Linear Algebra
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Lecture – 15
Vector spaces with finite generating system

Welcome back to this course on Linear Algebra. As you know, that last lectures I have been talking about the concepts of generating systems, linear independence, and basics. This lecture I will introduce a concept of dimension and we will prove that every vector space as a basis; and any two basis have the same cardinality, and this common cardinality will be called a dimension of a vector space.

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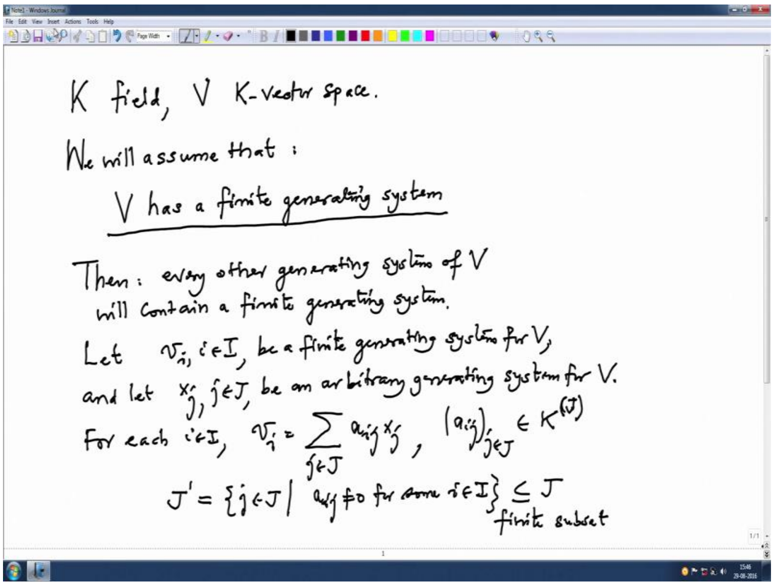

So, let me do it first. So, as usual K is an arbitrary field and V is a K-vector space. And as a first step we will assume first that V has a finite generating system. This we will make this assumption for a today's lecture and may be the next lecture we will deal with arbitrary vector spaces. So, first I want to prove that V has a basis and V has the any two basis have the same number of elements. So, we note that from the last lectures we note that if we have a finite generating system then every other generating system will contain a finite generating system. Then I will just note it here this will be used again and again every other generating system of V will contain of finite generating system.

Let me recall the proof of this very quickly. So, we have given V has some finite generating system. So, let v i, i in I, be a finite generating system for V, and let x j, j in J, be an arbitrary generating system for V. v i is the finite generating system in these i set is finite set, and this j could be finite j could be infinite, but we want to extract sub family from x j which generate V. Well, so it say generating system, so now, let us this is a generating system means every vector in v is a linear combination of this x j's.

So, in particular this v i's. So, for each i in I, v i has a representation in terms of this x j. So, that is a i j x j, j in J, and where this a i j tuple j is varying here this should be in k power round bracket J. Please recall the notation round upper round bracket j means only this a i j s are almost all zero for all j in J, for a fixed I, I is fixed. So, now, it is clear that if I take the subset j prime, j prime consist of all those j in J such that a prime not a prime all though j in J, such that a i j is nonzero for some i. This is subset of j and it is clearly finite subset of j, because if we fix and index i then only finitely many a i j's are nonzero. So, I am putting it all, so which clearly finite and also what is clear is I am going to next; I am not able to go next page.

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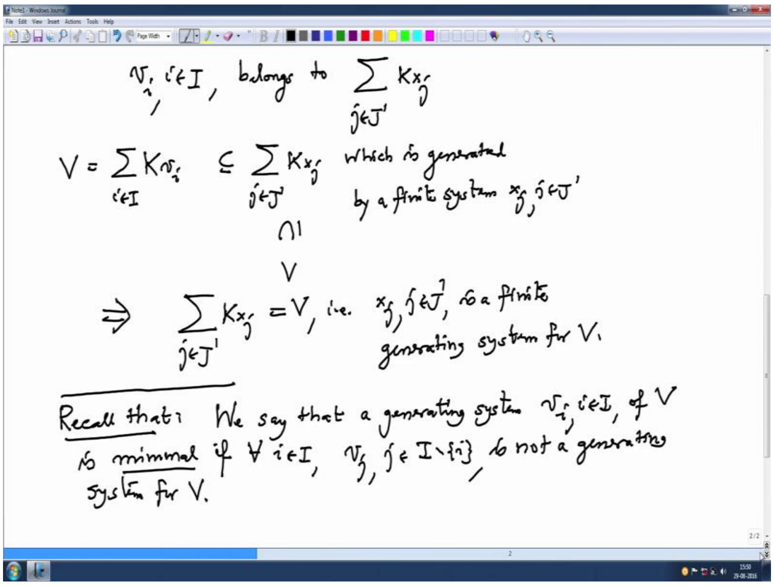

So, this j prime is a finite subset of J, and also it is clear that all this v i's, i in I belongs to the subspace generated by x j where j is varying in J prime. So, that means, K v i subspace generated by v i's which is the whole v because v i is the generating system this is contained in the subspace K x j, j in J prime when this subspace has a generating system finite generating system, which is generated by the finite system x j, x j varying in J prime. So, that means, it should be equal here because this is a subspace, subspace of V, but this already V. So, all together we conclude is subspace generated by k x j x j varies in j prime is the whole V, so that precisely means that is x j j in j prime is a finite generating system for V. So, we have justified our claim saying that if V has a finite generating system then every other generating system will contain a finite generating system.

And now also last time, we saw that which again I recall for the sake of completeness. Recall that we have seen that we call first of all let us recall the definition, we call we say that generating system v i, i in I, of V is called or we say that the generating system is minimal. If for every i in I, if I omit v i from this family, it will not be a generating system v j j is in I minus this i is not a generating system for V. So, we cannot afford to drop any vector from this generating system that is a minimal 1.

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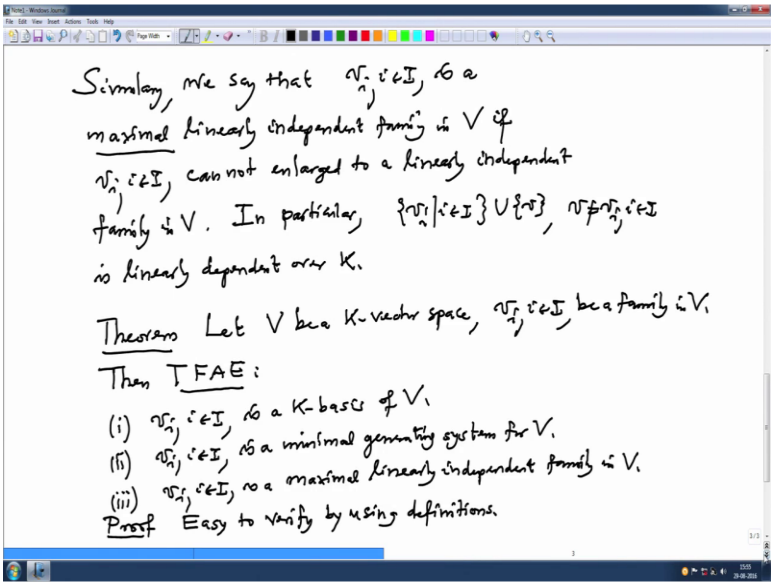

Similarly, we will say that family v i is maximal linearly independent family in V, if we cannot enlarge v i to a bigger linearly independent family. If v i, i in I, cannot be enlarged to linearly independent family in V. In particular, we cannot add one vector in v i family v i, i in I, union some other vector V, where V is not in v i is linearly dependent over K.

And also now it is very easy to see that I will just note in the form of theorem. Theorem, let V be a K-vector space, and v i, i in I, be a family in V. Then the following are equivalent, I will use this TFAE in the short form for the following are equivalent. Number one - v i, i in I, is a K-basis of V; two - v i, i in I, is a minimal generating system for V; and number three - v i, i in I, is a maximal linearly independent family in V. The equivalence of this is really very easy and I want to skip the proof, so proof I will just say easy to verify by using definitions.

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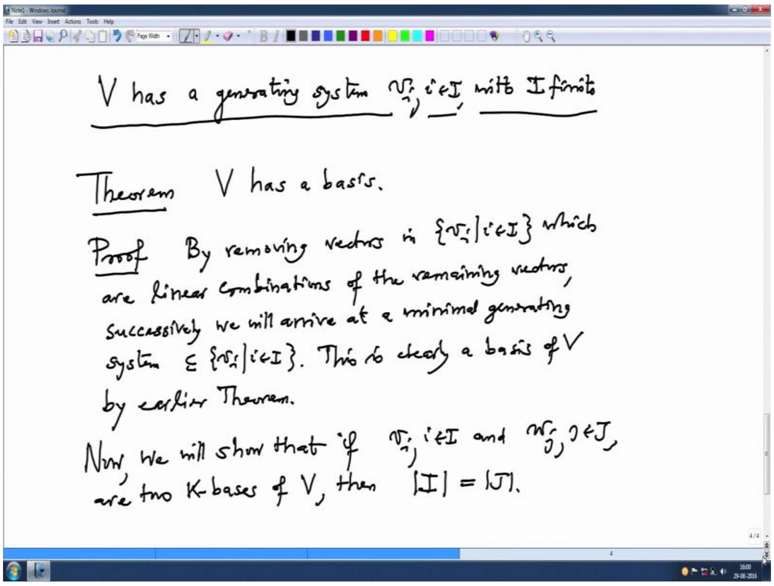

So, our assumptions, remember or what we are assuming is v i as the finite generating system that means, if we have V has a generating system v i, i in I, with I finite. Of course, one could also take i equal to 1 to n and write instead of writing v i, i in I, v 1 to v n, but there is no harm in writing in general index. So, this is our assumption. Now the earlier theorem to prove that; so let me write this as a theorem, theorem this is what we want to prove, theorem with these assumption V has a basis.

Proof, So, I want to I start with this finite generating system; there are only finitely mean elements in this family. So, I check that whether some vectors are needed in this family or not that means, whether vector is a linear combination of the remaining other once if it is so then I will remove it from this family. And continuing this process, we will arrive at a family which is a minimal generating system; and this is possible because this a either finite set we are only doing finitely many steps. So, proof is very simple by removing vectors in this family v i, which are linear combinations of the remaining once remaining vectors, successively we will arrive at a minimal generating system which is contained in this family v i.

And as we saw earlier theorem that minimal generating system is a basis. So, this will be this is clearly a basis of V by earlier theorem. Note that V, this process terminate because I is the finite set. So, it in finitely many steps, we will arrive at basis; however, we cannot do such a process for infinite set in general. So, we will have to use some more stronger form of sets theory to prove this things, which we will postponed till next time. But now our main problem is to show that now we will show that if v i, i in I, and w j, j in J are 2 K-bases of V then cardinality of I equal to cardinality of J. And note that both I and J are finite because we are assuming generating system V has a generating system which has finitely mean elements and therefore, any other generating systems will also have sub systems which are generating system finite generating systems.

So, we could concentrate on finite generating. So, V has a finite basis. So, all basis of V are finite. So, and we want to prove that now any two basis of the same number of cardinality. So, we will prepare for this.

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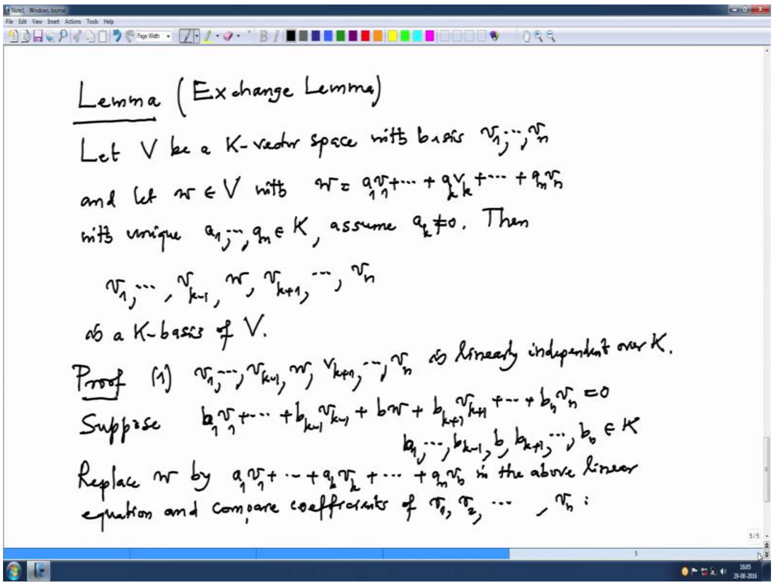

So, first I will prove a lemma, so that is very important lemma for many purposes. This is called exchange lemma. So, let V be K-vector space and with basis, now I will call it v 1 to v n. And let W be a another vector in V, so definitely because v 1 to v n is the basis this w we can write uniquely as with w, I can write uniquely as a 1 v 1 plus plus plus plus dot dot dot a k v k plus plus plus a n v n with unique a 1 a n in scalars. And if w is nonzero some scalar has to be nonzero otherwise w itself will be zero vector. And we will assume a k is the nonzero so that means, this v k occurs in the expression of W. Then v 1 to v k minus 1 and I want to drop v k from this basis; and instead of v k, I want to replace by w. So, I am exchanging the vector v k in the basis with the vector w and I want to claim then this is a K-basis of V and we shall make repeated use of this lemma, first let us prove this lemma.

So, we want to prove this v 1 to v k minus 1 w v k plus 1 v n is a basis so that means, we need to check two things. We want to check that they are linearly independent and it also generates. So, first let us proof that they are linearly independent. Proof, one - v 1 to v k minus 1 comma w, v k plus 1 v n this is linearly independent over K. So, to prove this, let us assume some linear combination of them is 0, and then we need to conclude that all coefficients are 0. So, suppose b 1 v 1 plus plus plus plus b k minus 1 v k minus 1 plus b w plus b k plus 1 v k plus 1 etcetera, etcetera plus b n v n is 0, where b 1, b k minus 1, b k plus 1 b n these are the scalars. And now we want to conclude all b's are 0.

Well, we have a expression for w in terms of v 1 to v n we will plug it in here, and compare the coefficients of v 1 v 2 etcetera etcetera v n to be 0, because it is a basis if a combination is 0 then each coefficient must be 0. So, replace w by its expression unique expression a 1 v 1 plus plus plus plus a k v k plus plus plus plus a n v n in the above linear equation and compare coefficients of v 1, v 2 etcetera etcetera up to v n what do you get let us see.

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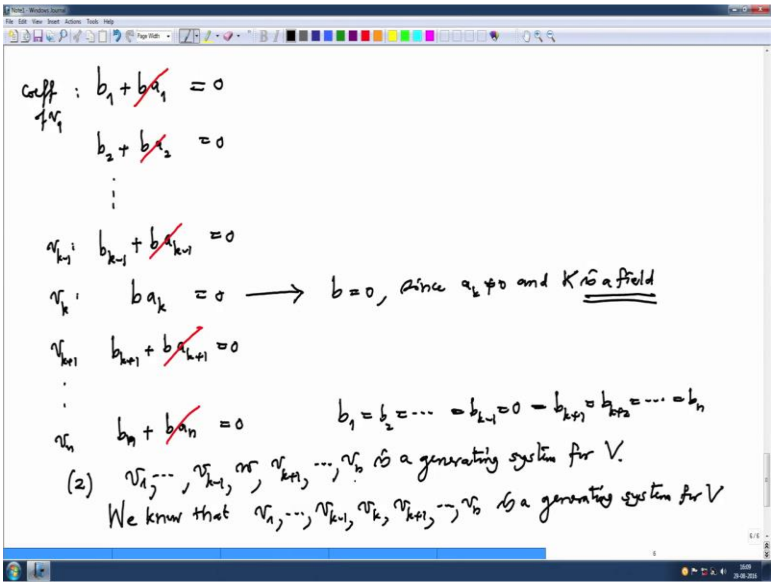

So, first coefficient of v 1, so if you put w equal to a 1 v 1 then the coefficient of v 1 will be b 1 plus b a 1. This is a coefficient of v 1 this is, so coefficient of v one is this which should be 0, because a combination is 0, and v 1 to v n is a basis. Coefficient of v 2, b 2 v

2 is b 2 plus b a 2 that is also zero this will go until that a k minus 1. So, b k minus 1 plus b times a k minus 1 is 0, since coefficient of v k minus 1 coefficient of v k will be b times a k this is also 0. And then again v k plus onwards same like v 1 to v k minus 1 that is b k plus 1 plus b times a k plus 1, this is 0 and so on coefficient of v n will be b n plus b a n is 0, this is n.

Now, from here we want to conclude all b as 0 and b is also zero, but look at this Kth equation that will tell you b is 0, since a k nonzero and k is a field that is why field is very, very important because the product of elements is 0, so and one of them in nonzero. So, the other must be 0, now b 0, once you go to b 0, then a look at the first equation this part is 0, so b 1 is 0. So, second equation b 0, so this is 0, k minus 1 this equation, this is 0. So, b k minus 1 0 a k plus 1 with the equation, this is 0. So, b k plus 1, this is 0, this is 0. So, we conclude all b 1, b 2, b k minus 1 are 0, and b is 0. So, therefore, from these we have concluded b 1 equal to b 2 etcetera equal to equal to equal to b k minus 1, they are all 0. And similarly b k plus 1 b k plus 2 etcetera etcetera up to b n, they are all 0, so that proves a linear independence of v 1 to v k minus 1 w v k plus 1 to v n.

So, now, second part we have to check that v 1 to v k minus 1 w v k plus 1 to v n is a generating system for V. So, this is also very easy because we already know we know already that we know that v 1 to v k minus 1 and v k v k plus 1 this is a basis that was given to you, in particular it is a generating system for V. So, to show that this along with w it is a generating system, I only have to check that the vector v k is a combination of v 1 to v k minus 1 w v k plus 1 and v n.

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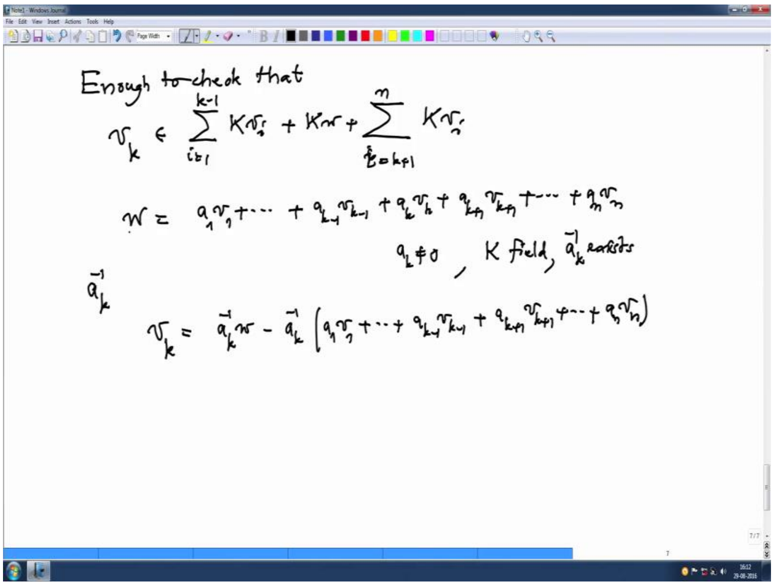

So, to check this, so it is enough to check that vector v k belongs to the subspace generated by v i i is from 1 to k minus 1 plus K w plus k i is from k plus 1 to n K v i. So, we have to check that this vector v k has a representation in terms of v 1 to v k minus 1 w and v k plus 1 to v n, so that is very easy because you know w has a combination like this a 1 v 1 etcetera etcetera that is our assumption. This was given to us. And this a k coefficient is nonzero. So, I will keep this term on one side and remaining terms I will shift it to other side, and multiply by a k inverse, so multiplying by a k inverse that is possible because K is a field. So, a k inverse exists because a k is nonzero.

And I can rearrange this and rewrite it as. So, I will get an equation v k will be equal to a k inverse w minus a k inverse a 1 v 1 plus plus plus plus a k minus 1 v k minus 1 plus a k plus 1 b k plus 1 plus plus plus plus plus a n v n. So, this gives us an expression for v k in terms of v 1 to v k minus 1 w and v k plus 1 to v n. So, that proves that v 1 to v k minus 1 w k plus 1 to v n is a generating system for V.

So, we will make a short break and do it after the break.