Linear Algebra
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Lecture – 14
More examples of a basis of vector spaces

Welcome to this lectures on Linear Algebra. Last couple of lectures, we have been seeing some examples about generating systems, linear independence and a basis of a vector spaces. I still want to discuss some more examples, which are very much useful for applications in physics, and mathematics and even computer science.

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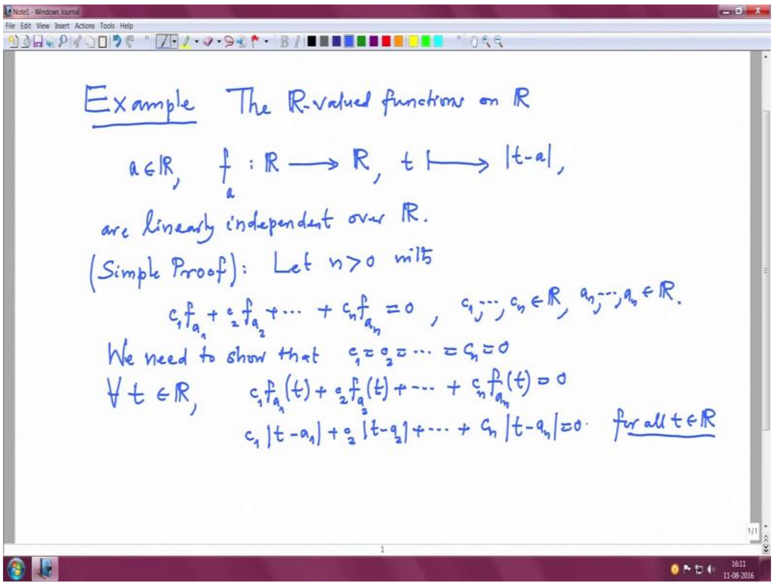

So, let us continue with the next example. This example deals with what one call in physics plane functions. So, the R-valued functions on R their functions from R to R and there it is a family. So, for each real number a in R, we are looking at the function f suffix a, which is defined by t goes to modulus of t minus a. So, we would like to check that these functions R linearly independent over R. So, this means, so let us try to prove this, this is really very simple proof, so simple proof. We need to show that if, so let n is natural number with c 1 f a 1 plus c 2 f a 2 dot dot dot dot dot plus c n f a n is actually the 0 function, where c 1 to c n are real numbers and a 1 to a n R are also real numbers. To show that these functions are linearly independent we need to show that if a finite linear combination is 0, then each coefficient is 0.

Now we need to show c 1 equal to c 2 equal to equal to equal to equal to c n all these real numbers are 0, but this function this is a left hand side is a function and it is 0 means, it is a 0 function. So, this means for all for every t real number when I will left hand side a get c 1 f a 1 t plus c 2 f a 2 t etcetera plus c n f a n t this is 0, it is 0, for all t that is a meaning of this equality. But then by definition of this functions, this is nothing but c 1 mod of t minus a 1 plus c 2 mod of t minus a 2 plus plus plus plus plus plus c n mod of t minus a n this is 0, and this equality is true for all real numbers t, this is important. So, this means this means I want to keep only one term on the left hand side and the shift all other terms on the other side.

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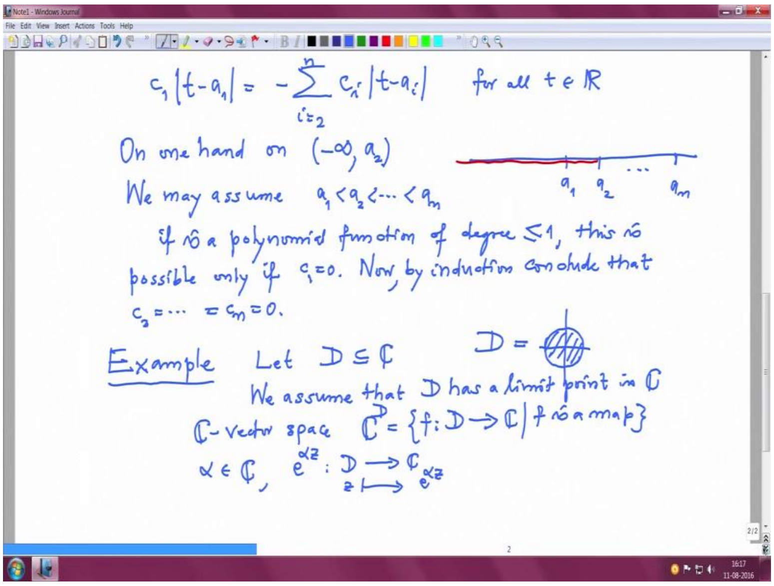

So, we will get c 1 mod t minus a 1 equal to minus I will dot summation i is from 2 to n c i mod of t minus a i and this equation is valid for all t in R. So, if you look at this equation these on one side, it is a polynomial; so on one hand, on the interval minus infinity to a 2. First of all, without loss without loss of generality, I want to assume we may assume a 1 less than a 2 less then less then less than a n. Otherwise, you renumber them. So, on the interval minus infinity to a 2, so they are numbered like this. So, is a real line and this is a 1, this is a 2, and so on, they are increasing, this is a n and.

So, I am looking at this interval open interval. On this interval, on one side, it is a polynomial function, one hand it is polynomial function of degree is equal to 1. And see on the other side it is this function. So, it is clear that this is possible only on the other side it is positive. So, this is possible only if the coefficient c when 0, because it is positive on this side, this side is a polynomial of degree less equal to 1. So, this is possible only when c 1 is 0, this is very easy to say you draw so that means, you approved c 1 is 0. And then by induction, now by induction conclude that c 2 etcetera all are 0.

So, we have done other example. This again this example is very useful. The next example is very useful when one studies complex analysis or differential equations and they are also used in physics. So, the next example, now let us take subset D, let D be a subset of complex numbers for example, you could take D equal to the unit disk for example, you could take this equal to D the disk. And I need to assume which; obviously, this disk as this property D as a limit point in C. We assume that D has a limit point in C.

I will indicate you when where do I use this property for which, and now I consider functions now I look at the vector space C-vector space of complex solid functions on D, these are all maps f from D to C. It is a map and we have noted earlier that this is a C-vector space. And I want to give countable many elements here in uncountable many elements in this vector space which are linearly independent, so that will also when we define the dimension etcetera that we will also prove that this vector space has uncountable dimension.

So, the functions we consider here are the functions for each alpha in C, we are looking at the function e power alpha z. So, this is a function this is defined from D to C, this is defined everywhere actually, but think of it is a function from D to C. So, these are functions from D to C is z going to e power alpha z exponential alpha z and I want to show you that these are linearly independent.

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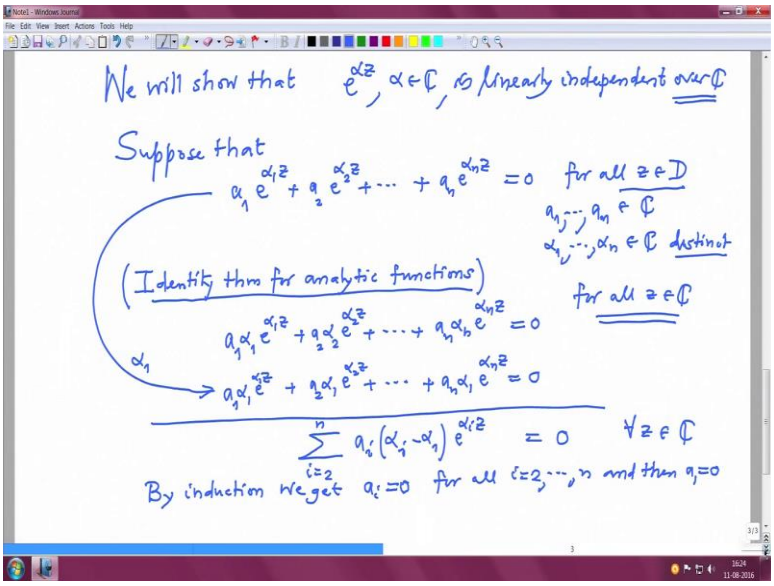

So, we will show family e power alpha z alpha varies in C is linearly independent over C that means what, that means, what if a finite C-linear combination of this function is 0 0 function then the coefficient should be 0. So, suppose that we have sum like this, a 1 e power alpha 1 z plus a 2 e power alpha 2 z plus plus plus plus plus a n e power alpha n z this is the 0 function that means, these equality is valid for all z in C. And this a 1 to a n are the coefficients, which are also complex numbers, because we want to move their linearly independent over C. And this alpha 1 to alpha n are fixed complex numbers, what is varying is z.

And now we differentiate this. First of all note that where I am using the fact that the subset D that we have assumed that has a limit point. So, either left hand side is a function on z function on D, if a function on D is 0 and D has a limit point then it is 0 everywhere in C. So, this is very important theorem in complex analysis. So, this is I will just note it this is called identity theorem for analytic functions. What does it say it say that if we have an analytic function on a domain D, where D as a limit point then it is actually the 0 function. So, by that this will be 0 for all complex numbers. So, I made a mistake here. First to start with it is 0 for all D all elements in D, but because of the identity function identity theorem for analytic functions, this equality actually holds for all z in C. This is where we are using our assumption that D has a limits point in C.

Now, when I differentiate this what do I get? When I differentiate this, and everybody is aware that when I differentiate exponential, what is the derivative that is a 1 first term that is alpha one times e power alpha 1 z this is a derivative of this term with respect to z plus a 2 alpha 2 e power alpha 2 z plus plus plus plus plus a n alpha n e power alpha n z this is also 0 function. This function is 0, then the derivative is also 0, this is one equation I get. On the other hand, this given equation I could have simply multiplied by alpha 1. So, if I multiplied this equation by alpha one what do i get a 1 alpha 1 e power z plus a 2 alpha 1 e power alpha 2 z this is alpha 1 plus 1 plus a n alpha 1 e power alpha n z So, this equation, we have multiplied by alpha 1, it is 0.

Now, I subtract if I subtract this first term will go and then what do I get then we will get summation from 2 i equal to 2 to n to some we get an equation that is a i alpha i minus alpha 1 e power alpha i z, this is 0. So, we have now new equation, and this is true for all z in c. So, we have reduced the length of this linear combination. So, by induction that is all false, by induction we get this coefficient 0, but alpha i is the distinct because we have to take different functions. Therefore, from here you conclude this coefficient there 0, but alpha i cannot be alpha 1 therefore, a i's are 0 for all i equal to 2 onwards and once that is 0 then you get only one equation which is 0. So, exponential function is never 0, so a 1 is also 0. And then a 1 0 because exponential function is never 0, so that conclude the proof of this.

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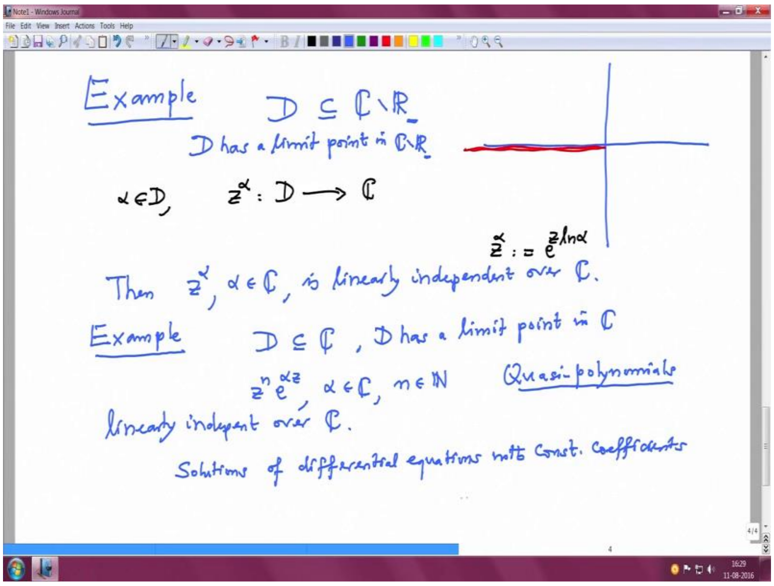

Now, one more think I will mention, but I will not take in detail, but the proof proofs of these following two assertions are same similar to that of the earlier two examples. So, the next one, I will just mention them with they are very, very useful for the solutions of differential equations, say that again which functions. So, again we have the assumption now D is a subset of the complex numbers minus the negative real axis. So, see here we have a complex plain, but I am omitting this negative real axis. This is omitted, because this is done so because I need we are considering this function z power alpha these are functions for each alpha in C alpha in not in C alpha in D and considering this function z power alpha, which are functions from D to C.

Remember the definition z power alpha the definition of z power alpha make sense only when this is the definition which is e power z log of alpha, this make sense only when alpha has no alpha is not a negative real number because so log does not make sense. So, for that reason we have to take a subset D of the complex numbers complex number minus this negative real axis, and also we need to assume now D as a limit point in this in C minus this negative real axis.

This assumption is like another earlier example we have used identity theorem for analytics function that means, if an analytic function is 0 on a subset, who has a limit point then analytic function is 0 everywhere. So, then the family this family, so then z power alpha, alpha varies in complex numbers this is linearly independent over C. The idea of the proof is similar, so I will omit it.

But now I want to combine this example and earlier example. So, the combination is if I take the functions again with the same assumption D is a subset of the complex numbers with D as a limit point these are the assumptions in C. Then I consider the function z power n e power alpha z, alpha is varying in complex numbers now you can take complex numbers and this n is varying in natural numbers. So, now, I did not have to bother about omitting this a negative part of the real axis, because I only took their integers natural numbers not in integers.

So, they are only in this side and I have this product functions. This family is linearly independent over C; proof is again early two examples combined. These functions are also called these s functions are called Quasi-polynomials; they are useful in describing the solutions of the differential equation with contra coefficients. I will not go in to these right now; I will not go in to these right now.

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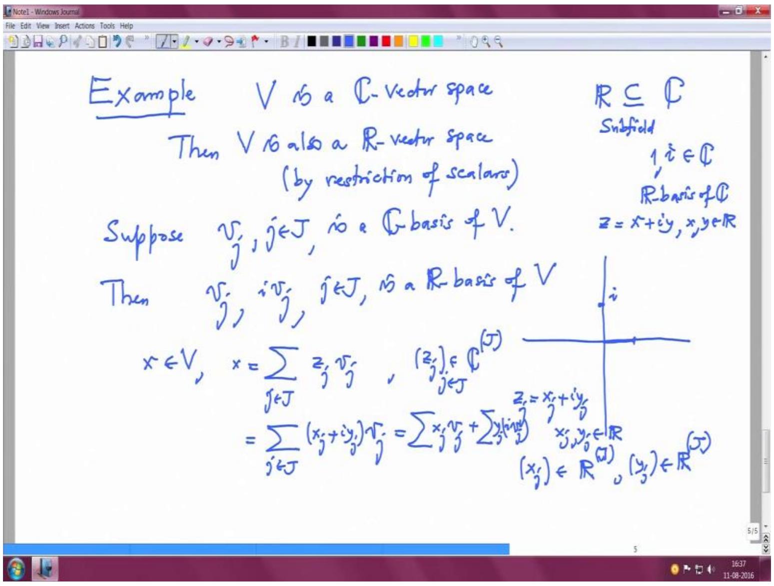

The next one is also very important for next example is also very important for when more than one scalar field is known. So, typically let us say suppose V is a complex vector space, V is a C-vector space, but you see this will C has a sub field R. R the field of real numbers is a sub field. And by very definition of C 1 coma i they are elements in C and they form R basis of C. This is clear because we know that every complex number z, we can write in the uniquely as x plus i y, where x and y are uniquely determined real numbers. And this i is a if we draw a picture it is like this. So, this is i, this is y-axis is called a imaginary axis so and 1 is here. So, these two elements form a R basis of C.

So, if you start with a vector space over C that is also then V is also a real vector space R vector space by restriction of scalars. And now if you give a C-vector space basis of V then how do we extract how do we find an R vector space basis of V, this is the problem that is addressed to in this example. So, suppose v j, j in J is a C basis of V, now which more than one field is involved, so it may be very essential to write what basis and linearly independent or what etcetera, etcetera. Then I want to check then v j and i times v j, j is varying in J is a R basis of V.

So, remember this i is, this is called com imaginary complex imaginary number. So, this is that i. So, I have avoided using indices i. Now, so this means in particular it was c vector space as a basis consisting of n elements then its R basis will consist of 2 R, 2 n elements. So, if the dimension will become two times. So, this will mean that dimension of v as a C-vector space and dimension of v as a R vector space they are related by one is two times i. This I will mention again when I have introduced actually the dimension, which is in the next lectures.

So, what do I have to prove to check, we need to check two things, we need to check that every element of V is a combination of this guys with real coefficients. And what is given to us is every element x of V, we can write it as a combination with coefficients in complex numbers. So, every x in V, we can write it as I do not want to use a. So, z j times v j, j in J where all but finitely mean z these are non j. So, z j belongs to c power round bracket J, j in J.

This is what we have given because this generates this generates v as a c vector space. And now I want to check that this generate v as a R vector space so; that means, I should have a expression for x is this. I should have expression for x in terms of v j's, but the coefficient in terms of v j's and i times v j's, so that the coefficients allowed are only real numbers, but this is very simple because each z j is a complex number. So, each z j i can write it as x j plus i y j, where x j y js are real numbers.

So, just plug it in there and separate it out. So, you will get j in J x j plus i y j v j. And if almost all z j's are 0 then also almost all x j's and y j's will also be 0. So, it will also be clear that x j belongs to R power round bracket J and y j belong to R power round bracket J. And when you expand this, this becomes summation x j v j plus summation y j and that I will push it inside i v j.

So, we manage to check that this is the generating system for v over R. Now, we also have to check that they are linearly independent over R, but we have given this is linearly independent over R, and the trick is the same. So, if some linear combination is 0 with real coefficients then do each complex number I did like this, each combine this two real numbers with this, and we get a complex numbers it will be 0.

So, I will stop and we will continue in the next half more I will now try to prove over that every vector space as a basis. And then the next step will be any two basis of a vector space have the same number of elements and that will give us a concept of a dimension.

Thank you.