Linear Algebra
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Lecture – 21
Existence of a basis [continued]

Come back to this linear algebra lectures, I will continue from the last lecture. Remember last lecture we have proved that if V is a K vector space; K is arbitrary field then we have proved that V has a basis.

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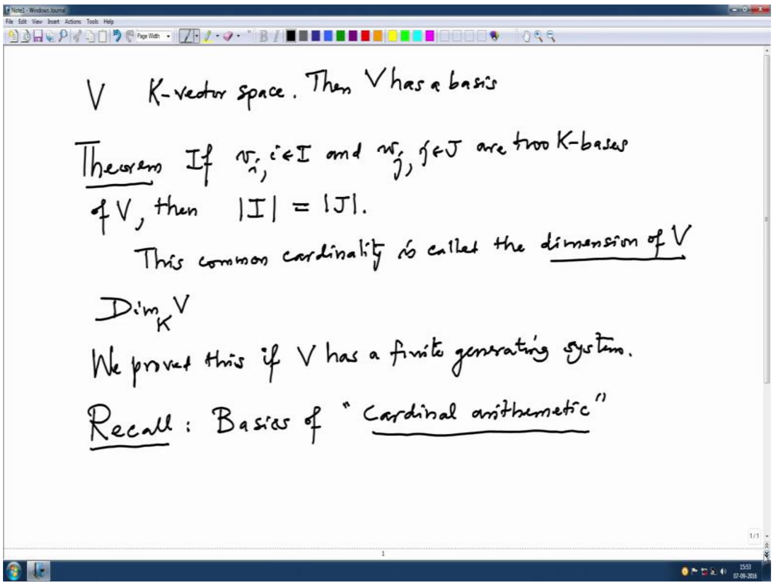

In case if V has a finite basis then we will say V is finite dimensional in case if V has infinite basis then we will say it is not finite dimensional or we will say it if infinite dimensional, but in this lecture, I want to prove that any 2 basis of V have the same cardinality. So, the theorem I want to prove the following theorem; if v i i in I and w j j in J are 2 basis 2 K basis of V then cardinality of I equal to cardinality of J.

And this common cardinality this common cardinality this common cardinality is called the dimension of V usually denoted by dim K V. We have proved this theorem in case when i is finite. So, we have proved this we have proved this if V has a finite generating system note that in that case we have noted that every basis is finite therefore, i is finite j is finite and further we have proved exchange theorem and there by proving they are cardinally there equal and that if the dimensioning case of if this finite generating system.

Now, today I want to prove this without this assumption, but before I prove that I need to recall precisely the result which concerns about the cardinalities of the sets in general this is necessary because when we want to prove things more precisely we should also know the definitions more precisely and their use. So, I want to recall basics of what is called as cardinal arithmetic this is what the basic; mostly I will tell the definitions clearly and the results I will not prove that many of the results we will need to use Zorn's lemma or axiom of choice.

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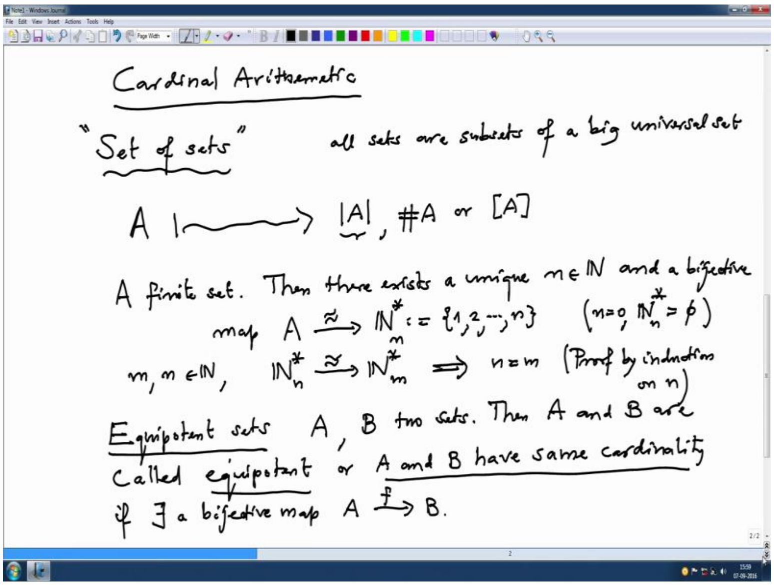

So, I will skip the proofs, but I will write the; proves in the notes. So, let us recall this is cardinal arithmetic. So, the idea is the following. So, first of all before I start talking precisely I will keep to a word the so called famous paradoxes in said theory. I will combine myself to saying that set of sets and this means this means you see if I talk generally then we might land into paradoxes. So, this means that all my sets are all sets are subsets of a big universal set which I will not mention what that begin also set is, but it is understood that all are subsets of some big sets. So, therefore, for that I will keep using the phrase set of sets.

So, for each element of this set say each element is a set and we have a collection of sets. So, to each element of set or each set a I am going to attach some object which either I will write like this several notations in the literature. So, either it is written like this or it is written like this or also written like this and so on, but I will use this notation. So, what is it first of all node that in a finite case if a is a finite set; that means, they are finitely mean elements and crudely speaking these number is called the cardinality of that finite set, but when you go to infinite case this crudeness will not be sufficient.

So, we will have to come up with better notation and also better definition. So, first of all you note in the finite case if a is the finite set then a there exist let me write precisely this statement then there exist a unique natural number N such that a to N star n; N star N is by definition 1 to N then there exist a unique N and a bijection bijective map that I will denote sometime like this. So, note that this needs approve and the proof usually goes like this that this proof is. So, to prove this what one need to node that if we have 2 natural numbers m and N and if we have a bijection form N star N to N star m bijection then we have to conclude N equal to m.

These will needs a approval and this one can prove easily by proof by induction on n. So, all finite sets are you can identify under bijection with this subsets 1 to N and note that when I say this if N where 0 N star N is by definition is N to set. So, this I want to imitate and defined concept of equipotency equipotent sets. So, you have 2 sets A and B 2 sets then A and B are called equipotent or they have the same cardinality or A and B have same cardinality if there exist a bijection there exist a bijective map from A to B f and in this case I will postpone this.

We call them equipotent now it is very clear the following.

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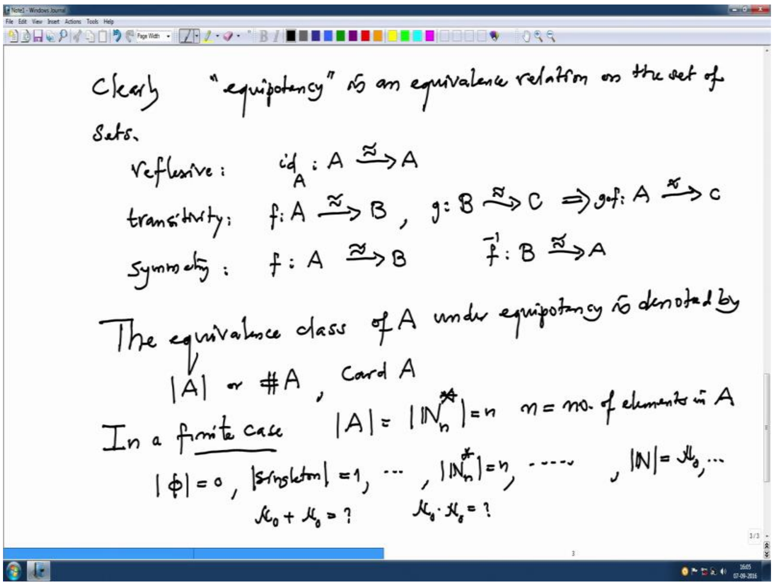

Clearly this equipotency is an equivalence relation on the set of sets; that means what. So, it should be reflexive reflexivities clear because identity morph of any set A to A, this is the bijection. So, A is equipotent to A transitivity, if A is equipotent to B; that means, if there is a bijection from A to B let us call it f there is the bijection g from B to C then we need a bijection from A to C, but that you can take the composition g compose f bijective 2 composition of 2 bijective f is bijective. So, they show that the equipotency the transitive relation.

Symmetry if a is equipotent to B; that means, if there is a bijective map from A to B then f inverse E exist and it is a bijective map from B to A so; that means, it is symmetry; that means, it is symmetry. So, this equipotency is equivalence relation on the set of sets. So, any equivalence relation we can talk about equivalence classes. So, the equivalence class of a under equipotency is denoted by either it is denoted by this symbol or also it is denoted by this or it is denoted by simply card. So, that is the equivalence class. So, now, let us see in a finite case what happens. In a finite case, this represents all the sets which i have in bijective correspondence with A.

So, in the finite case prototype of this equivalence class A, this is nothing, but one representative we know very well that is N star N where this N is the unique N is the uniquely determined in natural number who is say the number of elements number of elements in A, this is in a finite case. So, you start let us start few numbers. So, you see start with a empty set empty set is finite the number of elements is 0. So, it is empty set will give this the empty set this symbol in nothing, but 0 we can identify this with N for singletons, all singletons will fall in the same equivalence class and that this is 1 and, so on, all the sets which has precisely an elements will fall in one class and that class is same as this class and that is N and you can keep doing this as long as the sets of finitely mean elements.

The first time you come do a set which does not have finitely many elements that is the set of natural numbers and that this is usually denoted by aleph naught. So, aleph naught represents all the sets which I have cardinal which are in bijective corresponding with the N and then you can keep going it, so like this. So, these are called by abuse of language these are called cardinal numbers and we have studied the arithmetic in case of up to here you have studied arithmetic we have not studied for example, how do you add aleph naught plus aleph naught plus aleph naught whether what is the meaning of this or what is the meaning of aleph naught into aleph naught.

These we have naught these does not come in our arithmetic and let say finite arithmetic, but I want to now extend this arithmetic to arbitrary class. So, I should be able to talk what is their plus what is their product what are the rules like our usual arithmetic rules exponentiation and all other current cardinal arithmetic can be extended to this set up and the study this study is called cardinal arithmetic. So, this I want to digress not so much, but little bit. So, for example, also we want to also extend the order the usual less equal to order we want to extend to the; this arithmetic.

So, first of all I want to write; I want to first if you remember even when you studied the set of natural numbers even before addition and multiplication the concept of order wise introduce first and then it was use to study addition multiplication and there relation with the or order the natural less equal to monotonicity etcetera, etcetera.

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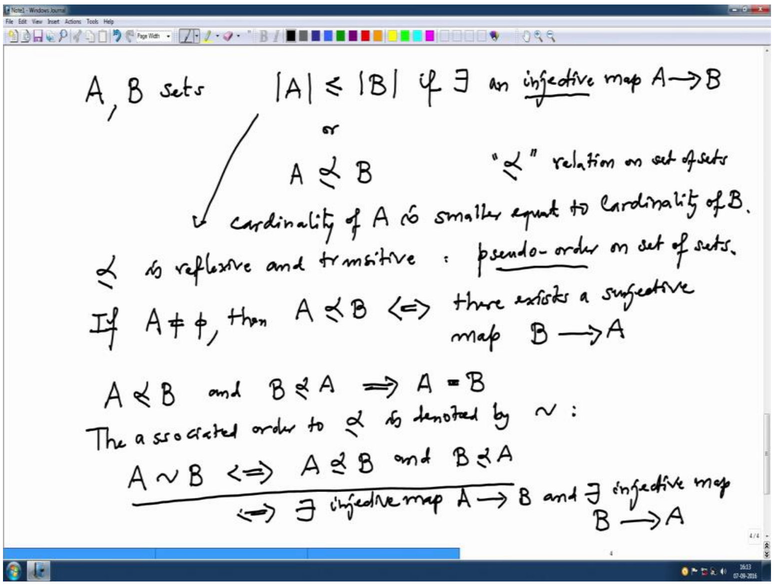

So, I am going to do; I am going to imitate the same thing. So, for example, if A and B are sets arbitrary sets then I will write this less equal to this if there exist bijective map, there exist and injective map; map from A to B infinite case it is clear then if there is an injective map the number of elements is less equal to number of elements in. So, this is this make sense.

And instead of writing this; I will also use the notation this. So, I would defined this is a relation on set of sets and in this case we will also said and the cardinality of A is smaller equal to cardinality of B. So, we will also say cardinality of is smaller equal to cardinality of B, now since we have done injective map we can also do the similar thing using the surjective map for example, we can translate this. So, note that this relation what I have said it is. So, what are the properties for this relation this relation properties are it is reflexive; reflexive is very clear identity map is injective and transitive because if we have injective map from A to B injective map from B to C then the composite map is also injective, but what is not true is symmetry is not clear symmetry is not true in fact.

So, such a thing such a relation is usually called a pseudo order this is a pseudo order on set of sets. So, one can also characterized this reflexive relation by using surjective maps for example, one can easily prove. So, if A is a non empty set and then a; this symbol B let us call it scrip less equal to if and only if there exist surjective map from B to A. This we also needs a proof, but this not to difficult, I am going to leave the proof all the proofs

I would add it in a notes. So, now, if we have a pseudo order then you can make it an order by pushing in more things. So, and the order we will corresponds to and equivalence relation. So, one minute; this is an order, but the order is not any (Refer time: 22:23) sorry, I have we have to make sure that it satisfy anti symmetry.

So, to make it anti symmetric we can we need like this A less equal to a script less equal to B and B less equal to a then we want a equal to B we need something like this. So, we push in see each pseudo order you can make it an order by pushing in this property. So, the associated order associated order to this pseudo order that is; so that I will denote the so; that means, what; that means, I have enlarge this relation and make it order. So, that one I want to denote by is denoted by this. So, what is the definition of this? Then even say that A tilde B if and only if A scrip less equal to B and B scrip less equal to A.

So, that is how we made we are extended this pseudo order to now order. So, this is our relation now. So, in other words what is let us spell it out what it means a tilde B if and only if there is a in injective map from A to B and also there is a injective map from B to A. So, this is if and only if there exist injective map from A to B this was the definition of this scrip less equal to and there exist injective map from B to A then we will call them A tilde B now; obviously, if we have this then A and B have the same cardinality that is that should be proved and this is precisely was conducted by canter, but he did not give a full proof in the proof was given by Bernstein; Bernstein's theorem.

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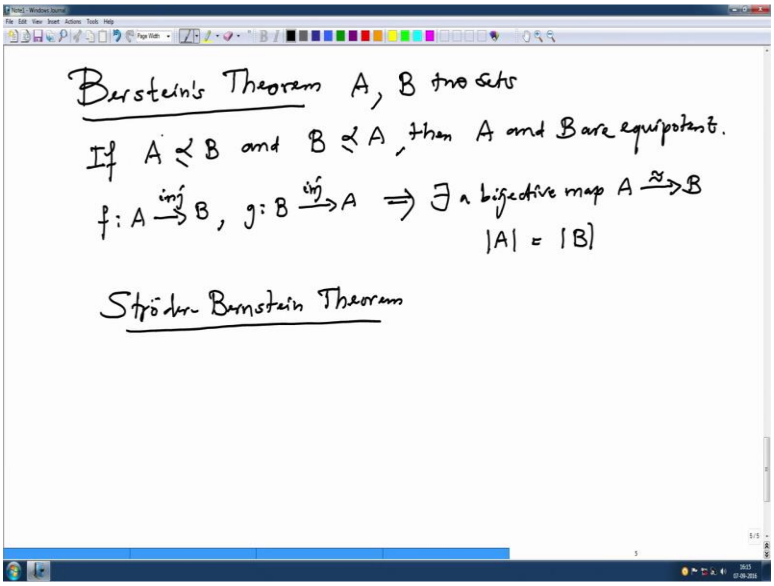

It says that if A and B are 2 sets; if a this scrip less equal to B and B scrip A then A and B are equipotent.

In other words if there the injective map f from a to B and then injective map from B to a then they the bijective map from a to B this means the cardinality of a equal to cardinality of B or a and B represent the same equivalence class this is very important theorem of Bernstein it is also called sometime some books give a Schroeder Bernstein theorem the same or Bernstein's equivalence theorem. So, this is what is very important it tells you how do we check that the 2 sets have the same cardinality to check that the 2 sets have the same cardinality what do we need to produce is an injective map from a to B and injective map from B to A then they have the same cardinality this is what we will need to use to prove our 2 basis of the same cardinality.

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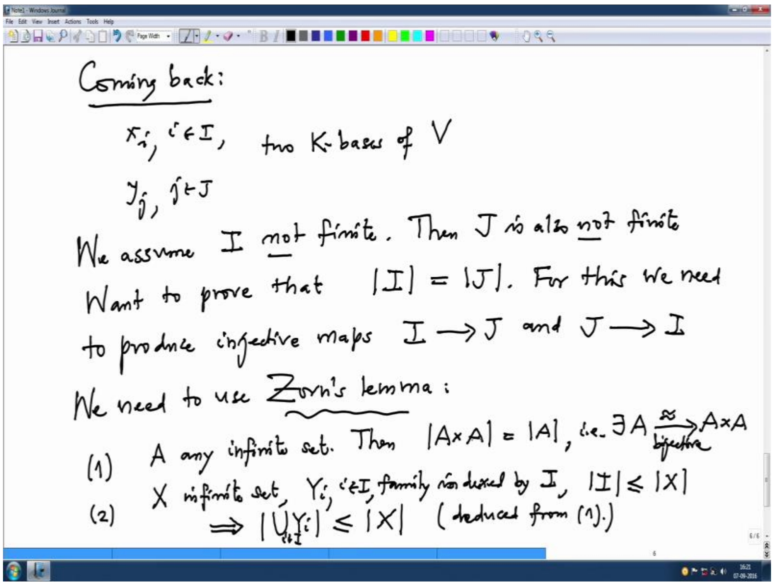

This Bernstein theorem there are many proofs the standard proof I will write in the notes and also slick proof which uses fixed point theorems etcetera that I will mentioned it in exercise let us come back to our proof that vector space basis. So, coming back remember we want to prove the following we have one basis X i i in I and y j j in J, there 2 K basis of V and what do are we assuming we are assuming we assume i is not finite. Remember when I is finite we have noted J is also finite and they have the same number of element this is precisely the finite case we have proved and that is proved by using the exchange theorem that is proved by using the exchange theorem.

Now, I is not finite. So, then first thing to notice then J is also not finite because if a vector space has a finite generating system then it will have a finite basis and one basis is finite then all basis are finite. So, if one basis you will infinite then all bases are finite. So, both I and J are not finite sets and we want to prove we want to prove we want to prove that cardinality I equal to cardinality J, but for this what we need to prove is. So, for this we need to prove that we need to produce injective maps from I to J and J to I because if there in 2 injective maps from to the then there is a bijective map also so; that means, there cardinalities are equal.

So, our job is to prove now there are injective map from I to J and they may injective map from J to I for this we need to use we need to use Zorn's lemma, but Zorn's lemma is applied not directly, but using Zorn's lemma we first prove the following assertion which I will not prove, but I will straight clearly the first assertion one proves by the Zorn's lemma is if A is any infinite set then the cardinality of a cross a is same as cardinality of A. So, this means the bijective map from so that is they the bijective map from A to A cross A, there exist a bijective map from A to A cross f. So, this is simple application of Zorn's lemma that I will not do the proof here then second observation I will need to use is if you have a X is infinite set and y I is a family of sets family indexed by set I where cardinality of I is not more than cardinality of X again this means there is a injective map from I to X then the assertion is cardinality of the union is less equal to cardinality of X and this inequality can simply we deduced from 1 this 1.

So, both this I will not give the proofs, but use this to prove this cardinality I equal to cardinality J let me finish this proof and then we will stop this is very simple.

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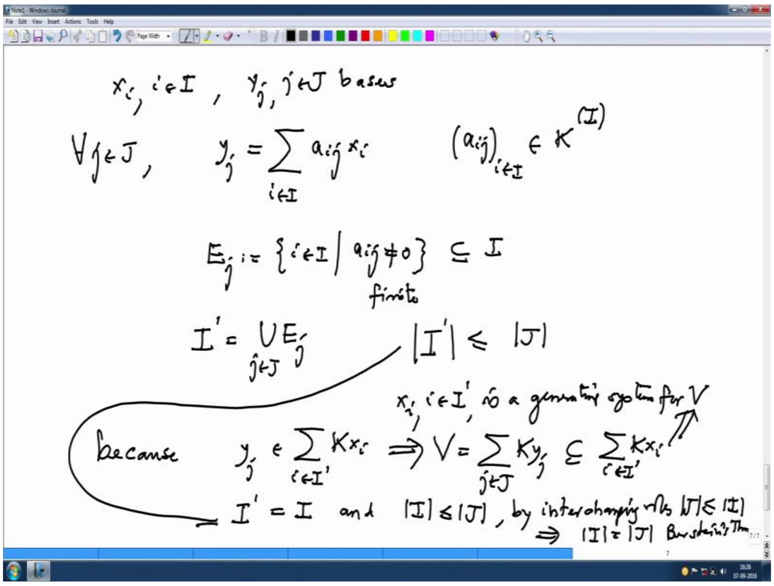

So, now what we have given? You have given that X i is the basis and y j is also basis now fixed J. So, for every j in j, this means for every j in J this vector y j, I can write it the finite combination of this X i family. So, this is summation a i j X I, this sum is running more i and tuple a i j where i is varying these tuple belongs to K power round bracket I; that means, only finitely many for if it j only finitely many as their nonzero.

So, you look at the set e j this is by definition although i is i in I such that a i j is non zero, this is finite subset of I, this is the finite subset of I since e j are there finite, we will know that all these are finite subsets of this and I take their union to be i prime, I prime is the union of e j's, j is varying now this union is indexed by j therefore, by where the observation to above which illustrate cardinality of I prime will be not more than cardinality of j this is one thing and another thing is if I just take only all X is where i varies in I prime this is a generating system because once I have this all this indices i then I capture all y j's.

So, subspace generated by these will contain all y j is and therefore, it will be a generating system. See this is because it is a generating system for V because y j belongs to the subspace generated by this X i i in I prime, but this therefore, subspace generated by y j s will be containing subspace generated by X i i in I prime; j in J, but this is already V because y j is the basis therefore, these has to be V. So, that imply this say generating system, but you see this is of the basis.

So, you cannot draw any element from the basis. So, that only say that that proofs that this I prime this must be equal to I; that means, this I prime has to be the whole I and cardinality I is less equal to cardinality I. We cannot drop any element from I to get a generating system for this. So, we have approved cardinality I less equal to cardinality J. Now I inter change the roles of the basis. So, by inter changing the roles we get cardinality J is less equal to cardinality I and then Bernstein theorem tell you then cardinality I equal to cardinality J this is Bernstein. So, this gives a complete proof of the fact that any 2 basis of the same cardinality modulo those 2 results which I coated which is a consequence of Zorn’s lemma we will take a break and come back after the break.

Thank you.