Linear Algebra

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Lecture - 16
Steinitzs exchange theorem and examples

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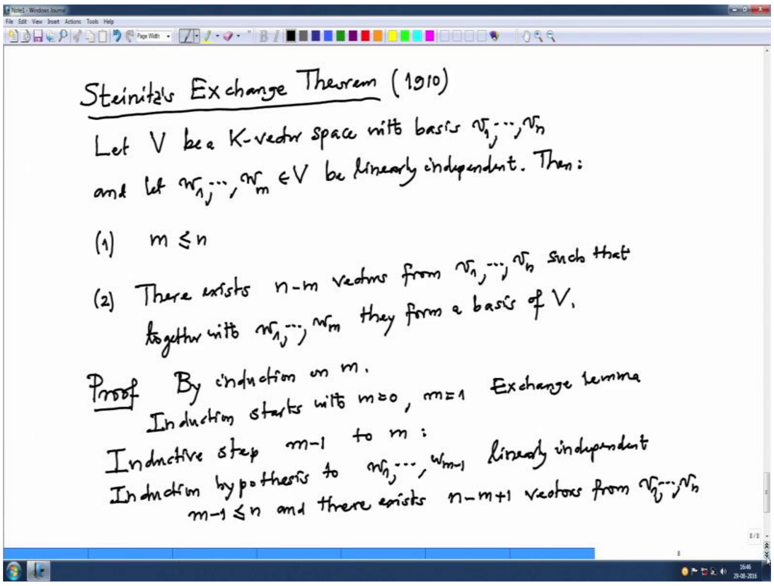

Welcome back to the second half of this lecture. Last lecture, we just finished the proof of exchange lemma. Now, I want to make a stronger formulation of this exchange lemma, which is very important theorem this is usually known as Steinitz's exchange theorem. This was proved by Steinitz's in 1910, which is stronger form of exchange lemma. Let us write down the statement precisely first. So, let V be a K-vector space with basis v 1 to v n. And let w 1 to w m vectors in V be linearly independent. Then part one - m is less equal to n. And part two, part two is a stronger part form of the exchange lemma; that means, I will exchange some m vectors in v 1 to v m and replace them by this w 1 to w m, and along with that it becomes a basis of V. So, to be precisely there exists n minus m vectors from v 1 to v n such that together with w 1 to w m, they form a basis of V. So, w 1 to w m and n minus 1 vectors on v 1 to v n they will form a basis of V that is the statement.

So, remember earlier exchange lemma was n equal to 1, so of this part two. Again the proof is not so difficult. So, let us write the proof, proof is by induction on m, induction starts with m equal to 0, because m equal to 0, there is nothing to prove. However, we can also see proof for m equal to 1, m equal to 1, it just exchange lemma. It is a vector w only one vector is given and its linear independent. So, w has to be nonzero vector. And when you write a combination of v 1 to v n because we want to n to basis at least one of the coefficient is nonzero and that whichever the coefficient is nonzero that V K, you can replace by w that is what we have seen in exchange lemma. So, m equal to 1 also which correct.

Now, inductive step from m minus 1 to m that means, you are assuming the session for m minus 1 and want to prove it for m. So, I am applying induction hypothesis to w 1 to w m minus 1, this is the part of this linearly independent set. So, therefore, part of a linearly independent, linearly independent. So, these are linearly independent and therefore, part one for the induction hypothesis will tell us m minus 1 is smaller equal to n; and part two will let us and there exists m minus n minus 1, so that is n minus m plus 1 vectors from v 1 to v n, which together with w 1 to w m minus 1 form a basis of V.

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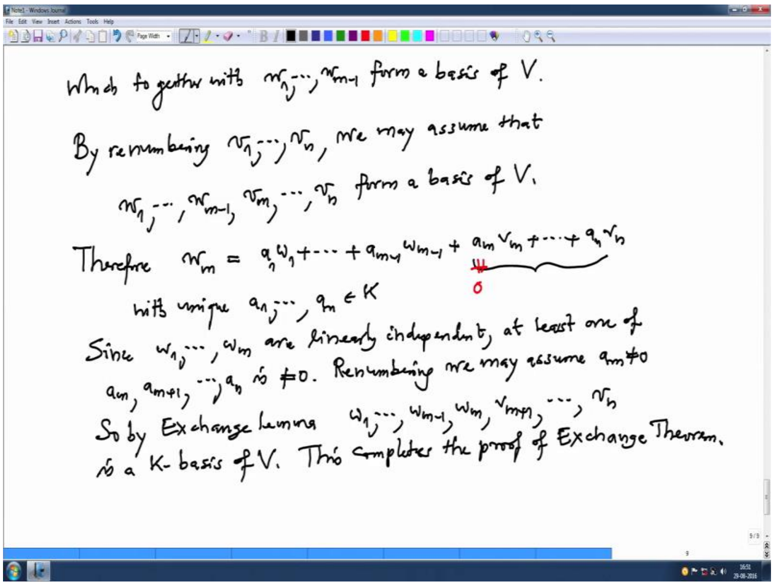

Now those m, those n minus m plus 1 vectors among v 1 to v n. By renumbering v 1 to v n, we may assume that w 1 to w m minus 1 along with v m to v n form a basis of V. Therefore if this a basis therefore, the remaining vector w m can be uniquely written as combination of these vectors as a representation like this a 1 w 1 plus plus plus plus plus a m minus 1 w n minus 1 plus a m v m plus plus plus a n v n with unique a 1 to a n in K, because it is a basis.

Now, now our assumption is you since w 1 to w m are linearly independent at least one of the coefficient from a m to a n is nonzero, if all these guys are 0, then we get a dependent relation among the w 1 to w m that is not possible because they are given linearly independent. So, one at least one of a m, a n plus 1 up to a n is nonzero. So, again without renumbering, we may assume a m is nonzero. Once a m is nonzero, now this is the basis w 1, w m minus 1, v m, v n and this coefficient is nonzero, this is nonzero. So, exchange lemma will tell you us that I can replace v m by this w m and get a basis. So, by exchange lemma w 1 to w m minus 1 w m v m plus 1 etcetera to v n is a K basis of V, this is what we wanted to prove. Because we wanted to prove that w m to w n along with some n minus m vector, they form a basis that is precisely the statement. So, this proves this completes the proof of exchange theorem that is all.

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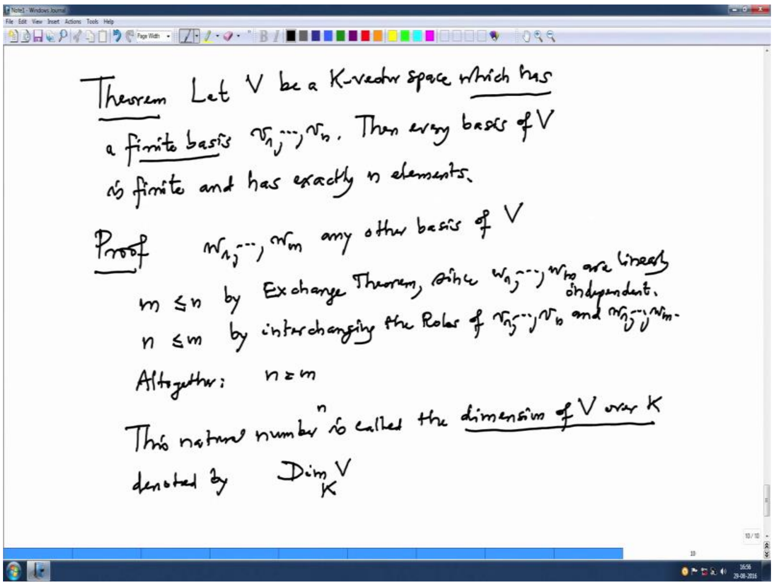

Now, we will deduce lots of consequences on this. So, this was the first one is, this is the theorem what we wanted to prove. Theorem let V be a K-vector space which has a finite basis; let us call it v 1 to v n. Then every basis of V is finite and has exactly n elements. So, any two in particular, any two basis of V has the same number, but we remember assuming that the finite basis. So, proof, we are given a finite basis and first we want to concluding every basis every other basis is finite. It is because if you have another basis then we know that it will contain every other basis, if it is not finite then it will contain a finite generating system. And finite generating system by removing unnecessary elements we will come to the minimum generating system and then it will continue finite basis, but that is not possible because basis a minimal generating system itself. So, first of all it prove that every basis is finite.

Secondly, now if you call the basis to be w 1 to w m any other basis of V, then we know in particular w n to w m are linearly independent. And then when we use exchange theorem part one that shows that m is to less equal to n by exchange theorem. Since w 1 to w m are linearly independent, but then I will interchange the roles of the basis w 1 to w m and v 1 to v n, I will take this as the basis and I take v 1 to v n as linearly independent vectors. And similarly again by exchange theorem, I will conclude n is less equal to n by interchanging the roles of v 1 to v n and w 1 to w n altogether n has to be equal to m, so that proves this assertion that every 2 n is two basis this same number of elements. So, this common number this integer this natural number is called n is called the dimension of V over K. And I will denote it by dim K V.

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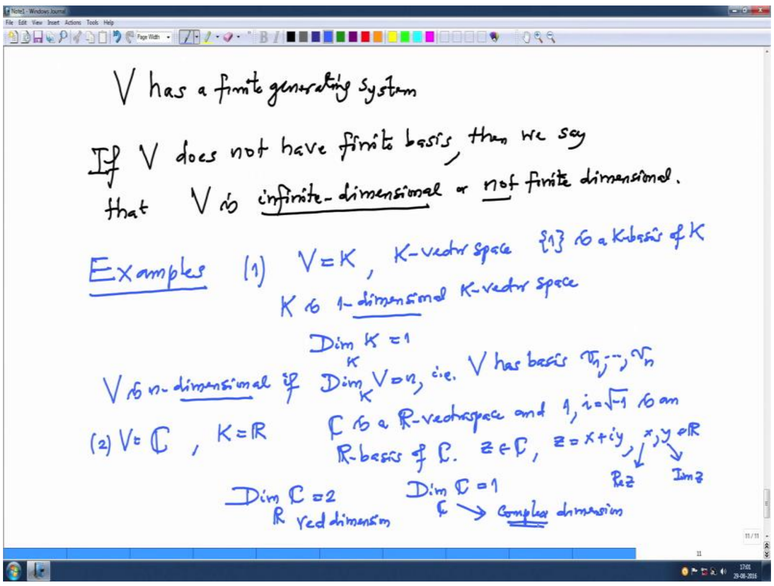

So, we will discuss some example now about the dimension. So, let me remind you that this we have done only with the assumption that V has a finite generating system. And in this case, we have proved that V has a basis - finite basis, and any two other basis have the same number of elements. If V does not have a finite basis, if V does not have finite basis, then we say that V is infinite dimensional or beta will be not finite dimensional. First we have we want to see some examples of both the cases - finite and infinite dimensional.

So, some examples, one - if I take V equal to K, then you may be it is a K-vector space and singleton one, this is a K-basis of K, so that simply means this K is one-dimensional K-vectors space. That means, dimensional K over K is 1. I will also keep saying V is n-dimensional, if dimension V is n so that means, V has a basis v 1 to v n consisting of n vectors. And if n-dimensional, we will call it n-dimensional. Second one, this is more familiar we have used from many years namely, you get complex members, this is my V, and K is real numbers. Then we know that C is a R-vector space; and 1 and i, i is a imaginary root this is root of minus 1 it is to know complex number this is an R-basis of C. We instead saying this many times we have used as a languages every complex number z can be written in the form z equal to x plus i y, where x and y are real numbers. They are immediately determined by z that precisely means 1 comma i the basis of r basis of C. And this x unique x is sometimes also called real part of z and this unique y is called imaginary part of z.

So, in this case, the dimension is 2 because basis as two elements. So, dim C over R is 2. Note however, C over C, if you look at C over C this is a complex dimension this is also I will call it as complex dimension, this is 1. This is a real dimension, and this is real dimension, and this is complex dimension. Complex dimension means on the complex the field of scale is complex numbers.

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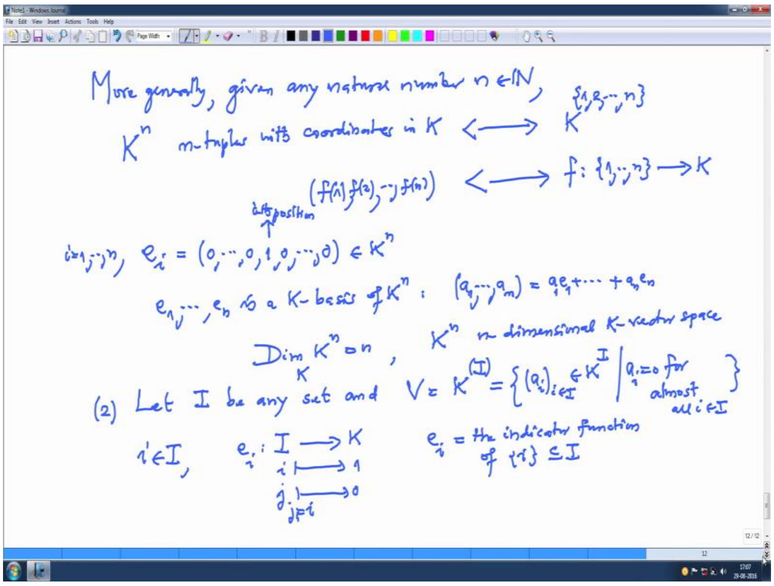

So, for there is a one-dimensional vector space, there is a two-dimensional vector space, in general I want to give example a n-dimensional vector space. So, given more generally given any natural number n in N look at K power n, these are whether you think n tuples, n tuples with coordinates in K or equally you can also think their functions K valued functions on a finite side 1 to n. These are same, because each function f from 1 to n these we have seen in earlier lectures, each function will be corresponding to the tuple f of 1, f of 2, f of n and each tuple will define a function uniquely. So, it is easier to think of about it.

And now remember this we have to defined e i for each in between 1 to n, we have defined these e i's they as tuples n tuples with entries 0 0 everywhere except the ith position, and ith position it is 1. So, this is ith position. These are relevant in this vector space, these are vectors there n 1 of them n of them. And we have seen that this e 1 to e n is a K-basis of K n, this is simply because any tuple any other tuple a 1 to a n, we have written as a finite uniquely finite linear combination of this vectors e 1 to e n, namely a n e n and this a n are really determined by this tuple. So, therefore, this is a basis.

So, and this basis exactly has n element. So, therefore, dimension of K power n over K is n. So, K n is n-dimensional vector space, n-dimensional K-vector space. So, for every natural number n will be have a vector space with dimension exactly n, we will see in couple of lecture that every vector space of dimension n will be isomorphic to this one.

So, we will have to develop a language isomorphism of vector spaces and so on that is you will do in couple of next lectures.

Continuing this example more generally in fact I want to a give a vectors space whose basis in a cardinality, where given set. So, start with let I be any set may be finite may be not finite and V is equal to K power round bracket I. Let me recall again k power round bracket I is those i tuples a i's in k power I. There you can think of then i tuples or think of the function of i to K, K valued for function on the set i such that a i's are 0 for almost all i in I. This phrase means for all but finitely any indices a i's are 0.

So, we have seen this is the vector space. And again same if given i in I, we have this tuple e i think of it is function its better I to K, the ith element goes to zero and all sorry ith limit go to 1 and all other j s go to 0, if j is not i. This is you can also think of this e i as indicator function of the singleton subset and we have seen in earlier lecture that this is e i form of basis of K I, K round bracket I.

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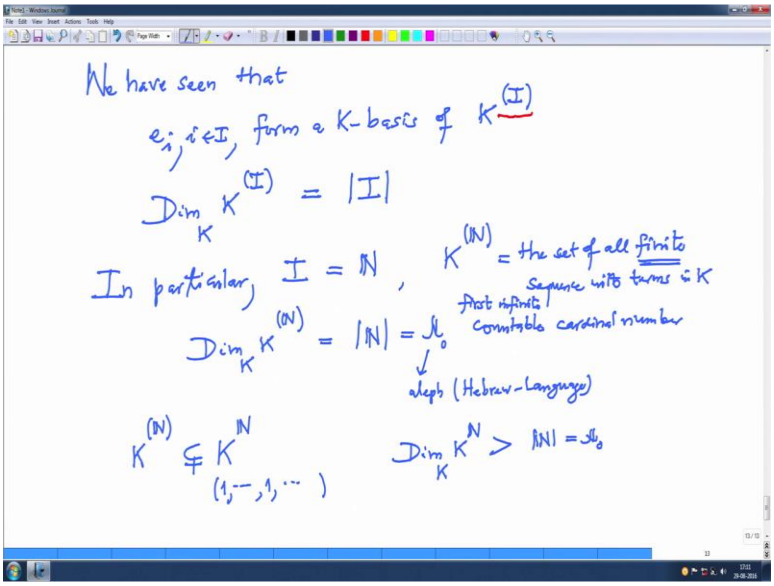

We have seen that earlier the family e I, i in I, form a K-basis of K power round bracket I. I would like to stress this round bracket I is very, very important because otherwise, we will not able to write tuples as a finite combination of this for instant the tuples constant tuples 1 1 1, and everywhere all the if I is infinite. So, therefore, we approved that dimension of the vectors space k round bracket I over K is precisely cardinality of I. This is of course, once we prove that any two basis of the same cardinality, and this is one basis whose cardinality is cardinality of I. So, any other basis should have the same cardinally, this proof we have postponed for the time being.

So, in particular, if I take the set I equal to the natural numbers set of natural numbers then we know this is a countable set, then K power N round bracket N, these are precisely of finite sequences with terms in K. These are set of all finite sequences with terms in K. And this vector space dimension of this over K is precisely the cardinality of I, now cardinality of n because I equal to N in this case. This cardinality is also has the name it is denoted by this, this symbol which is called aleph, aleph this is aleph naught this is alpha bits from Hebrew language, which is countable. This is countable the first countable cardinal number, first countable cardinal number first countable. I should say first infinite countable cardinal number.

Whereas if you take all sequences in K, K power N, we definitely know that this k power n contains properly this finite sequences K round bracket N. And this is definitely not equal because the sequence constant sequence 1 1 1 1 everywhere this does not belong to this subspace. So, definitely you know that the dimension of this sequence space is more than the cardinality N, this is alpha, because if it is equal, it will be this would you will say this. So, we will continue our examples and more conclusions in the next lecture.

Thank you very much.