Linear Algebra
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Lecture – 19
Existence of a basis

So, welcome to this lectures on Linear Algebra. Recall that last time, we have been studying vector spaces and subspaces and so on. And last time we have proved existence of a basis of a vector space under the assumption that it has a finite generating system. Today, I want to prove this theorem for arbitrary spaces. So, today's lecture we will prove the existence of.

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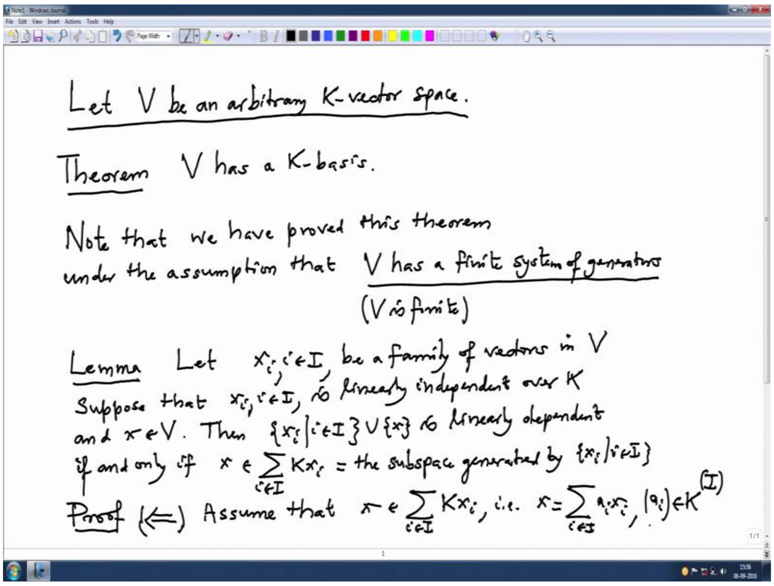

So, let V be an arbitrary space K vector space. And we proved that this lecture we will prove very, very important theorem that V has a basis. Note that we have proved this theorem under the assumption that V has a finite system of generators. In this case, I will also keep saying shortly V is finite does not mean that V has a set is finite, but it has a finite generating system. Now, we want to do it arbitrary, so and then in the second part, we will compare the size of the basis, size of any two basis. For that, we will need concept of cardinality etcetera, this I will recall when we will need it.

So, I am going to improve the lemmas which we have proved under the assumption that V is finite. So, first of this is, so let us call this has a lemma we say the following. Let I will not keep saying let V be an arbitrary vector space over a field K that is standing assumption. So, let x i, i in I be a family of vectors in V. Assume that suppose that x i, i in I, is linearly independent over K. And x is an another vector in V then this family x i, i in I along with x is linearly dependent if and only if the vector x belong to the subspace generated by x i's. Recall that this is our notation for the subspace generated by the family x i. And by definition, it is a smallest subspace which contain all the vector x i's, and elements of this subspace are precisely the finite linear combinations of x i's.

So, let us prove this proof. Proof is similar to the case when V is finite, but let us write down the proof for self continents. So, first I am proving this. I am proving this that means, I am proving the fact that if x belong to. So, assume that x belongs to the subspace then what do I want to do, I want to produce a nontrivial relation among the family x i along with x. So, this means x belong to this means x is a linear combination, a i x i, i in I where the coefficient tuple a i belongs to k round bracket I. And again recall that k round bracket I means only finitely many components in the tuple a i are nonzero.

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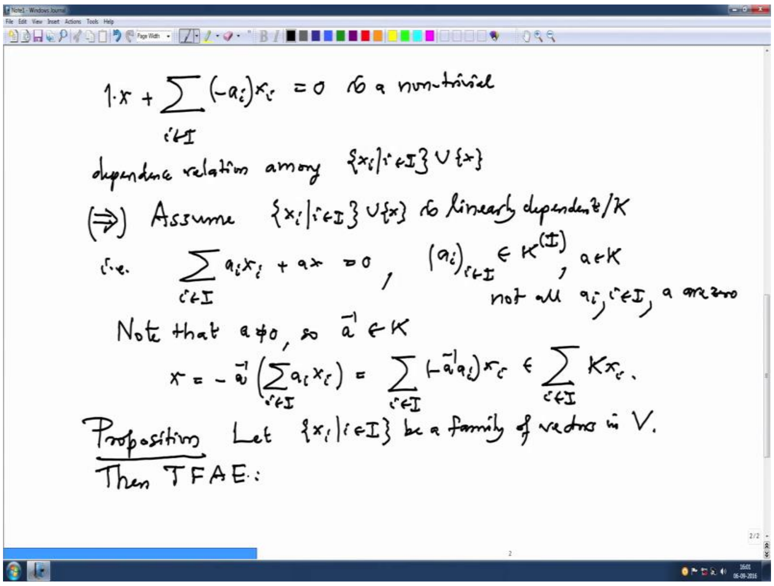

Then you keep you shift everybody to the other side and equal to 0, so that means, one times x plus summation i in I minus a i x i, this is 0. This is a nontrivial dependence relation among the family x i along with x a nontrivial, because these coefficient of one is coefficient of x is 1, which is not 0. So, therefore, we prove that this family this extended family when i join x 2 the family x i this is linearly dependent. Now, the other way rewrite this, now we are assuming the family x i these this family is linearly dependent so we are assuming x i, i in I union x is linearly dependent over K always over K so that means so that is we have a non trivial linear dependence relation; that means, we have a relation like this a i x i plus some a x is 0, i in I. And as usual this coefficient tuple a i this tuple even k round bracket I.

So, note that because these family x i's is linearly independent this a has to be nonzero. So, first note that a has to be nonzero. So, a is also in K and because this is nontrivial dependence relation not all a i, i in I and a are 0. At least one of them in nonzero, but I claim that that nonzero guy this a must be nonzero, because if a were zero then we will get summation a i x i 0 and because x i's are linearly independent then a i will be 0 then all these will be 0, but that is not true. So, a i is nonzero. So, a inverse make sense in K, because K is a field. And now keep rewrite this equation by shifting x to the other side multiplying by inverse. So, we will get x equal to minus a inverse a i x i, i in I, and by using usual vector space means we get this is minus a inverse a i x i, i in I. So, this right hand side clearly belongs to the subspace generated by x i's. So, that is what we wanted to prove. If this have extended families linearly dependent then x must be in the subspace generated by x space that was to be prove, so that proves a lemma.

Next proposition is similar to what I have proved in finite case, so that is a proposition. So, let x i, i in I be a family of vectors in V, and this proposition tells the equivalent conditions, how does when check this is a basis. So, then the following are equivalent.

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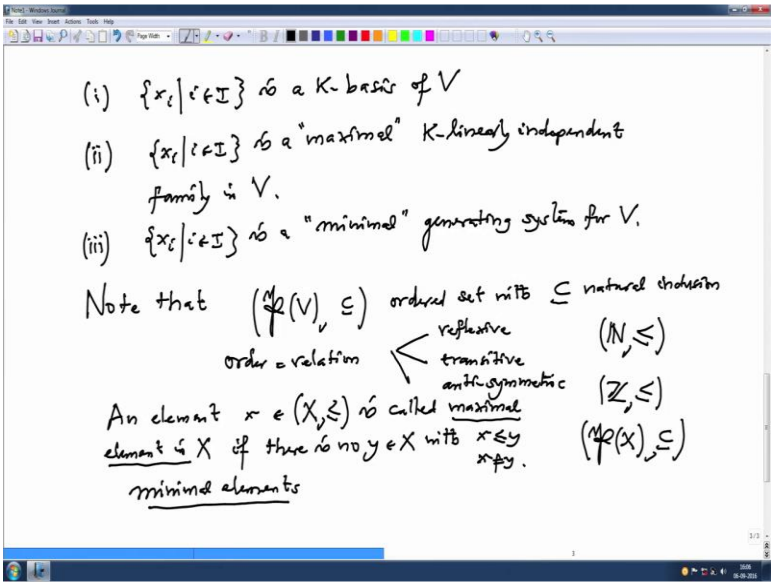

One - x i, i in I, is a K basis of V, this is what we are interested in finding the basis. Two - x i, i in I is maximal, I will just proved this in quote in maximal K-linearly independent family in V. Three - x i, i in I is a minimal generating system for V. So, now let me just repeat the maximal and minimum, because we are not in a finite case. So, everything is happening in the power set of. So, note that, so before I indicate the proof recall or note, note that we are power set of V with the naturally inclusion this is an ordered set, ordered set with this ordered naturally inclusion. An ordered set means three things order is a relation. So, order on a set is a relation which set satisfy three property, it is a reflexive, transitive and anti symmetry. A model prototype example is a set of natural numbers with the usual less equal to this is the prototype example or z with integers with the usual let say equal to these are prototype or another example is a power set of this set with the inclusion.

Now, when one says maximal that means, one says somebody is a maximal element in an ordered set, if no proper bigger set or nobody maximal then that is there. So, let me write formally an element x in the order set x less equal to is called maximal element in x, if there is nobody bigger equal to x. So, if there is no y in x with x less equal to i and x is not y. So, such a element is called a maximal element. Similarly, analogously we can define minimal element. And ordered set, can I have more than one maximal element, when there is only one maximal element that is called the maximum, and similarly, for minimal example.

So, this second condition say that this family is linearly independent and no strictly bigger family then this is linearly independent, so that mean if O add any extra element to this family then it becomes linearly dependent. Similarly, three that it is a minimal generating system means it is a generating system and if I remove any element from the generating system, it cannot be a generating system.

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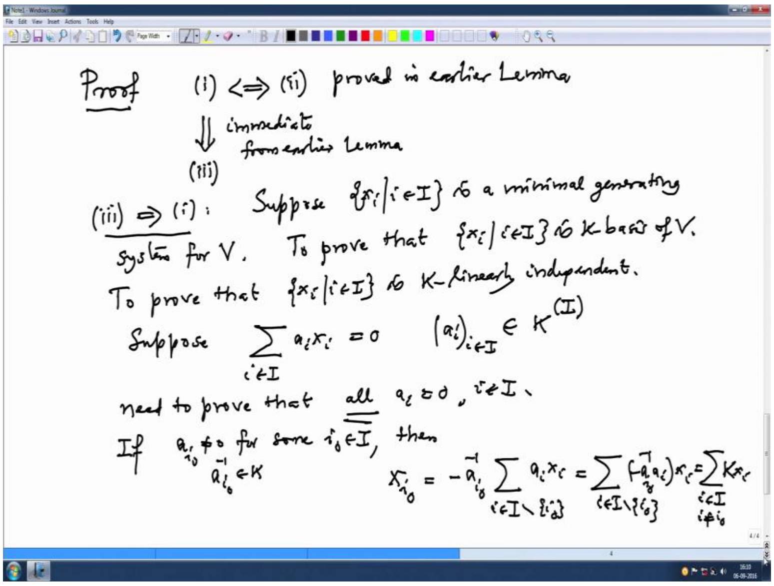

So, now let us quickly prove it, it is very simple. So, proof, for example, one if and only two actually is proved in the earlier lemma that is precisely earlier lemma. Similarly, one implies three, this is also immediate from earlier lemma. So, only to prove, therefore three implies one, this is a only implication we need to prove. And what is three, three is given. So, suppose this x i is a minimal generating system for V. And we need to prove to prove one that is prove to prove that it is a basis x i, i in I is a K basis of V. We need therefore, to prove that this family is linearly independent, because linearly independent generating systems are precisely called a basis.

So, to prove that this family x i, i in I is K-linearly independent so that means, we suppose there is a dependence relation and prove that each coefficient is 0. So, suppose we have a relation linear relation like this a i x i equal to 0, i in I, and the tuple i tuple a i should have only finite element in nonzero components, and we need to need to prove that all a i's are 0. We need to prove this all. So, suppose some a i's were nonzero, if let say a i naught is nonzero or some i naught then we need to get a contradiction. Then again the rewrite this dependence relation by keeping this term corresponding to i naught on one side, and shift to remaining terms to the other side, and because this a i naught is nonzero a i naught inverse exist an each and element in K.

So, I rewrite and multiply that equation by a naught inverse. So, we will get equation x i naught on one side equal to minus a i naught inverse summation a i x i i in I and remove i naught from there because i naught brought it this side and multiplied by them with that as usual vector space rule. This will tell you minus a i naught inverse a i x i, i in I and i naught, but this is a subspace generated by the remaining vectors other than i naught, this is K x i, i in I and i in not equal to i naught. So, this is a contradiction to the fact that it is a minimal generating system because this means you can drop x i naught, that is not possible because assumption three is says it is a minimizing generating system, so that proves three implies one.

So, we have proved the lemmas. Now, first I will state a theorem and maybe I will need to recall some something again from the ordered sets. And also I mean in this lecture we will need to use Zorn's lemma which is very important lemma which is also equivalent to (Refer Time: 21:42) and choice and so many other statements in set theory. But Zorn's lemma is the statement which ensure the existence of a external elements in an ordered set and these I will elaborate on these when we just before we prove the theorem.

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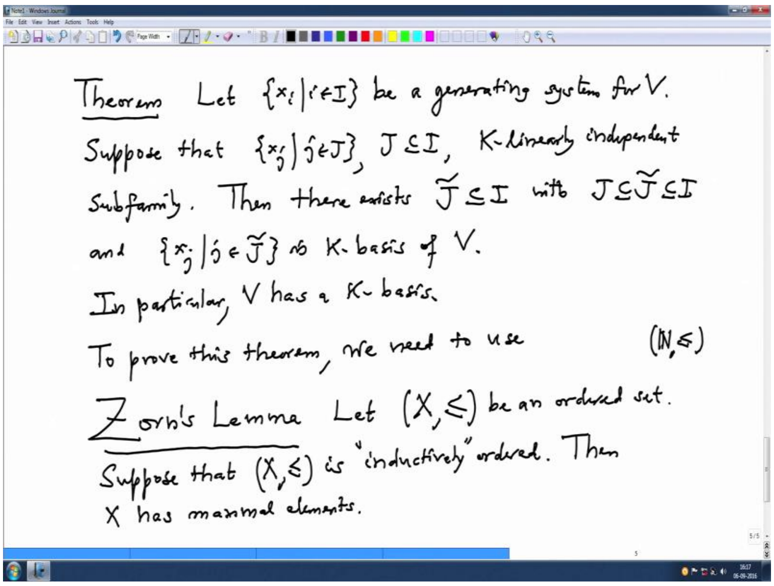

So, the main theorem I want to prove is the following and we will deduce over theorem from this theorem. So, this theorem says, so let x i, i in I be a generating system for V. And we want to process to either to minimize it or to maximize, the linearly independent sets. By earlier proposition, we know that minimal generating system with the basis or maximal linearly independent set is a basis. So, we have to find a method, so that either we can minimize the generating system or maximize the linearly independent sets. So, suppose that and we can always choose a generating system for any vector space because worst come to worst, we can take all element all vectors in that system that is clearly a generating system it may be too big. So, we want to have some process.

So, suppose that sub family of this x j j in j where j is a subset of i such a thing is called sub family, because we are taking the few elements on that given family. Suppose that this is K-linearly independent sub family, these also again possible. Worst come to worst, we can take empty set J equal to empty set is empty families always linearly independent. Then what is the assumption then we can enlarge with J there exist J tilde to subset of y with it should contain this given J and the sub family indexed by J tilde is K-linearly independent, now not only K-linearly independent, but maximal is for K basis of V, so in particular V as a basis. So, to prove this theorem, we need some mechanism to enlarge this family to a maximum possible so that means, we are looking for a maximal elements.

So, to prove this theorem, we need to use Zorn's lemma. Zorn's lemma as I said Zorn's lemma gives a condition on a ordered set, so that it has maximal elements. So, let me state Zorn's lemma first. So, let x less x less equal to this be an ordered set, we need some assumption to put on these ordered set, so that we can infer the maximal elements; obviously, you see the standard model what we know n less equal to as you know these does not time maximal elements. So, arbitrary ordered set will not working general. So, I will write the term and defined the term. So, suppose that x less equal to is inductively ordered, I will define these term inductively. If you have these assumption, then we can say that then x has maximal elements.

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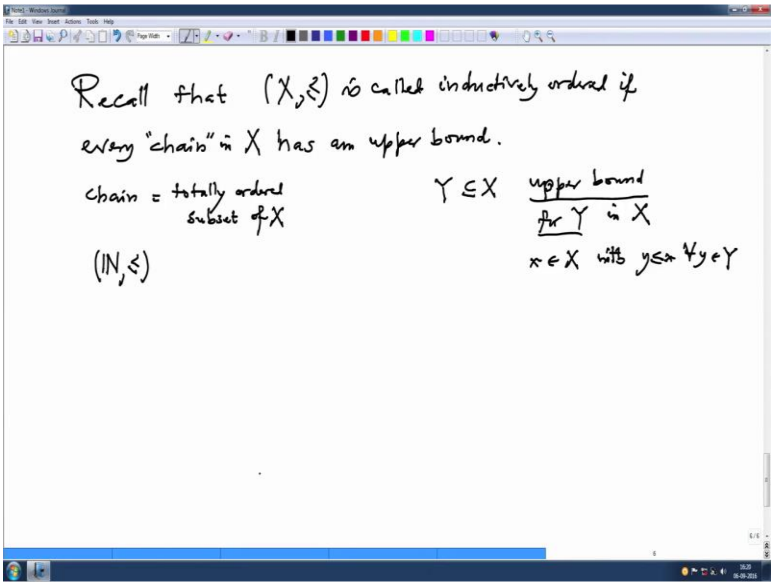

Now, let us let us recall what is inductively ordered. So, recall that X less equal to is called inductively ordered if every chain in X has an upper bound. So, again let us recall what is the chain in an ordered set. Chain in an ordered set is precisely a total ordered subset, totally ordered subset of X is called a chain. Or typically for example, if you take this N less equal to this already chain whole X is a chain in this case, but finite subsets is also chain and so on. Finite subsets of any ordered set is it will have obviously, a maximal element because you can compare. It may not be chain, but the association that it has a maximal element can be easily check for a finite ordered check.

What is an upper bound for a chain? So, in general, if we have a subset Y of X upper bound for X, upper bound for Y, for Y in X is an element x which may or may not be in y with the property that y should be less equal to x for all y in y. Then such a thing is called an upper bound for y in x. So, now I guess all the terms are clear in this. And we will continue the proof of the theorem by using this Zorn's lemma in the next half.

Thank you.