Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 20
Existence of a basis (continued)

Before I go to the proof of above theorem, I would also like to give couple of remarks about Zorn's lemma.

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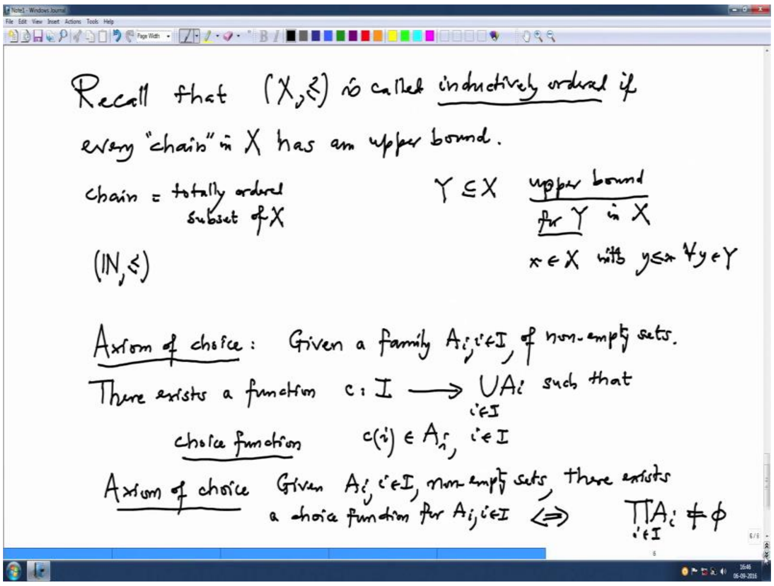

So, namely see one can proof Zorn's lemma by using axiom of choice. So, let me briefly real axiom of choice which is more intuitive statement which is very easy to believe, but it is also equivalent to Zorn's lemma. So, axiom of choice; so, let us recall if you are given a family A i of non empty sets, there exist function C from the indexing set I to the union A i such that the image of any I under C of I should belong to the corresponding set A i such a function is called a choice function.

So, we are making a choice and picking up elements from A i and then you mapping I to the A i. So, the existence of axiom of choice is precisely a statement which says given family of non empty sets, there exists a choice function for this family. So, this Zorn's lemma what I stated above if equivalent to this axiom of choice and this axiom of choice restatement is a precisely. So, this is equivalent saying the product say A i this is non empty that is axiom of choice because an element of the product set is the tuple and the tuple will give you the function.

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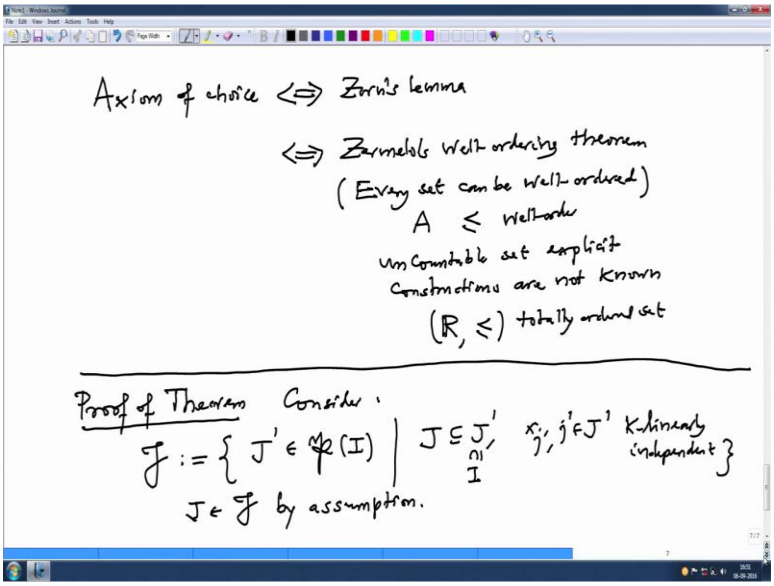

So, if the tuple here that will correspond to choice function. So, the normal course on abstract set theory will have such implications axiom of choice is equivalent to Zorn's lemma and also equivalent to another means full theorem which is called Zermelo's well ordering theorem. This Zermelo's well ordering theorem say that every set can be well ordered; that means, on every set we can put an order which is a well order; that means, well order means every non empty subset of that set as the minima.

So first, though this is an abstract theorem given any set a we can define n order on that we can construct an order which is a well order this construction is not explicit this is an abstract construction and even further non uncountable sets explicit constructions are not known even on the set of real numbers we know this usual let us equal to this is an order this is a total order totally ordered set, but this is not well order because now if you take a set like subset like open intervals 0 1.

For example this is a subset the minimum. So, or even whole set real numbers even if you take a set of real numbers there is no minimum now. So, this order this usual order is not a well order, but a Zermelo theorem say that there exist a well order on, but how to explicitly construct that is not known.

So, this so far, now I am going to use this Zorn's lemma. Zorn's lemma is the most handy way to check some order set as maximal elements or not and once you have maximal also we can infer about minimal elements by proving the theorems to the opposite order set. So, it is usually consider one only concentrate on the maximal elements. So, so let us come back to the proof of the theorem we stated proof of theorem remember what we need to do is we need to we have given a generating system and we have given a part of that generating system which is linearly independent and we want to maximize this we want to put a maximal. So, we have to come to a situation we have to extract from this we have to get a ordered set which is should be inductively ordered and then use Zorn's lemma to conclude in the maximal elements there and this maximal elements should be required basis that is the idea.

So, consider a set skip J, I will call it this is by definition, I collect all those J primes in p i on the power set of I; that means, all those subsets J prime of i such that J should be containing J prime and the family x J prime J prime in J prime. So, the J prime will remember this is in p i mean this is contained in i this is K linearly independent. So, J this scripts J is precisely all families of the given generating system which contain the linearly independent family sub family and it enlarge to a linearly independent family these are precisely J prime. So, by a assumption that these set J belongs to J, the script J because we have given the sub family indicates by the J is linearly independent.

So that means, this J belongs to J prime this J belong to script J this is by assumption.

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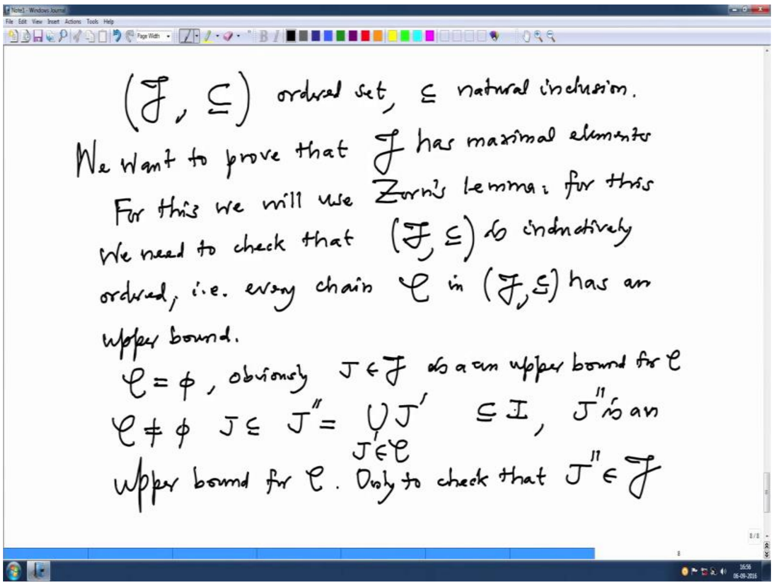

Now, I want; so this J a script J is; obviously, ordered by a natural inclusion this is on ordered set this J is this natural inclusion is this is natural inclusion and we want to check that these ordered set as maximal elements we want to prove that J has maximal elements for this we want to use Zorn's lemma for this we need to use we will we will use Zorn's lemma. But for this we need to check the assumption given in Zorn's lemma we need for this we need to check that these ordered set J inclusion this is inductively ordered; that means, every chain in this script J has an upper bound.

So; that means, we want to check that every chain C in this ordered set has an upper bound. So, first let us take suppose C is empty chain then; obviously, the J which was given to a C in assumption that is an upper bound is an upper bound for C, note that upper bound just means it should contain every element of C. So, elements of C elements of C are subsets. So, upper bound means subset which is in j, but it should contain all elements of C. So, that is an upper bound for C.

So, suppose C is non empty then I am going to look at the union of all elements in J this one note that because C is non empty this union makes sense C, C where empty union will have problem. So, because C is non empty this J prime make sense and it; obviously, contains it is a subset of I; obviously, contain J because each J prime is in C and C is a chain in this and all elements of J contains j. So, therefore, J is a subset of J double prime and J double prime is a subset of I.

Now, we also it is clear that J double prime is an upper bound for a C, but only to check therefore, only to check that the J double prime should also be in script J then only it is an upper bound for C in j. So, you have to verify now that J double prime is in the script j, but J is contained in J double prime is a subset of I. So, on therefore, only thing to check only real checking is.

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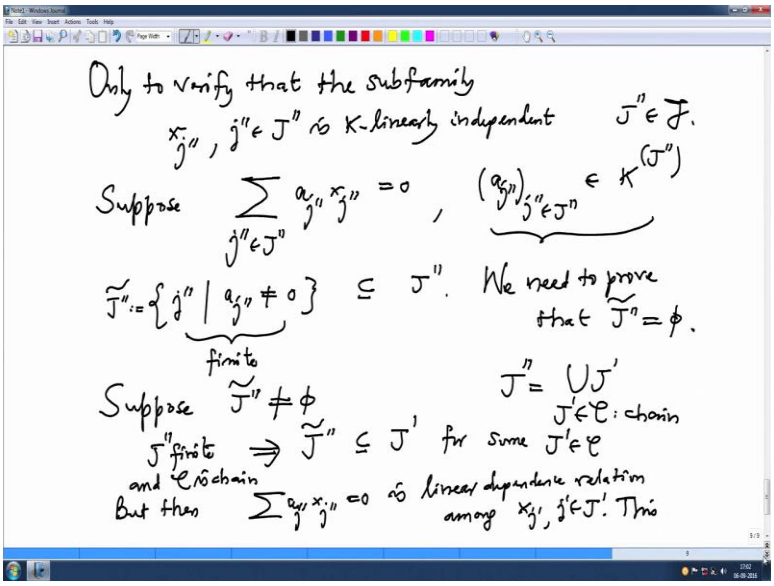

So, only to verify that the sub family x, x J double prime as J double prime varies in J double prime is K linearly independent because this will ensure that J double prime is indeed an element of script that was a definition of J script J.

So, we want to verify that this family is linearly independent. So, what does that means, that means if you have a linear independence relation then all coefficients are 0. So, suppose we have a independence relation like this a J double prime x J double prime is 0 this sum is varying over J double prime in J double prime and; obviously, a J double prime this tuple J double prime in J double prime this is in K open bracket J double p rime this as usual this simple means only finitely many coefficients are nonzero.

So, look at this set although J double prime for which a J double prime is nonzero this is some subset of J double prime and this is a finite set this is a finite subset of that J double prime because only finitely many of them are non zero and what we want to prove is this set is actually empty set let us call this set as what can I give name let us call this as J tilde or J double prime tilde. So, we need to prove that J double prime tilde is empty set;

that means, all a J double primes are 0 and then we would approved that they are linearly independent.

So, suppose it is not then we should lead to contradiction. So, suppose J double prime tilde is non empty what is a finite set and they are all elements of J double prime and J double prime. So, now I want to use a fact that this J double prime is a union of J prime this J prime is varying in C and I want to use this fact that C is a chain; that means, if i given a finitely many elements in J double prime they will belong to this union finitely many elements, but among the J double prime in J j prime is varying in C. So, they have a inclusion relation among them.

So, by without loss without loss of generality this J double prime tilde will be contained in one of the J prime not by this is a wrong word without loss. So, because J double prime is finite and C is a chain it follows that this J double prime tilde will be in one of the J prime because if for example, if J double prime as only one elements this is obvious if J double prime as more than one element to they will belong to 2 different J primes, but the J primes are comparable. So, I will choose a bigger one. So, definitely this, but then this relation this relation will become a dependence relation so, but then, but then this a J double prime x J prime equal to 0 is a dependence is a linear dependence relation among x J prime J prime is in J prime which contradicts this contradicts these contradicts the choice the J prime belongs to script J.

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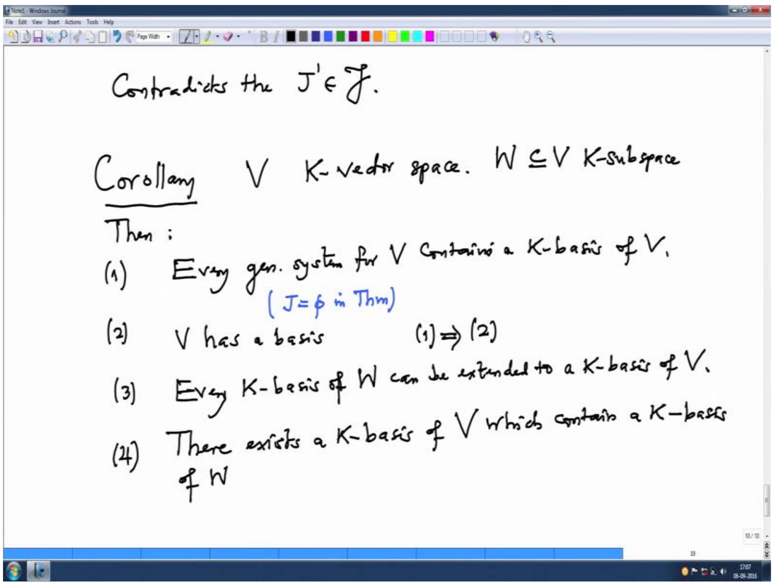

So, that proves what we wanted to proof that prove that I can enlarge given linear independent family to a maximal linear independence family and by then previous lemma it is a basis. So, that that proves the theorem that we wanted to prove. Now I want to write one corollary. So, one corollary, I want to write corollary is as usual V is a K vector space this corollary in this corollary I have collected a special cases of the above theorem which we handed to use. So, and W is a subset of V then one every generating system for V contains a basis K basis of V because take the generating system and find maximal linearly independence of set by the above theorem and this much be a basis.

So, second; so this I will write here this is a special case, I will write the; I will write the blue in the bracket here this is a case where J is empty set in a theorem because that empty set is linearly independent and you make it maximal above the theorem. So, this thing exists and that is a basis. So, V as a basis; V as a basis, so these 2; 1 implies 2 is clear. So, 1 is the special case J equal to empty set in the theorem third one every basis every K basis W K basis of W can be extended to K basis of V.

So, K basis of V is linearly independent family of w, but if it is linearly independent family of W it is linearly independent in V also. So, apply again the theorem to this. So, for there exist a basis K basis of V which contains a K basis of W, so this is also this statement of 3. So, now, I just want to give one remark here. So, remark.

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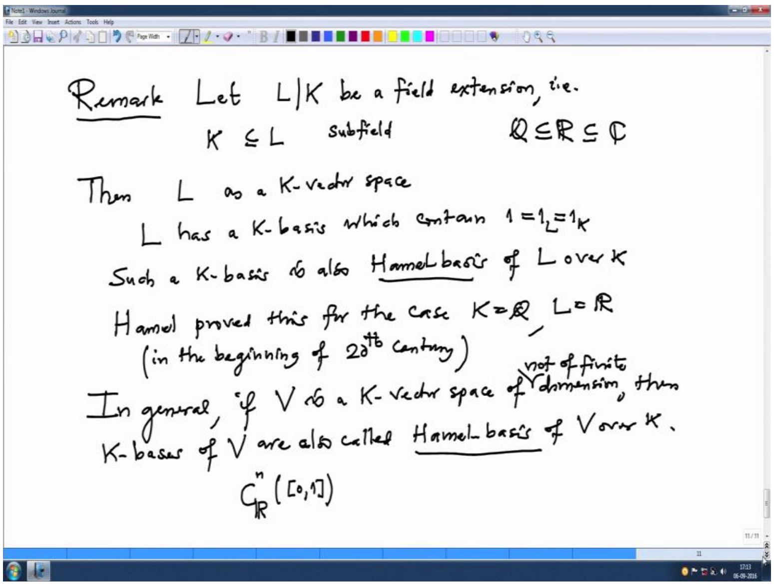

So, suppose I have a field extension. So, let capital L or capital K be a field extension that simple means. So, that is K is a subset of L and it is a sub field that mean the operation in the plus and product multiplication operations in K are restriction of the additions and multiplication in L.

Now, typically example is Q contained in r. So, R is a field extension of Q similarly C is a field extension of R or C is a field extension of Q. So, field extensions are very important in the study of theory of equations and for field extension you also need to use a vector space argument many times. So, then what we have just proved above that L if you look at L as a K vector space it as. So, L as a K basis which contain 1, remember one is same thing as 1 of L or 1 of L is same thing as 1 of K multiplicative identities are same because it is a field.

So, we can extent this means we can extend this set one remember one is linearly independent over K and this linearly independent set we can extend it to basis by the earlier theorem this is very very important. So, such a basis; such a K basis in the case so in the; is also called Hamel basis of L over K Hamel where the first Hamel proved this for the case K equal to Q and L equal to R. This was proved in the in the beginning of 20th century.

So, with this also people in general in general now this is very common language in specially in function analysis in general where the not finite dimensional vector spaces are involved in general if V is a K vector space of not a finite also of dimension not finite dimension not of finite dimension then we; I have just put that V as a K basis the K basis of V are also called Hamel basis of V over R V over K remember lot of spaces, we have done we have did in the example above namely n times continuously differentiable real valued functions on say closed interval this is a R vector space and this is not finite dimension.

So, we know by (Refer Time: 30:06) theorem, this as a basis and the one can also prove that the basis actually is not countable. So, such basis are called Hamel basis. So, now still we need to prove the following theorem. So, like a finite dimensional case. So, we now need to prove I will just say it orally we now need to prove that any 2 basis have the same cardinality now for this we will do this in a next lecture, but for this, I will have to recall little bit more precisely the concepts of cardinalities and how does one prove the 2 sets have same cardinalities or not this we will do it in the next lecture.

Thank you.