Linear Algebra
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Lecture – 10
Generating system, linear independence and bases

Come back to the second half of this lecture.

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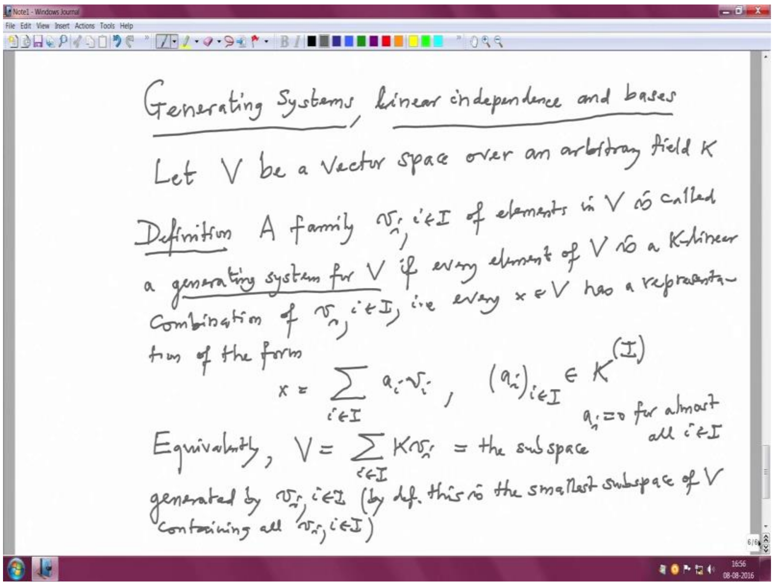

This time I am going to start a new section on generating systems, linear independence and bases. So, as usual let V be a vector space over an arbitrary field K. Definition - a family small v i, i in I of elements in V is called a generating system for V if every element of V is a k linear combination of v i s so that means, every x in V, we can write as representation of the form x equal to summation a i v i, i in I and only K-linear coefficients are nonzero. So, that means, in notation from our clear lectures this i tuple a i belongs to k power round bracket I. We call that only those i tuple belong to k power round bracket I or which a i is equal to 0 for almost all i so that means, this sum will really finite sum.

Equivalently V is K v i. Remember from one of the earlier lecture this is the notation the right hand side is the notation for the sub space generated by the family v i; and by definition this is or by definition this is the smallest subspace of v containing all the elements v i. So, if any time, one wants to check that somebody the generating system for V, we need to check that V is the only subspace which contain all this guys.

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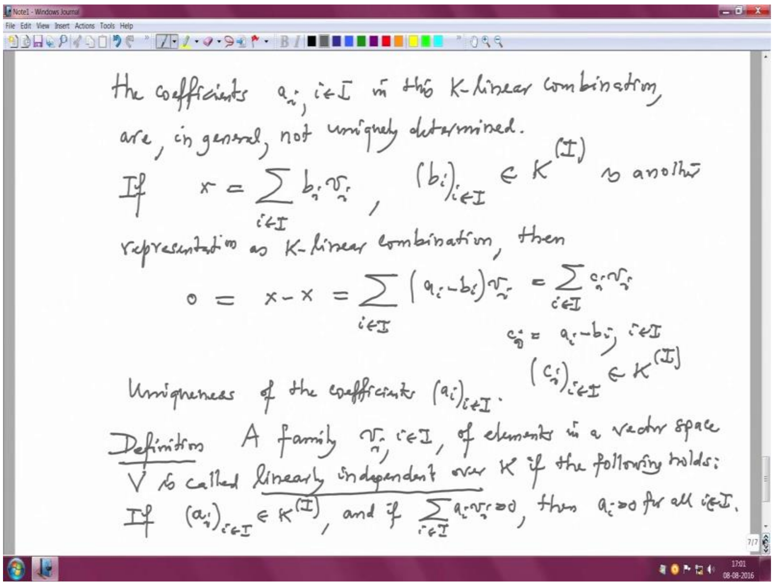

So, the coefficients in this representation, the coefficients a i's in this linear combination K-linear combination, are, in general, not uniquely determined. For example, if x are the another representation submission b i v i where the tuple b i is in K power I - round bracket I is another representation as K-linear combination, then 0 will be equal to x minus x, and when we collect the coefficients you can rewrite the difference as i in I, a i minus b i v i, which will be equal to c i v i. I am just calling c i equal to a i minus b i i in I, and this will also be c i will also b in k power round bracket I.

If c has a nonzero then you get a representation of 0 as a linear combination of v i nontrivial. So, if at all there is a different representation then it leads to the following new definition. So, the uniqueness of the coefficients a i demand the following condition on the family v i. What condition, so that is let me put it in a definition form. So, definition a family v i, i in I of elements in a vector space V is called linearly independent over K if the following holds. If a i is an i tuple which is in K power round bracket I that is almost all a i's are 0, and if the linear combination a i v i is 0, then a i should be actually 0 tuple a i equal to 0 for all i in I. If the family v i satisfy this property which we call it linear independence over a, then the uniqueness will automatically follow.

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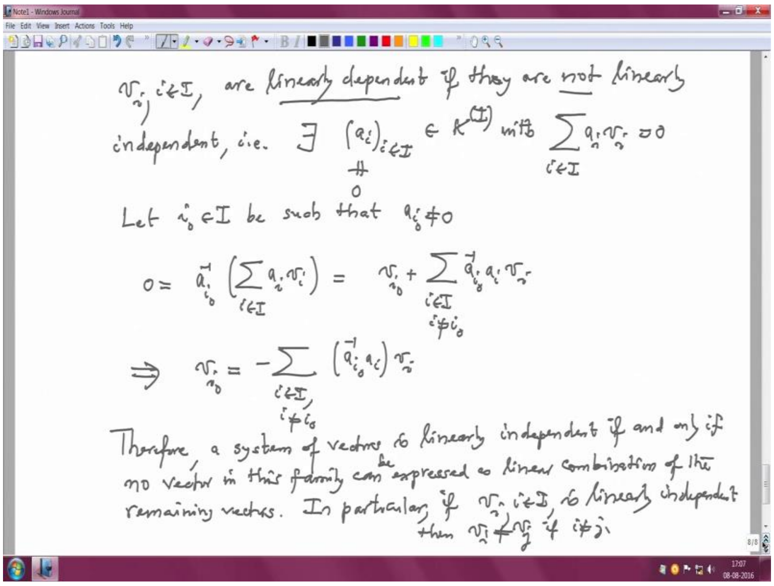

We will say that family v i, i in I are linearly independent, if they are not linearly independent. So, linearly dependent means so that is there exist an i tuple a i and K power round bracket I with the combination linear combination a i v i with 0, while there exist a nonzero, this is nonzero tuple, tuple is nonzero means at least one of the component is nonzero. So, if such a thing exist then let us take let i naught v in I is such that a i naught is nonzero, then you multiply you solve this equation. So, you keep multiply this by a i naught inverse, this is also 0, because this sum was 0 and we have multiplied this sum by a i naught inverse a i naught inverse exist because a i naught 0 is nonzero element in the field, and therefore, inverse exist.

And this one is now i can write it as v i naught plus the remaining guys, a i naught inverse a i v i this is i in I and i different for i 0. And I will transfer this sum to the other side. So, then so that will imply v i naught equal to submission i in I i naught equal to I naught coefficients are now a i naught inverse a i v i so that means, the element i naught element in this family is a linear combination of the remaining ones. So that means, in a generating set we do not need these v in I. So, this says that. Therefore, I will write a consequence therefore, a system of vectors sometimes I call it vectors; sometime I call it elements in a vector space. System of vectors is linearly independent if and only if no vector in this family can be expressed as linear combination of the remaining family or the remaining vectors. In particular, if v i family is linearly independent then v i cannot be equal to v j if i is different from j.

So, in linear this will linearly independent family no vector is repeated.

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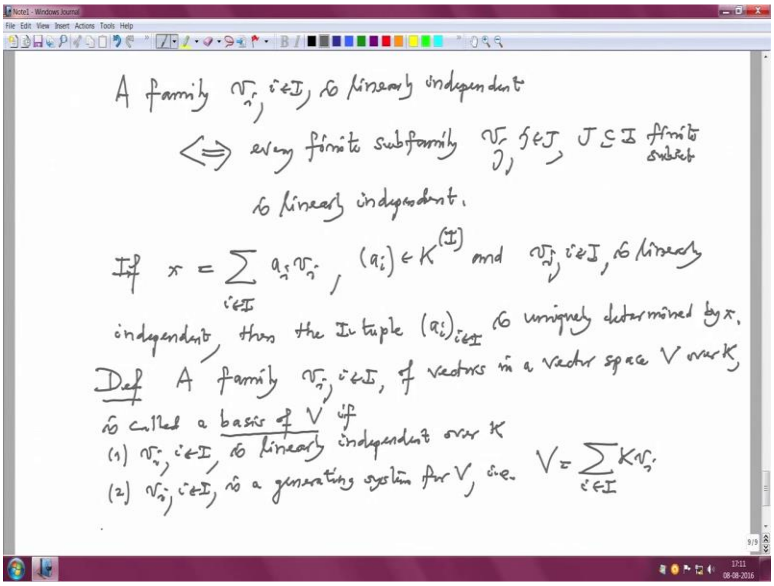

Also in the trivial remark a family v i, i in I, is linearly independent if and only if every finites subfamily v j j in J, J is the finite substitute of it. This finite family is linearly independent. So, now going back to the earlier remark, if x is a linear combination a i v i, i in I, a i in k power round bracket I, and the family v i is linearly independent then the tuple a i is uniquely determined by x. So, this leads to the following definition.

So, definition, a family v i, i in I of vectors in a vector space V over K is called a bases of v if one v i, i in I, is linearly independent over K. Over K is sometimes I keep it when K is a fixed field and there is no way to confuse it, but it is a better habit to write always. And two, v i, i in I is the generating system for V, that is V equal to the sum of the subspaces which means every element of V has a representation of a K-linear combination among of v i is not x i, v i.

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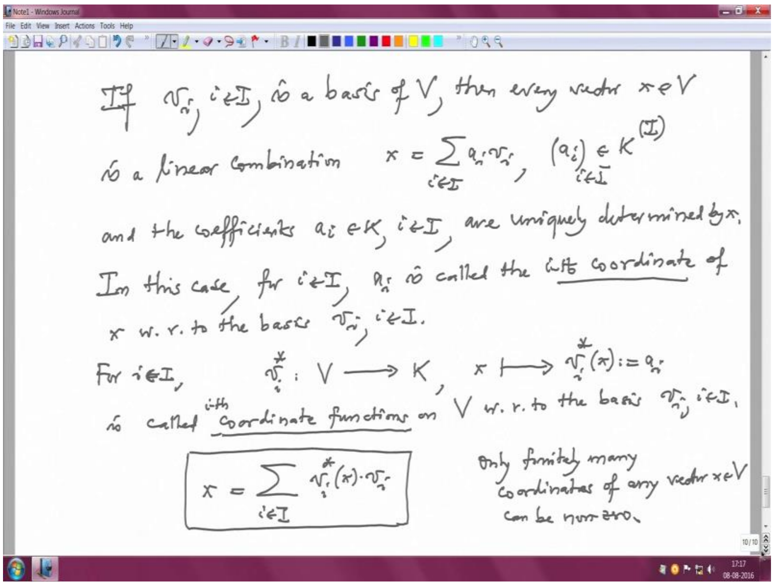

So, if v i is a basis, so I can do it in next page. If v i, i in I is a basis of V, then every vector x in V is a linear combination x equal to a i v i a i in the i tuple in the K power round bracket I and the coefficients a i which are relevant in the field are uniquely determined by x. They are only finite remember they are only finitely many as could be nonzero. So, in this case, for i in I, a i is called the i eth coordinate of x with respect to the bases v i. Sometimes these are also called the i eth components, but I would use i eth coordinate, so that means so to put it in a different way.

So, for each i in I you get the functions from V to K, these I will denote these functions depend on this given bases. So, I will write it v i star. This function is defined arbitrary vector x in V goes to v i star x that is by definition it is a i; this makes sense because a i is are uniquely determined. So, this functions are called coordinate functions on V with respect to the basis v i. Of course, we have noted this no I should clean up a little bit, this is only one for each i, v i there this is called i eth coordination function.

So, we have as many coordinate functions as there are basis elements. Of course, I have not did address the question whether a arbitrary vector space has a basis or not. This is under the assumption that if it has a basis then this definition makes sense. So, in the next two lectures, I will also be concerned in general question, if V has a basis or not and if there are two bases of V then what is the relation between their cardinalities. And we will prove that any two bases of v are the same cardinality, and this common cardinality will be called the debentures, but before I do that I want to get used to this function etcetera so on. So, therefore, in this notation each vector x, we can write it as linear combination of a i's and this v i star x v i.

Note that only finitely many coordinate of any vector x in V can be nonzero. So, the only difference between these and these, I just showed the value of that these a i x i have replaced by v i x. So, the reason being you see the right hand side is also now it involves only a bases and x. When you look at this sum, you one may wonder what is a i, how do we calculate a i, but if you write this expression then it is clear from the way of writing that it is depends on x and the v i.

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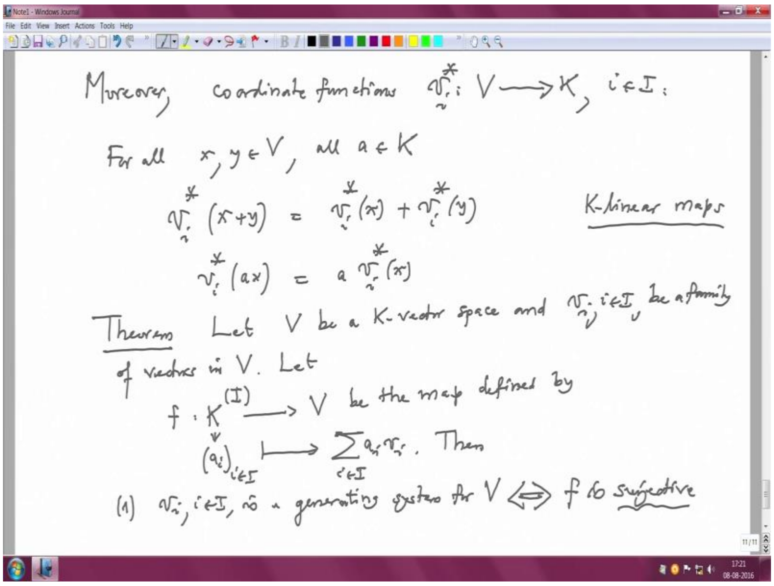

So, moreover this functions v i coordinate functions v i star, these are functions on V with values in K and there as many as the number of bases elements, they are numbered by the same set. And they are may respect addition of V and scalar multiplications, so that is for all x and y vectors in V all scalar a in K. These two equations v i star x plus y equal to v i star x plus v i star y, and v i star a x equal to a times v i star x.

These two equations hold that means, this v the first means v is an abelian group homomorphism, v plus is an abelian group K plus is also an abelian group and this first equation means its abelian group homomorphism. And second equation means that it respects the scalar multiplication; that means, whether I multiply by a scalar and apply a coordinate function or I multiply scalar later after applying the coordinate function these two are same.

So, these in general these are called K-linear maps, but overall I just now I do not want to get into this because I still have to deduce some more consequences from the definition of a bases etcetera. So, I will come back to these one of two lectures little late, but right now I want to note what we have observed in the form of theorem. So, theorem, let V be a vector space, V be a K vector space, and let v i be a family of vectors in V. I want to translate all the definitions what we have defined so far generating system linear independence and a bases in terms of the maps.

So, let from K power I round bracket I to V be the map defined by any i tuple here, who has only finitely many nonzero entries, map these two to the combination a i v i. This makes sense this is the finite sum because this is an element here, and its each tuple will uniquely determine this combination. Then these are the statements I am writing they are merely the translations of the definitions that we have above; number one v i, i in I is a generating system for V if and only if the map f is subjective. So, subjective, let me just recall. So, subjective means every element of V is in the image of this map I call this map as f that was indeed the definition of a generating system.

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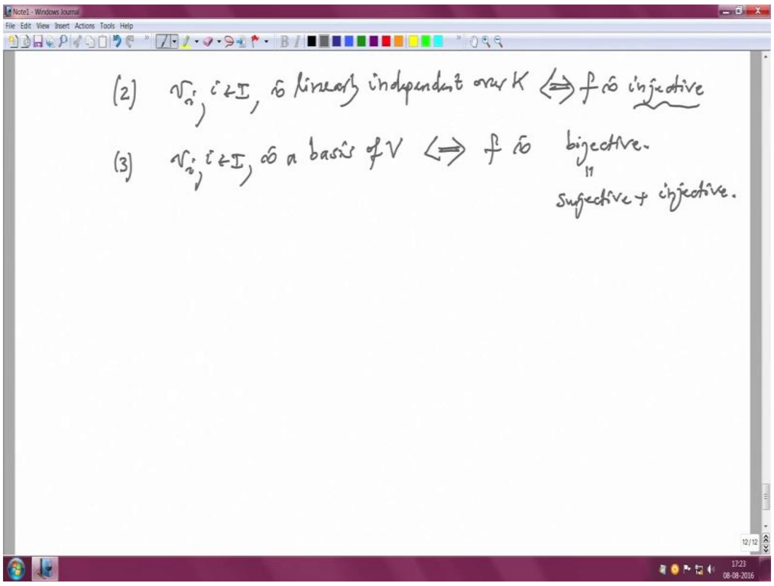

Second one v i, i in I is linearly independent over K if and only if f is injective; injective means two different elements they will have different images. So, this is also clear because a linearly independent family and vector cannot have two different linear combinations the coefficients are uniquely determined. Third one, combine one and two this side, so v i, i in I, is a bases of v if and only if f is a bijective. Bijective means injective and subjective, subjective and injective. So, these are merely the restatements of the definitions. So, from next time, I want to consider many examples which will make this concept more and more clear.

So, thank you till next time.