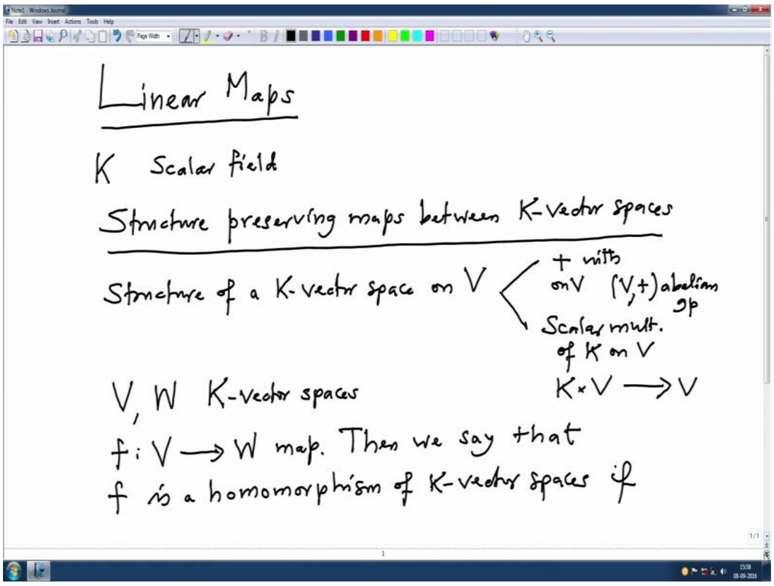
Linear Algebra
Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 22
Introduction to Linear Maps

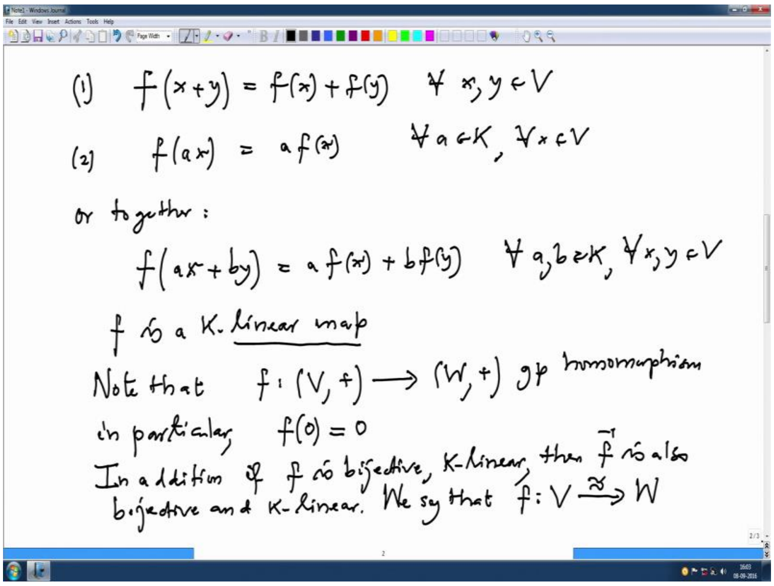
(Refer Slide Time: 00:27)



So, welcome back to this course on linear algebra. Now today I am going to start a new topic called linear maps. So, as usual K is our fix scalar field and what we want to do is actually we want to study maps of vector spaces which preserve structure. So, structure preserving maps; maps between K vector spaces this is what we are going to study.

So, recall that structure of a K vector spaces on V consists of 2 things; one is the addition such that with addition on V which makes V plus as an Abelian group and a second one is the scalar multiplication of K on V which is a map from K cross V to V. These 2 together form a vector space structure on V and if we have 2 vector spaces V and W both the vector spaces over the same scalar field and suppose you have a map f from V to W map then we say that f is a homomorphism of K vector spaces if it should preserve the addition and also it should preserve a scalar multiplication or it should be compatible with this structure.

(Refer Slide Time: 03:34)



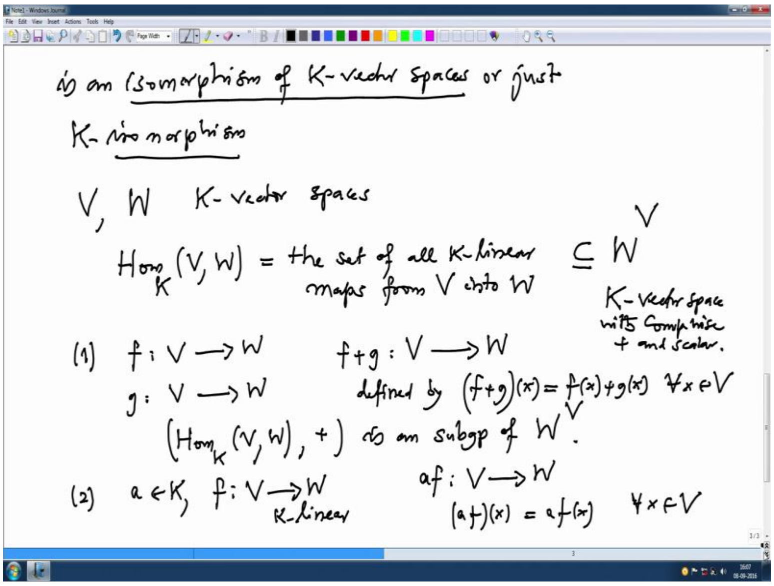
That means; first f of x plus y equal to f x plus f y for all vectors x and y and V and remind you that whether you add the vectors in V first and then apply the map f or apply the map first and then add the vectors in W.

So, in principle this 2 operations are different, but it is clear from our writing where we are operating. Secondly, f of a x should be equal to a f x where for all scalar say and for all vectors x. If both these equations are satisfied for all scalar and all vectors then we say that f is a homomorphism of vector spaces if you one want to put both together or together only in one condition we can rewrite it as f of a x plus b y equal to a f x plus b f y for all scalars a b in K and for all vectors x y in V then we call it f is a homomorphism of vector spaces or we will also just say that f is a K linear map and in this section we are going to study the set of all K linear maps from one vector space to the other vector space and soon we will see some examples.

So, first note that f will be if f is linear then f will be a group homomorphism of the Abelian groups. So, group homomorphism. So, therefore, in particular as you would have seen in the case of group homomorphism identity if this group which is 0 vector will be necessarily the 0 vector of the vector space w. So, this is must also one thing is note is if f is bijective in addition if f is bijective in addition if f is bijective and K linear then inverse f inverse is naturally bijective and K linear this needs little checking the K linearity of f prime needs a little checking, but which I leave it you.

So, in this case we say that f is an isomorphism of K vector spaces and sometimes write like this Hom.

(Refer Slide Time: 07:09)

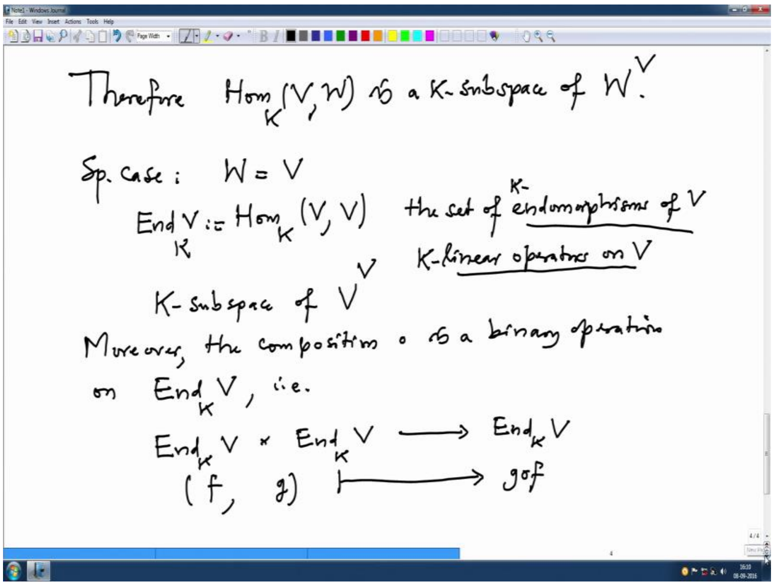


We say that f is an isomorphism of K vector spaces or just K isomorphism. So, in some couple of notation which we will keep using throughout the course if V and W 2 K vector spaces then Hom K V W this is a set of all K linear maps from V into W and note that this is a subset of W power [vocalized-noise], remember W power V is the set of all maps from V t o W and this is among them, we have chosen only the K linear maps and that we are denoting by Hom K V W just t o remember that K linear for this and Hom is short form for the homomorphism and we are going to study this object a more intimately.

So, the first thing to note is we can add 2 linear maps if f is a linear map from V to W; W g is a linear map from V to W then we can define new linear map from which we will call a call f plus g from V to W this is very natural this is defined by f plus g evaluated at x is same as f x plus g x for all x in V this is same thing as saying that remember on V power W; W power V is also vector space because its power of this vector space and this is a K vector space with component wise structure with component wise addition and scalar multiplication.

So, this means that this Hom K V W is actually a sub group of this. So, with this Hom K V W with this plus which is inherited form this point wise plus is a sub group of W power V second thing to note is I can also define for any given scalar and any map K linear map V to W I also have a very natural definition of the scalar multiple of f a f is again a map from V to W defined by a f evaluated at x is a f x for all x in V.

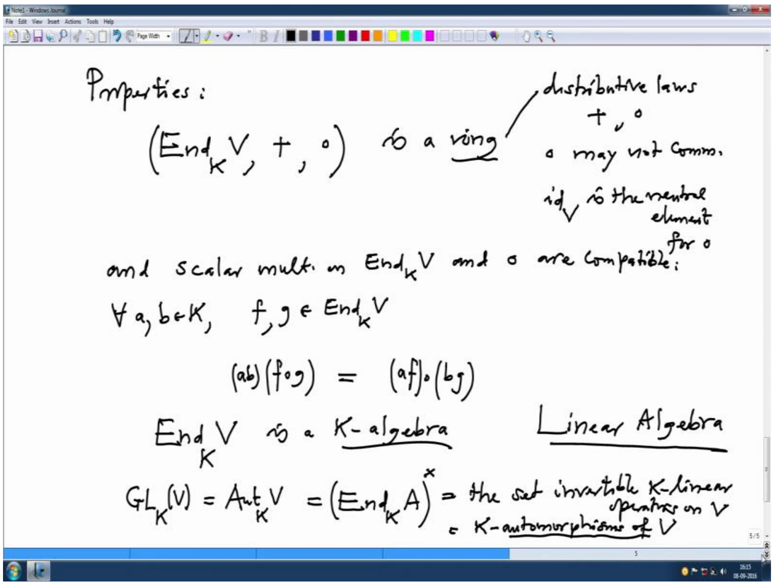
(Refer Slide Time: 11:17)



So, together we this 2 observations which it shows that it shows that therefore, Hom K V W is a K sub space of W power V. So, now, some particular notation when I means suppose special cases when W is also V then Hom K V the linear maps from V to V this I am going to give a special name its n K V these are also called the set of endomorphism of V K endomorphism or they are also called K linear operators on V and one of the main objective of the course on linear algebra is the study of K linear operators on a vector space also these this is K; K sub space this is a K sub space of V power V it has a more more structure.

So, more over the composition circle is a binary operation on endomorphism the set of K endomorphism; that means, we can compose 2 endomorphism and get another endomorphism; that means, we have such a map namely f compose g we sign it to g compose f. So, this is a binary operation and it as nice property namely.

(Refer Slide Time: 14:06)



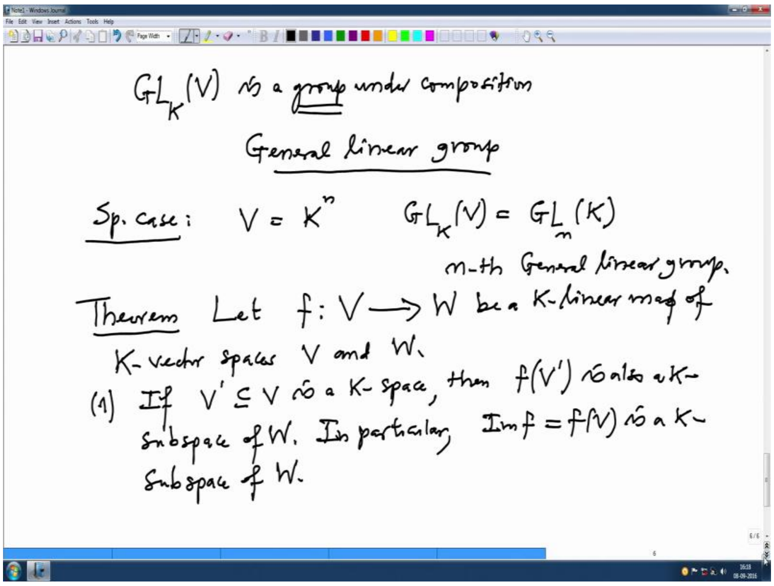
So, properties along with the addition of this vector space of K endomorphism that is a point wise addition or component wise addition and this move composition along with that it is a ring remember we have to check several things here which I am not going to check one this the distributive laws which connect plus and composition and now this composition may not be commutative. In fact, most often it is not commutative.

So, we have to check really 2 distributive laws left distributive law and write distributive law and also the existence of identity. So, identity map of V is clearly K linear and is the neutral element for the composition. So, all this together it forms a ring more over it also is a K vector space. So, the K vector space structure; that means, the scalar multiplication and this composition they are compared with each other. So, and scalar multiplication on endomorphism and composition are compatible; that means, whether for any scalars a b and any operators any 2 operators f g first we take the composition of 2 operators and multiple by the scalar a b or you multiple a f; f by scalar a and b by scalar; scalar b by endomorphism and then compose them these result is same that is a compatibility of scalar multiplication of endomorphism and the composition whether you scalar multiple first and then compose or the other way the result is same

So, such a structure is very important and generally such a thing is called an algebra and therefore, endomorphism of K; K endomorphism of a vector space V is for K algebra. So, we are going to study this K algebra in this throughout this course more and more intimately and that is a reason why the elements of these are K linear maps and this is a K algebra that is why we call this subject as linear algebra more general concept of K algebra is it is a ring and along and also it as a scalar multiplication of a field such that the addition of the ring and multiplication, and the scalar multiplication they are all tied together with all this compatibility condition that is called a K algebra, but I will not go into more general now right now our aim is to study this K algebra more intimately.

So, recall that whenever you have a ring then you invertible elements with respect to the multiplication and that remember we have denoted by the cross on the top. So, n K a cross this is the set of invertible K linear operators on V they are precisely the K automorphism of V invertible therefore, they are bijective and K linear and just now I have commented that inverse is also K linear. So, they are precisely the K automorphism of V and therefore, this is also usually denoted by K V sometimes I will also denote this by G L K; G L K V.

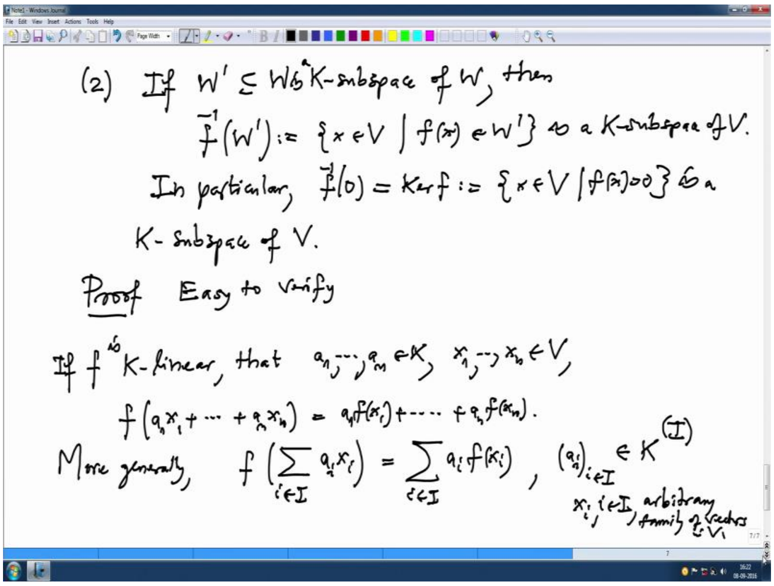
(Refer Slide Time: 19:17)



A important thing to note about it is G L K V is a group under composition and we are going to study this group more intimately along with the endomorphism G L is a short form for the general linear alright. So, let us see in the case. So, special case if V is K power n is vector space then usually G L K V also denoted by G L n K and it is called n th n nth general linear group before I see the more concrete examples I would like to recall the following theorem this proof is very easy I will I will try to say it overly, but we will write the statement very precisely.

So, let f from V to W be a K linear map of K vector spaces V and W part one this concerns about the sub spaces of V and the images of the these images of the sub spaces of v. So, if V prime is a K sub spaces then its image under f then f of V prime is also a K subspace of W in particular image of f image of the map f which is by definition f of V is K subspace of w.

(Refer Slide Time: 22:40)

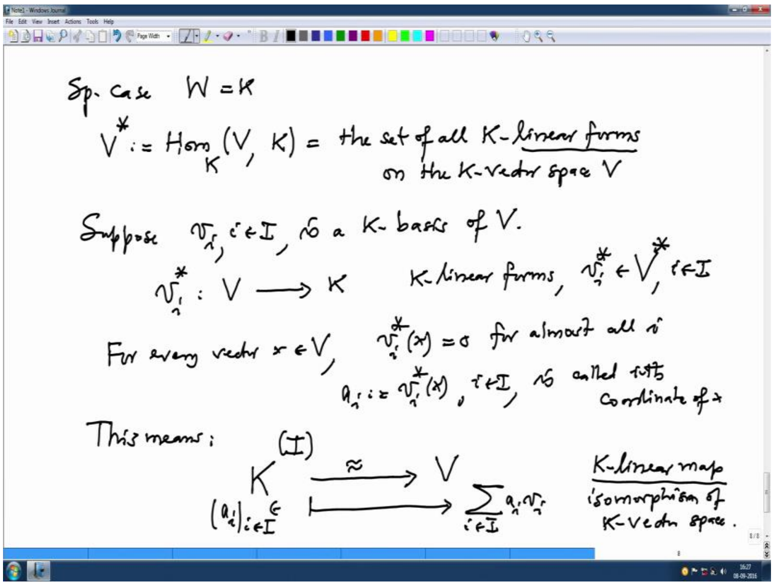


Second part second part the other way if W prime is a subspace of W the K subspace of W then the inverse image of W prime this is by definition all the vectors x in V such that f of x belong to W prime is a K subspace of V in particular the inverse image of the 0 subspace which is also called kernel of f which is by definition all those vectors x in V such that f of x is 0 is a K subspace of V. So, this is this is very easy to verify.

So, I will not write the formal proof easy to verify note that how do you check somebody a subspace you have to check 2 things it is an Abelian subgroup it is a subgroup of V plus and also it is close under scalar multiplication. So, both these conditions are nearly verification. So, I will not do it first of all note that if f is K linear then for any number of scalars finitely many a 1 to a n scalars and n vectors x 1 to x n in V which I give b y induction f of a 1 x 1 plus, plus, plus, plus a n x n equal to a 1 f of x 1 plus, plus, plus, plus, plus a n f of x n this is very simple checking by apply induction on n do to at a time in so or even more generally more generally f of the linear sum i in I a i x i equal to summation a i f of x i where a i is an the tuple a i; i tuple a i is belongs to K power round bracket I.

So; that means, almost all a i are 0. So, this sum makes sense and even this sum makes sense and x i i in I when arbitrary family of vectors in V this is nearly the earlier statement.

(Refer Slide Time: 26:21)



So, further another special case which I want to note is W equal to K when Hom K all linear K linear maps from a vector space V into K this is also called V star I will denote this by V star this is the set of all K linear forms on the K vector space V, usually the word form is used when the values are in a scalar field or their (Refer Time: 27:13) people will also used K linear functional their functions because the values are you know field.

So, remember that also when we already proved V as a vector V, every vector space as a basis. So, suppose V i i in I is a K basis of V then we have defined coordinate function V i stars their functions from V to K and they are also K linear maps K linear forms actually because the values are in field. That means, that is V i star are actually elements of V star for every I and V i star is a additional property namely for any for every vector x x in V; V i star of x is 0 for almost all I and wherever it is non 0 that is called the ith coordinate of v. So, a i defined by this V i star x i in I are called i th coordinate of x is called and there are at most finitely many coordinates non 0.

So, this means the following this is actually is nice this means this means we have a vector space this K round bracket I and we have a we have a natural map from V namely the map is very simple any tuple a I, i in I in K power round bracket I goes to summation a i V i this makes sense because only finitely many are non 0 and the fact that this V i is a vector V i is a basis of V means this map is injective and also surjective generating system will mean this map is surjective and linear independence will be this map is injective. So, this map is actually bijective and it is very easy to check that this map is actually K linear map; that means, this is an isomorphism of K vector spaces. So, I am not going to check that this is a K linear map because it is really easy one need to write only one line, but I will leave it for you to check it is a linear map.

So that means, we have approved that every vector space is a isomorphic to K power round bracket I, this is very very useful, but also coordination will depend on this chosen basis. So, when we change a basis the coordinates will also change and this will allow us to do more general thing in a more simpler way. So, we will take a break and continue after the break.