3=8 so substitute k_3 is here y_2 equal to 2 has k_3 upon y3 upon k_3 and y_3 is 8 upon 2 that is 4 so this implies y_2 is 2 why y_2 is 2 because y2 is under root of k_2 okay you substitute 4 here so y_2 is 2 and similarly k_1 is k_2 upon y_2 so 4 upon 2 is to y_1 is 2 because y_1 is equal to k_1.

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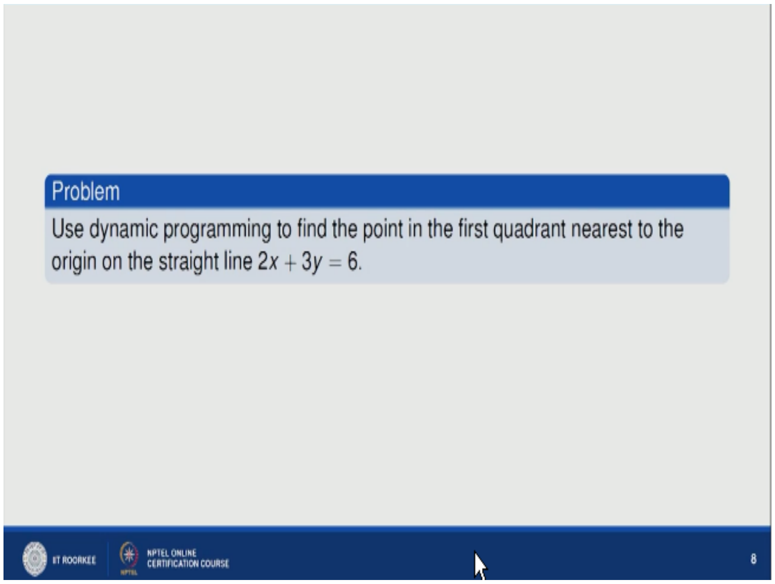

So in this way that means the optimal value of this objective function is obtained at when y1 = y2=y3=2 okay now let us come back to this problem now let us come to this problem using dynamic programming how to find point in the first quadrant nearest to the origin on the straight is here so let us try to find out the solution of this problem so what is the problem, problem is to find out the point which lie in the first quadrant and nearest to the straight line this.

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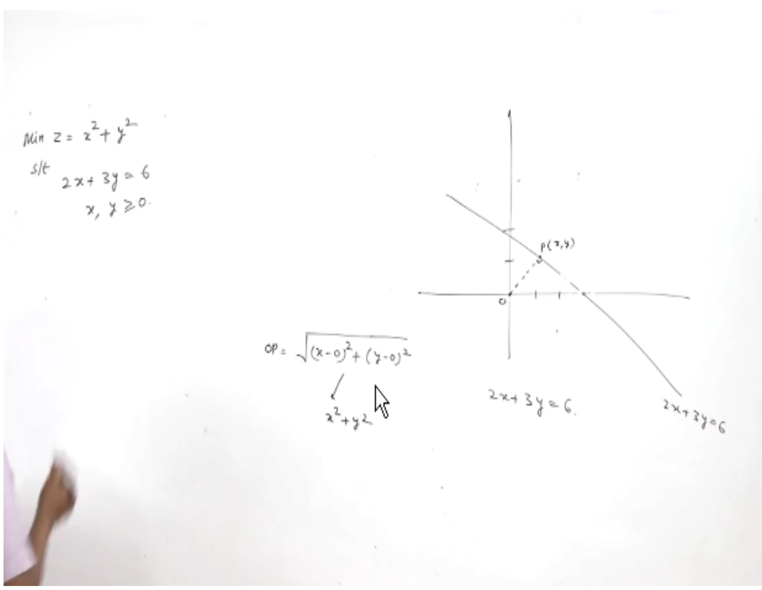

So what is the straight line, a straight line is 2x+3y=6 that is 1, 2, 3 and 1,2 so this is straight line okay it is 2x+3y=6 so we have to find out the point on this straight line which is nearest to the origin okay so of course that point will be the foot of the perpendicular of origin if you draw our foot of the perpendicular form this point on this line will be a point on this line which is shortest we can easily find out this point this point P which is shortest distance of origin on this line using quadrant also we can simply find out the equation of this straight line okay and point of intersection of the straight line with the straight line gives the pint P.

So subject to this condition and we want point should be in first quadrant that means x y both should be non negative okay so we have to minimize this is same as minimization of this is same as minimization of square of this value minimization of this quantity is same as minimization of inside the value inside the under root.

So simply minimize this expression okay so what is the problem, problem is minimization of z which is x^2+y^2 and subject to this point must lie on the line that is 2x+3y must be equal to 6and x y must be non negative because point should lies in the first quadrant okay so this is the problem this is the formulation of problem.

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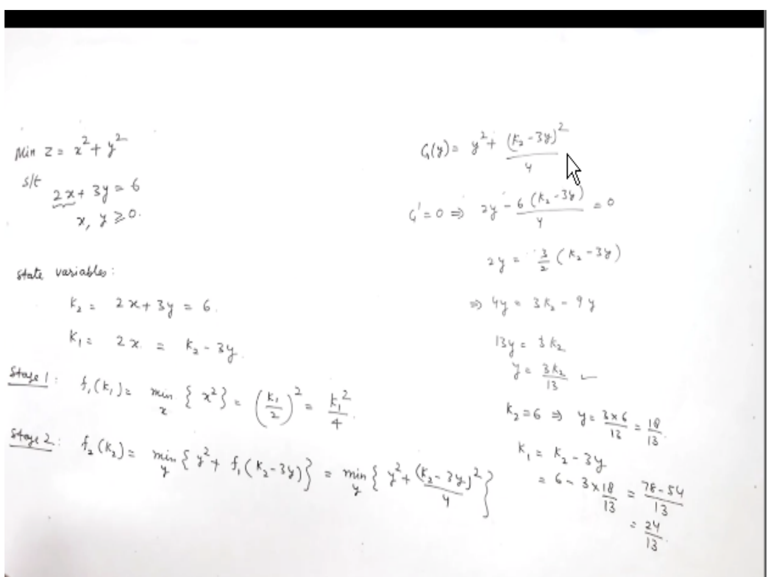

Now how you can solve this problem using dynamic programming again we first differentiate variables okay so state variables are first is k_2 only two variables are here x and y so we will be having two status k_2 and k_1 okay so k_2 will be 2x+3y which is equal to 6 and k_1 will be equals to 2x okay so this 2x and this 2x is equals to k_2- 3y okay from this equation.

Now start from the stage 1 stage 1 is f_1 k_1, f_1 k_1 is minimum of the first value that is x^2 and this is the 2x=k_1 I mean over x and what is x, x is k_1 by 2 okay so minimum will be $k_1^2/4$ now come to next stage or last stage because it is only two variable problem must be simple illustration to solve such type of problem using dynamic programming approach okay.

So it is f_2 k_2 which is minimum of y^2+whatever we obtain in the previous stage that is f_1 k_1 and k_1 is k_2-3y so it is k_2-3y okay and it is further equals to minimum of y^2+now f_1 k_1 is $k_1^2/4$ so f_1 of this quantity will be this square by 4 that is k2 – 3y/ 4^2 okay, now again it is minimum or y okay, now it is the single variable function of y only, you again differentiate this to y and put it = 0 to find out the minimal value of y, minimum value of this objective function sorry okay, so differentiate it with respect to y that will give.

Suppose Gy is y^2+ k2 – 3y whole square by 4 derivative of y put it = 0 so this implies 2y and – it is 6 times k2 – 3y/ 4 and put it = 0 okay so that will be, 2y = 3/2 k2 – 3y, this implies 4y = 3k2 – 9y and it is 13 y = 3k2 and y = 3k2 / 13 oaky, this is y, if you take the 2nd derivative of this quantity, so 2nd derivative will be negative you can check okay, now y is this. so what will be now k2 is 6 okay, so this implies y = 3x 6/13 which is 18 upon 13 and now k1 is k2 – 3y so it is 6 – 3 x y is 18 up on 3.

Now it is = that is 78 – 54 up on 13 okay that is = 24 up on 13, so that will be k1 and what will be x? x is k up on 2, so x will k up on 2 that is 12 / 13, so the optimal point is 12/ 13 and 18/13 so this is the point which is at the nearest distance from origin and the straight line this, so that is how we can solve this type of problem using dynamic programming. So these are some simple illustrations of dynamic programming that how we can solve few non linear programming problem using dynamic programming approach. So in the next lecture we will see how we can solve non linear problem by using 1st techniques, so that will be our next lecture so thank you very much.

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