Linear Algebra

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Lecture – 18
Dimension formula and its examples

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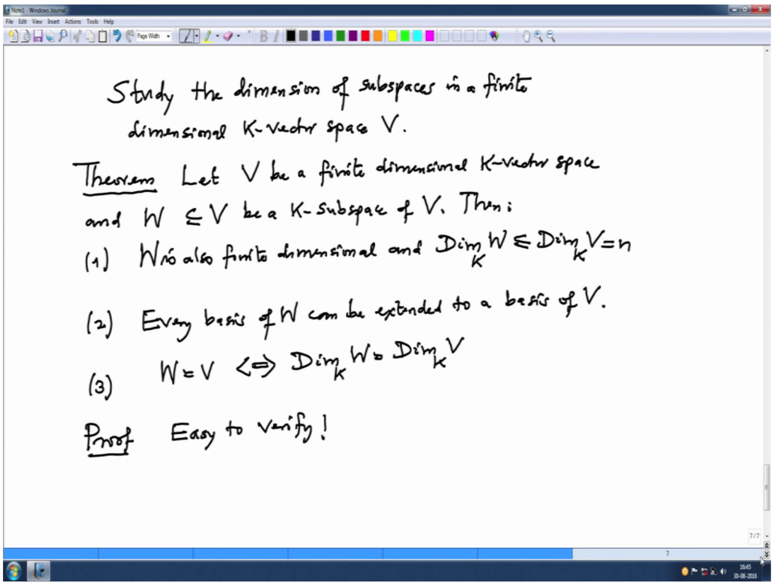

Now, in the second half of this lecture, we will study dimension of subspaces in a finite dimensional vector space K-vector space V. So, let us write in a form of theorem. Let V be a finite dimensional vector space K-vector space and W be a subspace of V K subspace of V. Then one - W is also finite dimensional, and dimension of W cannot be more than dimension of V. Is this clear? Because you will remember in earlier part, we have noted that, if we want to check somebody finite dimensional and dimension is less equal to N, then we need to check that every N plus 1 vectors are linearly independent or linearly dependent.

So, if you have N vector, so if you call this dimension of V as N if we take N plus 1 vectors in W in there also vectors in V and because dimension of V which will be N then will linearly dependent. So, whether they are in V or W linear dependency does not matter. So, second - every basis of W can be extended to a basis of V, this is also direct consequence of the exchange lemma exchange theorem, because you pick up a basis of W because this is a basis of W, they are linearly independent. And we can add some more elements in these to get a basis of with this was precisely the content of the exchange theorem.

Third one - if you want to check V equal to or W equal to the full space V, so W equal to V, if and only if the dimensions are equal. This is again follows from the fact that if we have correct number of vectors, you have to check that there linearly independent or basis or generating system that is a saying equivalent. So, this is also clear. So, proof, I will just say easy to verify.

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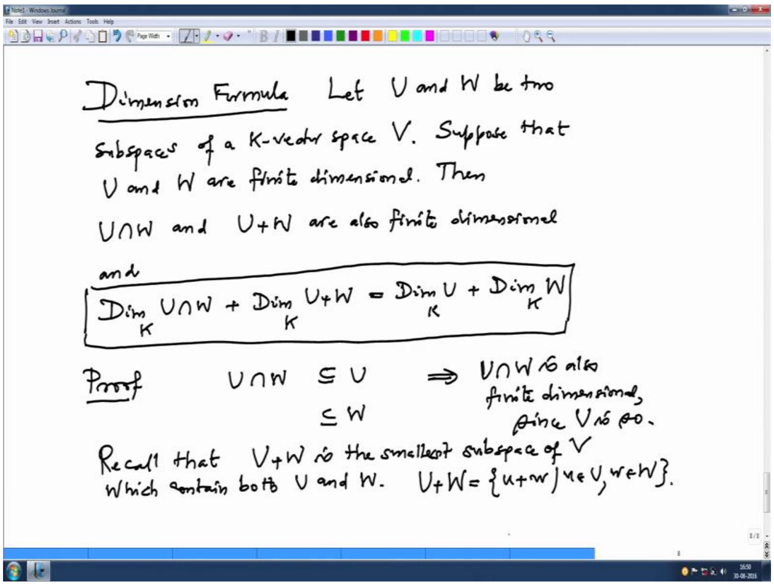

Now, I come to a very important formula, which we will use many times in future that also this also shows us how one can work with basis. So, this is easily called a dimension formula. It connects the dimension of intersection of two subspaces dimension of some of two subspaces and dimensions of them. So, let us write it very precisely, let U and W be two subspaces if a vector space K-vector space V we do not need to assume is finite dimensional we only need to assume that U and W a finite dimensional, suppose that U and W are finite dimensional. Then U intersection W and U plus W are also finite dimensional. And dimension of U intersection W plus dimension of U plus W equal to dimension U plus dimension W. Nice formula, this is called dimension formula. Later on, I will show you how it is use for many proofs.

So, let us try to proof this. So, first of all, proof. Note that this intersection U intersection W is a subspace of U also subspace of W and U is finite dimensional given and we have just check above that the subspace of a finite dimension is also finite dimensional. So, that prove that so first that proof that U intersection W is also finite dimensional since U is so. Now, to prove that U plus W is finite dimensional and this formula we will do together. I just want to recall you definition of U plus W, you remember recall that U plus W is the smallest subspace of V, which contain both U and W. This is a definition of U plus W.

And immediately after definition one can check that U plus W is precisely this sums U plus W, where u varies in U and w varies in W. So, to check these if we remember what we need to check the right hand side is a subspace and it contains both U and W and it is the smallest one. So, this is very useful too. So, now, let us try to prove this formula and together you this is finite dimensional.

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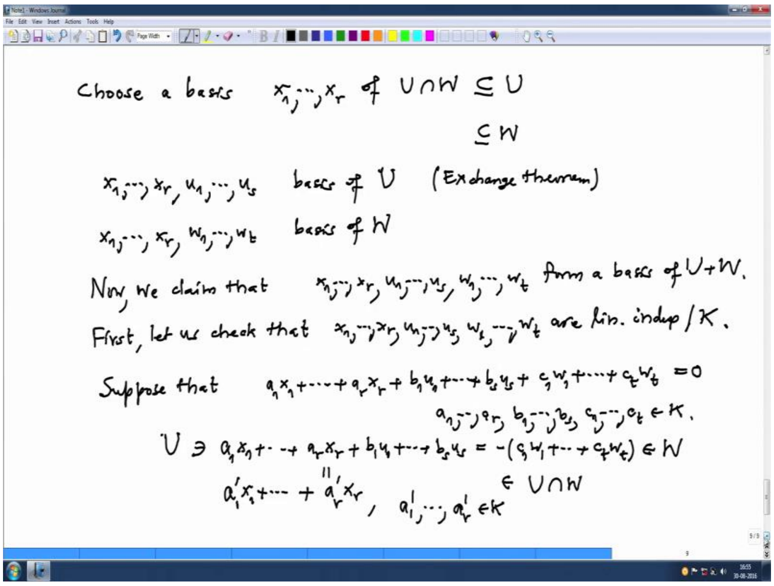

So, first what we do is choose we know every vector spaces basis. So, choose a basis x n to x r of U intersection w this is a subspace of U also subspace of W. Now, this is basis of a this subspace of U. So, I will extend this two basis of U by adding few vectors in that. So, x n to x r along with u n to u s basis of U, this I can do it by again by exchange theorem. Similarly, I do it for W also x n to x r w n to w t basis of W. Now, I put all of them together. Now, we claim that all together x n to x r u n to u s w n to w t form basis of U plus W. For that we need to check that they generate U plus W and they are linearly independent.

So, first let us check linear independence first let us check that these vectors x n to x r u n to u s w n to w t are linearly independent over K. So, suppose there is a linear relation, suppose that a 1 x 1 dot dot dot dot a r x r b 1 u 1 plus plus plus plus b s u s plus c 1 w 1 plus plus plus plus c t w t is combination is 0. Where a 1 to a r, b 1 to b s, c 1 to c t are scalars. And now our job is to prove that all a is all b is and all c is are 0. So, first I want to shift, shift these to the other side. So, that is a 1 x 1 plus plus plus a r x r plus b 1 u 1 plus plus plus plus b s u s equal to minus c 1 w 1 plus plus plus plus c t w t. Now, left hand side belongs to all these x 1 to x r u 1 u s it is a basis of u. So, it belongs to U the right hand side belongs to W, and there equal so that means, this they belong to U intersection w that means I can write these are the linear combination of x 1 to x r.

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Oh, they let us, ok I will try bigger. This means the left hand side, I can write as a linear combination of x 1 to x r so that means, I would write this as a 1 prime x 1 plus plus plus plus a r prime x r for some other constraints a 1 prime a r prime in k. But then these equal to these w s, but I know the x 1 to x r and w 1 to w t is a basis. So, they cannot have linear relation among them unless all coefficients are 0.

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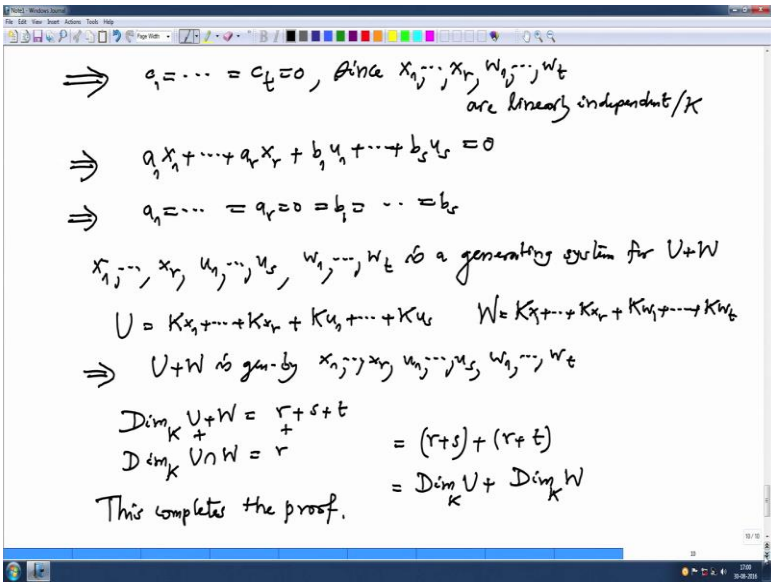

So, that implies a 1 equal to equal to equal to equal to c t equal to 0. Since, x 1, x r, w 1, w t are linearly independent over K. So, we have proved c’s are 0. Now, put back this in given linear equation K. Now, therefore, we get a 1 x 1 plus plus plus plus a r x r plus b 1 u 1 plus plus plus b s u s this is 0, because you already proves cs are 0. Now, but now x 1 to x r and u 1 to u s are linearly independent by our choice because that that form a basis of u, so that shows all the coefficient must be 0. So, a 1 to a r are 0, and similarly b 1 to b s are 0. So, all together we have proved that all a's are 0, b's are 0, and c's are also 0, so that proved that linear independence of x 1 to x r, u 1 to u s, and w 1 to w t.

Now, we want to check that this is a generating system. So, we want to check that x 1 to x r, u 1 to u s, w 1 to w t is a generating system for U plus W that means, you want to check that if subspace contain all these then it contains U and W also. But you know U is by our choice U is generated by x 1 to x r along with u 1 to u s, and W is generated by x 1 to x r and w 1 to w t. So, if some subspace contain all these guys, then it contains U and it contains W also.

So, therefore, it will contain U plus W also by definition, so that checks that U plus W is generated by x 1 to x r and u 1 to u s and w 1 to w t. So, we have proved that U plus W is generated by it is guy, they are linearly independent. So, it is a basis. So, dimension of U plus W is r plus s plus t, and dimension of U intersection W was r, because which it has a basis which as r elements. And what do want to prove you wanted to prove that some of their dimension equal to. So, the sum if I add here, I have to add here. So, the sum is therefore, r plus s or r plus s plus r plus t, but dimension of U is r plus s, and dimension of W is r plus t, because U has a basis x 1 to x r why u 1 to u s and W as a basis x 1 to x r, w 1 to w t, so that prove this formula also. So, it proves both together U plus W is finite dimensional of dimensional r plus s plus t, and the dimension for (Refer Time: 18:39). So, these completes the proof.

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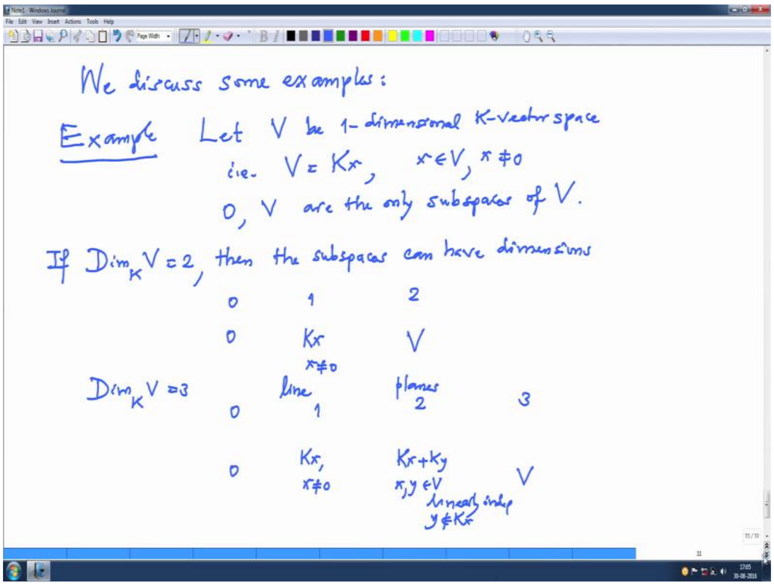

Let us I want to discuss till some more examples. We discuss some examples. So, example 1, suppose V is one-dimensional vector space, V is V be one-dimensional K-vector space that means, we have the basis consisting of one element. So, this means so that is V is K x, where x is any nonzero vector, 1 0 is a nonzero vector. It has to generate a nonzero subspace, and the only we know subspace is will have smaller dimension to the dimension of the subspace can either be 0 or 1. So, in this case, dimension 0 means 0 vector 0 subspace, and V they are the only subspaces.

Now let us take little bit bigger case. So, assume dimension V is 2. So, if dimension V is 2, then what are the possibilities for dimension of a subspace? So, then the subspace is can I have dimension the 0 or 1 or 2. If it is a two-dimensional subspace then the only two-dimensional subspace is V, because we have not reading somebody subspace of dimension equal dimension to ambient vector space they need has to be equal to that. One-dimensional subspaces are precisely generated by 1; it has a basis consisting of one element. So, it is of the form K x, where x is a nonzero vector; and 0 dimension is the 0 vector space, 0 subspace. So, these are the only possibility.

Now, these I want to generalize further to arbitrary dimension. So, for example, just to get a feeling, if dimension is 3, then what are the possibilities furthers dimensions of the subspaces either 0 or 1 or 2 or 3. That dimension 3 we know it is it has to be the whole V. Dimension 0 also we know it is the 0, 0 subspace. Dimension 1, it is generated by 1; it has a basis consisting of one elements. So, it is a basis, it is like this K x, x nonzero. Dimension 2 case, it we will it will have a basis consisting of two elements. So, if x, y is a basis of that then it is of the form K x plus K y, where now x y are vectors, and not arbitrary vectors they are now linearly independent.

So, you choose a pair of vector which are linearly independent and that will give you two-dimensional subspace. And how do you check to it is linearly independent or not just check that y for example, cannot be in the subspace generated by x, when it has to be linearly independent. So, this is the way one checks are two-dimensional vector space. Also the usual language which I will use later, that if it is one-dimensional subspace one usually calls it a line, two-dimensional subspaces are called planes and so on. Three-dimensional subspaces will be called three-folds and so on that will be the language, which we will use later on in the geometric situation. I want to generalize this into arbitrary, so that is I want.

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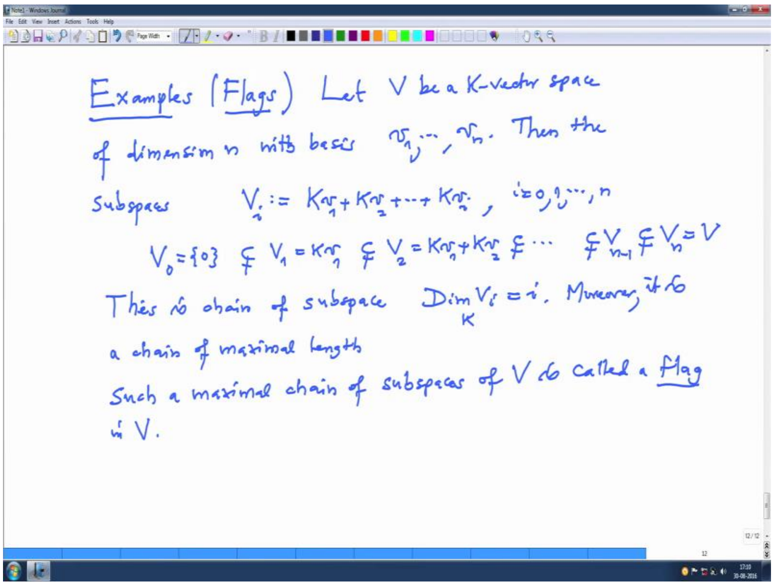

Let us write it another example. So, these are also called flags. So, let V be a K-vector space of dimension n with basis v 1 to v n. And let us look at the subspaces then the subspaces v i, v i is the subspace of V generated by v 1, v 2 etcetera to v i; and i is running from 0 to 0, 1 up to n. Note, when i 0; that means, this is a 0 vector, this is 0 subspace. So, v 0 is 0, v 1 is K v 1, v 2 is K v 1 plus K v 2 and so on, v n minus 1, v n is the whole V because v 1 to v n is a basis. And i teaches this inclusion is proper because this v 1 to v n are linearly independent. So, v i cannot be belong to the earlier subspace generated by the earlier case because otherwise v 1 we will get a linear dependence relation between v 1, v 2 etcetera v i that. So, this is a proper. So, this is a chain of subspaces and I teach it is dimension of v i is precisely i, this is a sending chain at each stage subspace will have dimension i, at each i the subspace of dimension i and this is a maximal chain, more over it is a chain of maximal length.

This is a chain of length n, and we cannot in set anywhere any other subspace, so that it becomes longer then n be is just because if you insert then what we will be the inserted subspace dimension, you see it should be bigger equal to i and it should also be smaller equal to i plus 1. So, there in the possibility, so this is such a thing is called a flag. So, such a maximal chain of subspaces of V is called of flag in V.

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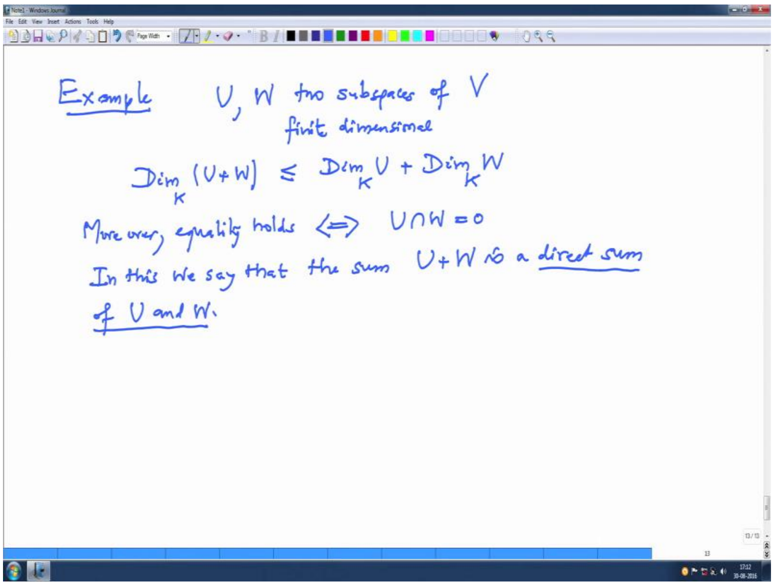

So, some more examples, so let us also look at next one. Suppose, U and V, U and W are two subspaces of V, and suppose U and both are finite dimensional. Then we know from the dimension formula, it follows there the dimension of the sum is bounded by the sum of the dimensions, because what we proved is this place dimension of the intersection equal to this equality. So, any case and I am now concern when will this equality hold. So, this equality, equality more over equality holds, if and only if the intersection should be 0 subspace, because 0 subspace will have dimension 0, and then will be equality. In this case, we say that the sum U plus W is a direct sum of U and W. We will use this concept more in later lectures. So, I think I will stop and we will continue in the next lecture.

Thank you.