Linear Algebra

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Lecture – 11
Examples of a basis of a vector space

So, let us continue our next lecture.

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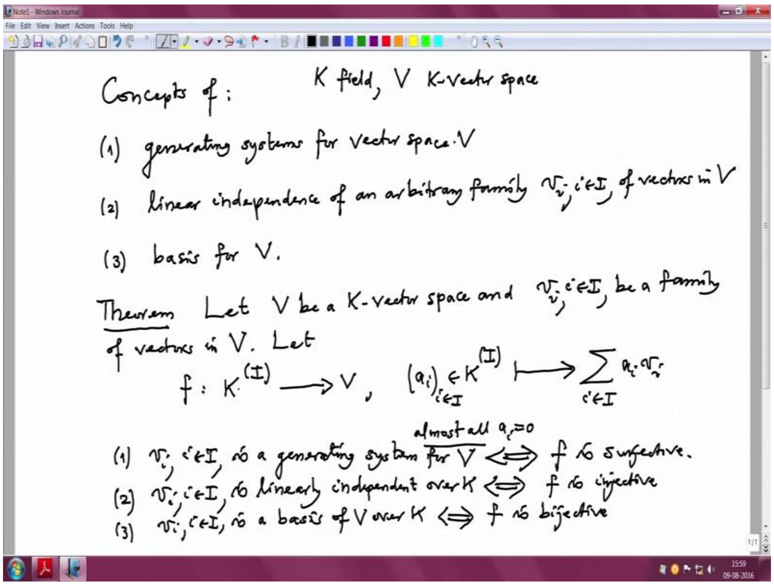

Let me just recall last time we have defined concepts of; 3 concepts we have defined last time; one generating systems for vector spaces. Let me also fixed a notation today also like yesterday that K be always field arbitrary and V K vector space and last time we have defined this 3 concepts generating system for vector space V 2 linear independence of an arbitrary family V i i in I of vectors is V and 3 basis for V and the most important thing we will keep using is the following theorem what we stated last time that. So, I will write again completely because this is the most important statement for this subsection let V be a vector space and V i i in I be a family of vectors in V, let f be a map from K power round bracket I to V defined by any tuple a i i in I which is in K power round bracket I goes to the linear combination of this a is a i V i.

So, first of all, let me remind you that when we write K power round bracket I we consider those tuples for which almost all a i is are 0; almost all means all, but finitely many tuples are components are 0 and therefore, this sum when we write it make sense because in this sound really only finitely many a i's are nonzero. So, some make sense. So, we have a map from K power I to V then we have translated this concepts in terms of this map. So, one V i i in I is a generating system for V if and only if the map f is surjective this is simply a restatement of the definition of a generating system because surjectivity means every element X in V is coming from some tuple; that means, every element takes as a representation of linear combinations in terms of V x.

Second V I; the family V i is linearly independent over K if and only if f is injective, this is also immediate from the definition of linear independence because linear independent means if a linear combination is 0 then each coefficient is 0. So, if some tuple goes to 0 then the tuple it is a; it should be 0 and tuple 0 means all components are 0; third one now combining 1 and 2 by definition basis is a generating system which is linearly independent.

So, V i i in I is a basis of V over K if and only if f is injective and surjective; that means, f is bijective and most of the time we will use this theorem and beside that this will also given later on way we consider isomorphism etcetera it will tell us that every vector space V is a isomorphic 2 on of this type.

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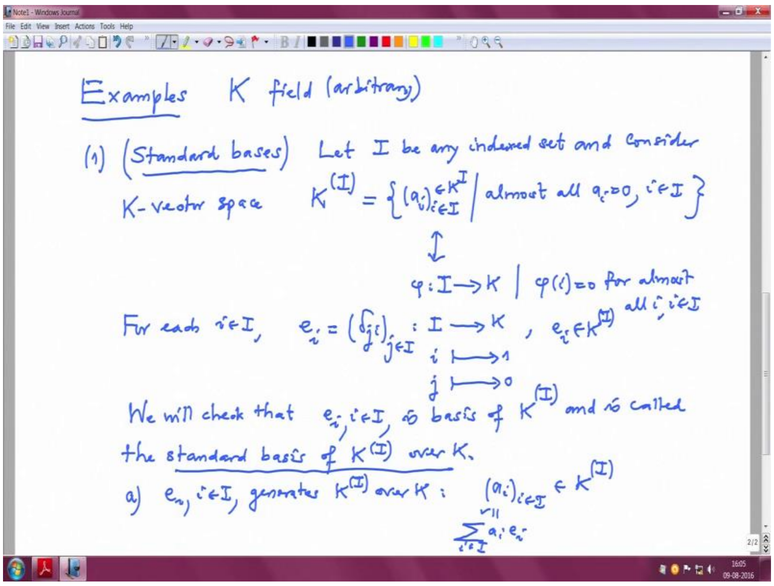

But I will repeat this when time comes. So, I want to see some examples. So, in these examples let us; K always denote field arbitrary when I take this special cases of field I will indicate that term and so first one this is standard basis. So, let let I be any; I will call it indexed I, usually indexed means the set which is used for indices for enumerating the elements of some set and consider the vector space K vector space K power round bracket I once again I will write here these are all tuples such that almost all a I's are 0 these are the tuples in K power I 1 can also think they may as functions also think they may as functions f naught, I do not want to call f, now functions phi from i to K such that this condition will get translated phi of i equal to 0 for almost all i in I.

So, we are seen in earlier lecture that this is a K vector space and now I want to give some family here which we will test whether it is a basis linear independence generating system it is a; so for each i in I, we have this standard tuple E i; E i is if you like delta a j a delta j i j is varying here j is varying in I. So, think of this has a map from i to K, where i goes to 1 and any other j goes to 0. So, either you think as a tuple or you think as a map and it is clear by the definition of V i only i goes to nonzero entry. So, E i's are elements in this K power round bracket I and I want to prove that. So, we will check that E i i in I is a basis of K power round bracket I and this basis is called and is called the standard basis of K power I over K.

So, to check it is a basis we need to check 2 things, it generates and family is linearly independent. So, first we will check that we are checking that E i i in I generates K power round bracket I over k. So, this means we should write, we should check that elements of arbitrary tuples a i i in I, K power I round bracket I is a linear combination of this, yes with coefficients in k, but note this tuple a i is a sum a i E i i in I this is obvious because when you the first of all sum on the right side make sense because only finite many as could be nonzero and secondly, when you expand it and compare the components it is clear that i th component in this right hand sum is E i which is i th component in the tuple a i also.

So, therefore, these are equal and this precisely means that any tuple in K power round bracket I is a combination of linear combination of the elements E i therefore, it generates K power I. Now, second thing we have to check that they are linearly independent.

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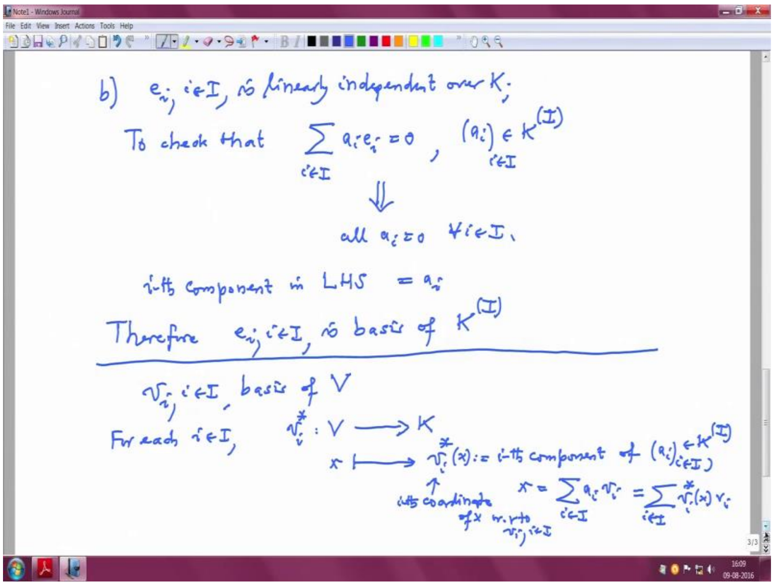

So, this is b we have to check that the family E i i in I is linearly independent over k; that means, we need to check that. So, we need to check. So, to check that if a linear combination of E I, a I, E i is 0 and again I will; so, this a i tuple is in K power round bracket I. So, this means its indeed a finite linear combination then we want to check that from these we want to check that all a I’s are 0 this is what we need to check, but if you take we have given this sum is 0.

So, if you check what is the i th component, i th component in l h s is precisely a i and so, if this tuple; this is a tuple if this is 0 then i th component if 0, but then i th component is a i. So, a i is are 0. So, this follows from that. So, we have check that it is linearly independent and therefore, therefore, E i is family E i i in I is a basis of K power I round bracket I. So, you remember last time also I would define concepts of coordinate functions with respect to a basis, I just want to recall. So, if V i as a basis of V then we have defined for each i i in I we have defined a functions from V to K by using this basis and that we have called it V i star. So, any vector X here goes to V i star X this is by definition i th component i th component of a tuple a i i in I where this tuple a i s uniquely determined by this equation X equal to summation a i V i i in I this tuple is in K power 1 bracket I.

So, because this is a basis each vector X can be written uniquely in this form. So, this tuple a i as only finitely many nonzero entries f four it is here and this coefficients a i’s is this is what map h goes to a i. So, this called the coordinates of this V i X is called i th coordinate of X with respect to this given basis and then we could have also written this as summation i in i V i star X V i. So, you say this is the idea of writing this is it only depends on X and basis V i.

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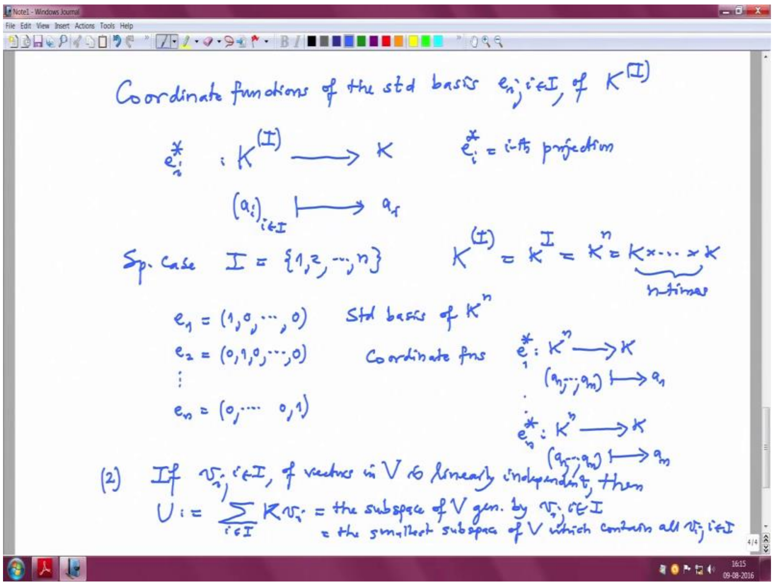

So, now in this in the above case what are the coordinate functions with respect to this standard basis? So, coordinate functions the standard basis E i i in I of the vector space K power round bracket I are what maps they are precisely. So, the in; they are maps from K power i to K and these are E i star and these are precisely the projection map this is map to a i this is. So, E i star is precisely the i th projection.

So, let us take a special case for when I is a finite set special case when I is a finite set and that to one to n in this case K power round bracket I is same thing as K power I which is also sometimes we are denoted by K power n which is K cross K cross K n times this is the Cartesian product of K n times and e 1. Now we have e 1, e 2, e n. So, e 1 is a first position one and then the remaining 0s, e 2 is second position 1 and the remaining 0s, etcetera, e n is the last position is one that is n th position 1 and everywhere also it is 0. So, this is a standard basis of K power n and coordinate functions E i star or e 1 star let me say e 1 star from K power n to K this is the first projection. So, a 1, a 2, a n goes to a 1 etcetera e n star is this n th projection that is this a 1, a 2, a n tuple goes to the last component a n that is n th cord; n th projection.

So, this is the standard basis. The next example; example 2, if family V i of vectors in V is linearly independent it may or may not generate V, but the linearly independent means if a linear combination of V i is 0 then each coefficient of 0, in this case, we cannot say it is a basis of V because we may not know that it is generating set for V or not, but surely what we can do is you can take now us subspace U, U is a subspace generated by this family V i is that is remember this notation, we have; we denoting like this K V I, this is the subspace of V generated by this family V i this means by definition it is a smallest subspace; the smallest subspace of V which contain all V i's and we have seen earlier in earlier lecture that such a subspace has an nice description that every element of that subspace is a linear combination of V i's.

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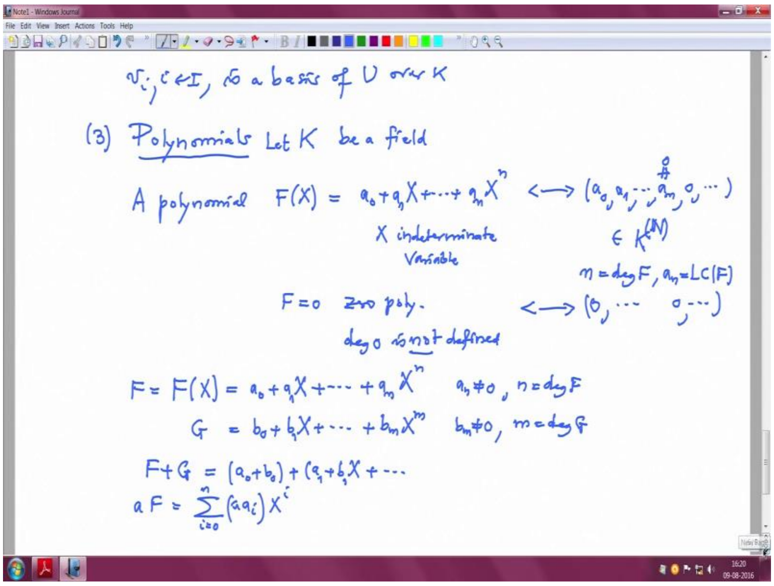

So, definitely, now we can say that this V i i in I is a basis of U over K because by definition now U is generated by V i's and to start with, V is a linearly independent family. So, this is a basis of U I, now next I want to recall third one; I want to recall for future also couple of remarks about polynomial form polynomials first about polynomials after this, I will come to polynomial functions after this I will come to polynomial function and after that I will even come to rational functions and after that even I come to many more interesting functions which will be very useful in analysis courses for example.

So, let K be a field and this actually one can do more generally that they can even a do not even need to assume that K is a field, but I will restrict it in these course only the field. So, everybody know what a polynomial is polynomial f in 1 variable X is an expression like this a 0 plus a 1 X plus plus plus plus a n X n where these X 1 usually calls it indeterminate or also call it variable and these are the formal expressions one can also think of them as a vector a 0, a 1, a n and 0, 0, 0 because I want to think this as an element n K power n. So, if they are round bracket n note that only finitely many terms are nonzero could be nonzero and the last one where it is nonzero if this is nonzero then one calls that n as a degree of F.

If everybody is 0 then the degree is not defined. So, that is the 0 polynomial. So, this is 0 tuple corresponds to the 0 polynomial, this corresponds to F equal to 0 polynomial; 0 polynomial. So, the degree of a 0 polynomial is not defined, this n is called; this a n is called the leading coefficient of F L C of f leading coefficient of F and we will now how do we add polynomials add it component ways. So, if you have one polynomial f X and normally even drops this X in the notation of the polynomial I will just call it capital F, this is a 0 plus a 1 X plus plus plus, plus a n X n G is another polynomial b 0 plus b 1 X plus plus plus plus plus b m X m a n is nonzero b m is nonzero, this is n is the degree F m is the degree of F degree G and then you F plus G same as we have been doing it in this add it component wise and put that a corresponding coefficient.

So, this is a 0 plus b 0 plus a 1 plus b 1 X and so on. So, it will depend on whether m is smaller or bigger and so on. So, this is very well also you can multiply by scalar a F is clear out a multiply you push that a inside everywhere. So, that is this is summation this also one can write summation from i equal to 0 to n a; a i times X power I. So, that is how we add polynomials we also have learned to multiply a polynomials now multiplication of polynomials is rather complicated in the sense we first multiply the the for multiply the powers of the variable X i X j X power i plus j and then use distribute to law to expand.

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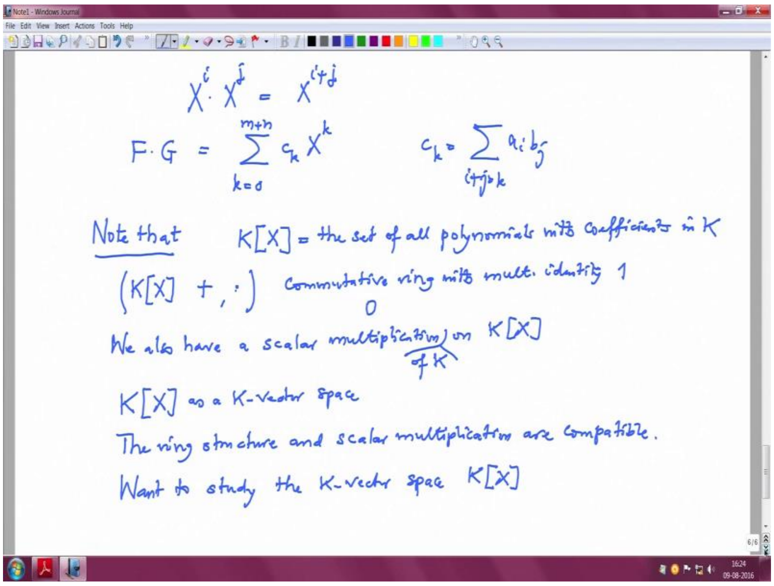

So, that is how we X extend this multiplication to the polynomials. So, this will be summation c K X K, K is from 0 to wherever m plus n here and C K is are defined by summation a i times b j this summation is learning more i plus j equal to K.

So, these are the usual formulas for the polynomials over a field and so, therefore, important thing to notice note that this K the set of polynomials this I denote by K square bracket X this is the set of all polynomials with coefficients in K. So, on this set we have 2 operations we have defined addition multiplication. So, with that becomes a commutative ring with multiplicative identity the constant polynomial one and the additive identity with respect to plus the Newtonian element is the 0 polynomial.

More than that we also have a scalar multiplication we also have scalar multiplication all K X scalar multiplication of K on K n. So, this makes this K X as a K vector space and it is very it trivial to see that the commutative ring structure on this vector space structure are compatible with each other. So, I will just say that the ring structure and scalar multiplication are compatible with each other. So, this means whether; so when one says ring structure; that means, this binary operation plus and the multiplication and scalar multiplication is the scalar multiple. So, whether you add first and then scalar multiply all scalar multiply add and multiply all these operations are compatible with each other.

So, for us very important here now I want to study we want to study first I want to study the K vector space of polynomials all polynomials.

So, for example, extracting generating set or whether is it linearly independent whether is it a basis or any elements it as etcetera, etcetera and many subspaces which will arise from this space of polynomials.

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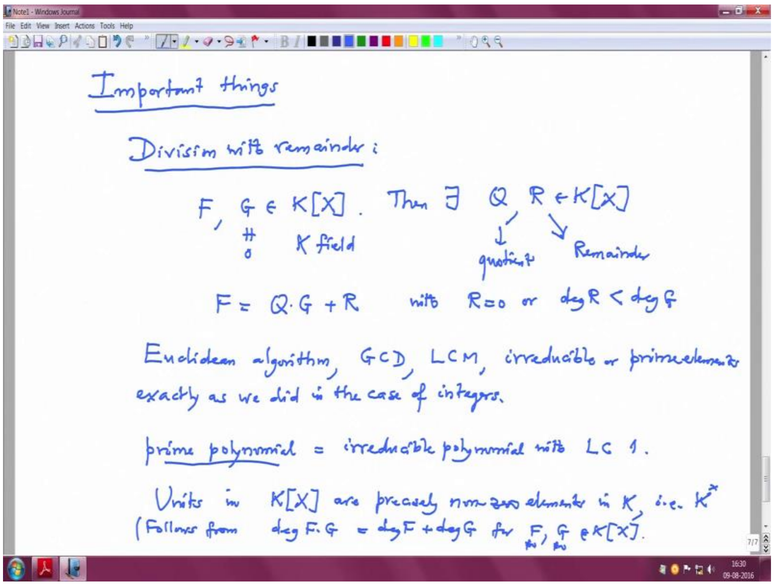

And, before I end with the polynomials, important things to note here which I will assume without proof this is exactly similar to that of ring of integers like we have seen in case of ring of integers fundamental theorem of arithmetic prime numbers division with reminder Euclidian algorithm GCD etcetera, etcetera that also make sense in the polynomial ring and the most important thing is like a what is called division with reminder this is the route of many things.

So, that is if I have 2 polynomials F and G with coefficients in the field K. So, K field is very important here and if G is a nonzero polynomial then I can divide f by G and write coefficient and reminder. So, then there exist polynomials Q and R with coefficients in the field this Q will be called coefficient and R will be called reminder. So, that the polynomial f we can write it as Q times G plus R with either R is 0 or degree of R is strictly smaller than degree of G. Note, here I have we can assume carefully and I have to say this separately because we have not defined degree of a 0 polynomial. So, therefore, this is necessary to mention and also degree G make sense because our assumption is G is 0 and now from this Euclidian with we can go on from these division with reminder we can go on to define what is Euclidian algorithm we can define what is the GCD, we can define what is the l c m also we can define what are irreducible or prime elements.

Exactly, exactly as we did in the case of integers, so when I say prime; prime polynomial; that means, irreducible polynomial with leading coefficient one leading coefficient one the last remark and we will have a break the units in the polynomial ring K X are precisely nonzero elements in the field that is we are denoting by K cross and this follows this follows which follows this follows easily from what we call it a degree formula if we have 2 polynomials degree of F time G equal to degree f plus degree G for F G in K X both nonzero this is very easy to prove because we look at the product and look at the coefficient of X power n plus n and it be will a coefficient of X power n in f and coefficient of X power m in G and. So, both are nonzero. So, the product will be nonzero because we are in a field. So, let us have a break and then we will resume soon.

Thank you.