Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 17
Examples of finite dimensional vector spaces

Welcome back to this lecture on linear algebra. Last time, we have been discussing some examples of dimension of vector space. I will continue this at least half of this lecture to discuss more examples on the dimensions.

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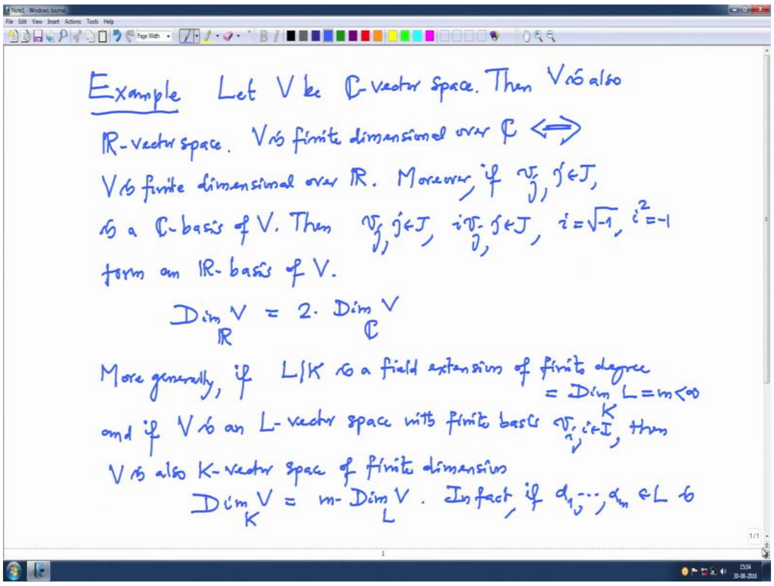

So, if let V be say it is C-vector space, C is a field of complex numbers then we know that V is also R vector space by restriction and scalars; more over V is finite dimensional over C if and only if V is finite dimensional over R. In fact, from a basis of V over C, we can also write down basis of C over R. So, more over if v j, j in J is a C-basis of V then take this vectors v j, and also take i times v j, where i is the imaginary complex number which is a square root of minus 1 or better we put is say i square of minus 1. When v j along with i times v j they form an R basis of V. So, therefore, dimension of V as R vector space is two times dimension of V as a C vector space that is a twice. These we can do it actually more generally.

More generally, if l over K is a field extension of finite degree. In the degree of a field extension is by definition dimension of a bigger field over a smaller field if this is m which is finite and if V is an L-vector space with basis with finite basis v i, i in I, then V is also K vector space of finite dimension. And the dimension as a K-vector space dimension of V as a K vector space is m times dimension of V as a L-vector space. This is very simple, again what you do is you take a basis of l over K and multiply each basis element by this basis and put together they will generate V as a K-vector space.

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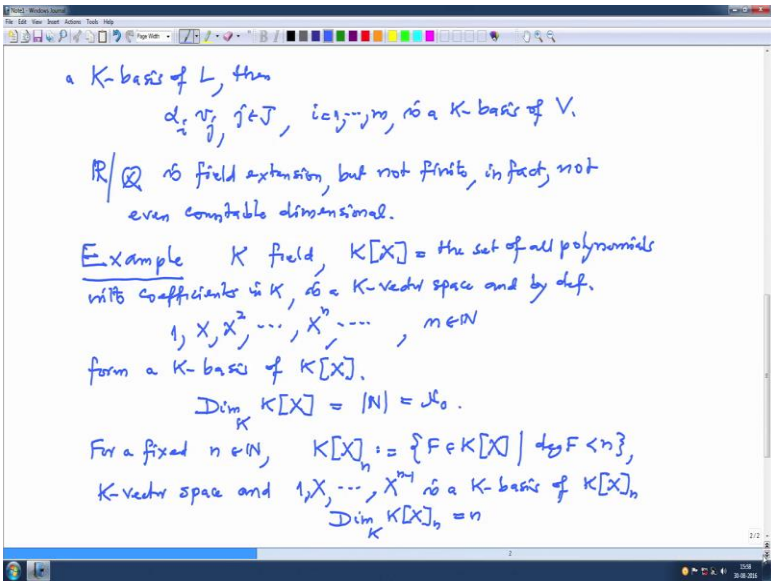

So, I will just write it down in fact if alpha 1 to alpha m in K is a K-basis of L then alpha i times v j, j in J and now i is varying on 1 to m is a K-basis of V. This is same proof as real vectors. So, however, these arguments do not work for a particular field extension R over Q, R over Q is also field extension, but not finite, in fact, not even countable, countable dimensional. In view of this studying vector spaces over Q is much more intricate then studying a real vector spaces or complex vector spaces.

The next example, if you have a field K, then we will see in the polynomial. So, the polynomials the set of all polynomials with coefficients in K is a K-vector space. And by definition of polynomials 1, X, X, square X power n and so on as N varies natural numbers from of K-basis of K X, therefore dimension of K X as a K-vector space is cardinality N which is (Refer Time: 08:26), which accountable dimensional vector space.

If for a fixed n natural number, if I restrict myself to the polynomials of degree strictly less than n, so these are all the polynomial F with degree of F strictly less than n. And this is clearly K-vector space, this is a K vector space which is a subspace of the vector space of polynomials. And 1, X, X power n minus 1 is a K basis of K X n that means, dimension K X n over K is precisely n. Now, next is if you know basis of a vector spaces and if you take the direction mode direct product, then you get a basis for the direct product vector space. So, this is a next example.

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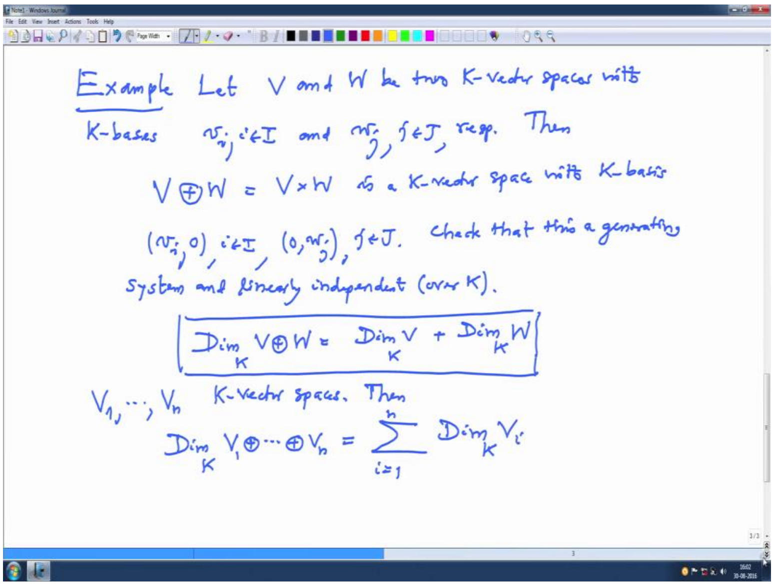

So, let us do it for two vector spaces first V and W be two K-vector spaces with K-basis v i, i in I, and w j, j in J, respectively. Then as we have seen earlier lecture V direct sum W, this is the product set with the component wise vector space structure is a K-vector space with K-basis. You put 0, first is v i in the first component then 0, i in I, and 0, w j, j in J. To check it is a basis, we need to check that it is the generating system and linearly independence, but that is almost clear from the definitions, because the vector space structure on the product, so it is component wise. So, one has to check that this is a generating system and linearly independent that I leave it for checking for.

So, in particular what we get is we get a dimension formula that is dimension of the direct sum equal to dimension V plus dimension W, this is dimension formula. And nothing special about two vector spaces, one can do it for finite limit. So, let me just mention if v 1 to v n K-vector spaces then dimension of the direct sum v 1 direct sum direct sum v n is summation i is from 1 to n, dimension K V i. Note that this direct sum I want to not talk more about these now infinite case now, when enough machinery, I will discuss more about that.

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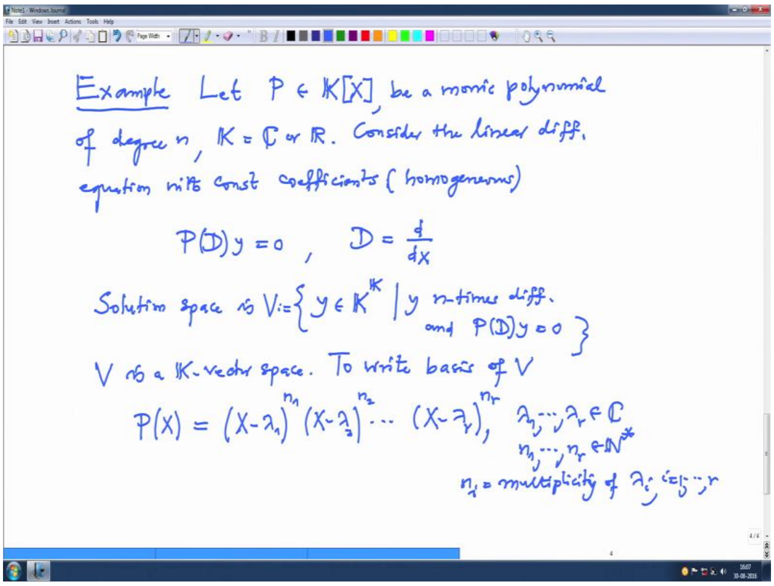

Now, one more example, this example is useful for engineers, only the differential (Refer Time: 13:52) that the differential equations. So, let P be a monic polynomial with coefficients in either real numbers or complex numbers be a monic polynomial of degree n. And K is either C - field of complex numbers or field of real numbers. And what we are looking is whether differential equation consider, the linear differential equation with constant coefficient P D y equal to 0 - homogeneous. Here D is the differential operator d by dx. And we are looking for functions. So, what are we looking for solution space is precisely all those functions y which are functions in one K to K and also K y should be m times differentiable, and it should satisfy this equation P D y 0. So, let me call this solution spaces V.

First, I will note that this V is a K-vector space. This is immediate from the fact that the differentiation is a linear map if the differentiation is specs addition and also the specs the scalar multiplication. So, it is a K-vector space and actually one can write down explicit basis for this vector space that is what I am going to do. To write down to write basis of V, what we need to do is you look at this polynomial P, and this is your definitely either real or complex coefficient polynomial, but over this complex numbers definitely it will split into linear factors. Because C the field of complex numbers and we know it is algebraic close field, therefore every polynomials splits into linear factors.

So, this polynomial will look like X minus lambda 1 power n 1 X minus lambda 2 power n 2 X minus lambda r power n r, where this lambda 1 to lambda r are complex number some of them could be real, complex numbers. And these n 1 to n r are natural numbers nonzero natural numbers, they are precisely the multiplicities. So, n i is a multiplicity, n i is a multiplicity of lambda i or i is 1 to r. In terms of this lambda and in terms of the n i we are going to write down a basis of V and in that and then we will know what is the dimension of this vector space. We will explicitly know how many linearly independent solutions are there.

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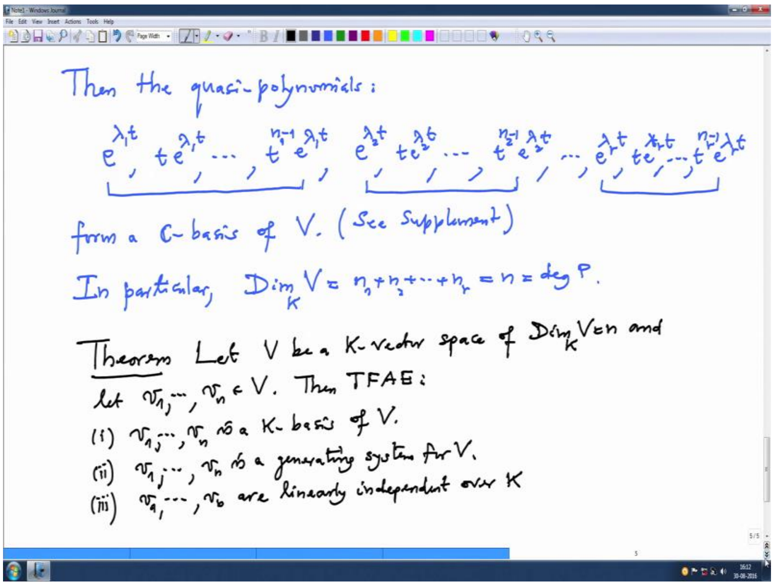

So, then the Quasi-polynomial. These functions which below I will write down, they are called Quasi-polynomials. What are they start with the exponentially e power lambda one t this is exponential function. Then t times e power lambda 1 t go on till t power n 1 minus 1 times e power lambda 1 t. Then start with lambda 2 e power lambda 2 t t e power lambda 2 t go on till t power n 2 minus 1, e power lambda 2 t and then keep doing this for all lambda. So, e power lambda r t, t e power lambda r t, last one is t power n r minus 1 e power lambda r t.

So, this is a block each root will give you this block, there r blocks these quasi-polynomials from a C-basis of this solution space V. This is very easy to check we have in fact, we have checked in earlier lectures that these functions are linearly independent. And they generate is a typical theorem one two is an analysis course. So, I will not prove it here, but maybe i will write the proof in supplements. So, in particular, the dimension, so see supplements. In particular, the dimension if the cardinality of a basis, dimension of a solution space is therefore, n 1 plus n 2 plus plus plus plus plus n r, but this is nothing but the degree of n equal to degree of p. So, what we are noted is the solution space of a linear differential equation with constant coefficient equal to the degree of the polynomial that is a set.

So, now, I want to give couple of remarks about how do we minimize our checking to check somebody a basis of vector space or not, because to check it is the basis of a vector space, we need to check two things one is the generative system and the other is a linear independent. So, somehow if you know earlier or by some other means that the dimension is known, then how do you decide given vectors from a basis or not. So, this is precisely the content of the next theorem.

Theorem says so let V be a vector space, and V be a vector space of dimension n. And let v 1 to v n are n vectors given in V. Then the following are equivalent 1, v 1 to v n is a basis; two - v 1 to v n is a generating system for V; and three - v 1 to v n are linearly independent over K. So, this theorem says if you know that they are correct in number then you only have to either check generating system or linearly independent then it will identically linear basis.

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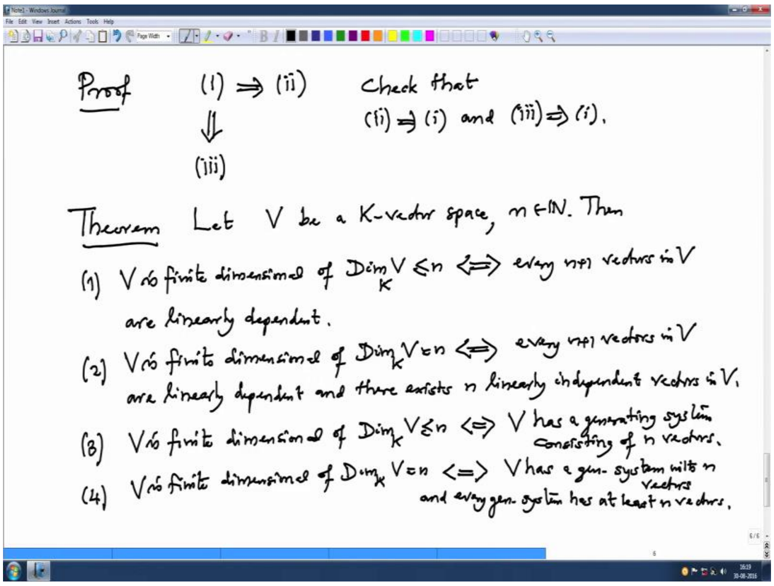

For example, the proof is very easy I just show on to indicate only. The proof, one implies two, and one implies three, they are trivial, because basis should be generating system also and basis also should be linearly independent by definition. So, to prove for example, two implies one that means, is a generating system we have given and the number is correct then want to prove it is a basis. But just note that we have noted earlier given any generating system by throwing away the unnecessary elements, we can arrive at minimal generating system, and minimal generating system is a basis that is what we have proved earlier. So, if at all a generating system is a not a basis then we can cut down, but that is not possible in this case because when we will get a basis who as a fewer elements, but we have checked yesterday in last lecture that any two basis have the same number of elements, so that is not possible.

Similarly for the linear independence, so check that I will just mention here check that 2 implies 1 and 3 implies 1. Also I would like to mention one more theorem like this which is very useful sometimes to check to compute dimension exactly, so that is a next one which use proof also I will omit, because it is similar to the theorem you have just mentioned.

So, let V be a K-vector space, and n is a given natural number, then first statement V is finite dimensional of dimension less equal to n if and only if every n plus 1 vectors in V are linearly dependent. This is because we know if you have a basis consist of n elements less equal to n elements, and if you take any n plus 1 vectors, they cannot be linearly independent because if they where linearly independent then we could extend it to a basis of v by exchange theorem. But then the dimension of V will be more than n, n plus 1 bigger equal to n plus 1 that is not possible. Similarly, we can write down equality here. So, V is finite dimensional of dimension n, if and only if of course, this part will be same every n plus 1 vectors in V are linearly dependent, and there exist n linearly independent vectors, vectors in V. So, the proof is similar to that of one. Now, these are the statements about linear independence and dimension.

Now, similarly I will write down two analogous statements, which will involve generating system. So, third - V is finite dimensional of dimension less equal to 1 less equal to n, if and only if V as a generating system consisting of n vectors. If it has a generating system consisting of n vectors then by cutting down we make a dimension smaller equal. One more and in we will make a break. Again V is finite dimensional of dimension equal to n, if and only if of course, V has a generating system with n elements n vectors and every generating system has at least n vectors, and in every generating system should have more bigger equal to n vectors. So, we will stop for a break, and we will continue soon.

Thank you.