Linear Algebra

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Lecture – 23
Examples of Linear Maps

Let us continue in the second half, I want to give some examples of linear maps.

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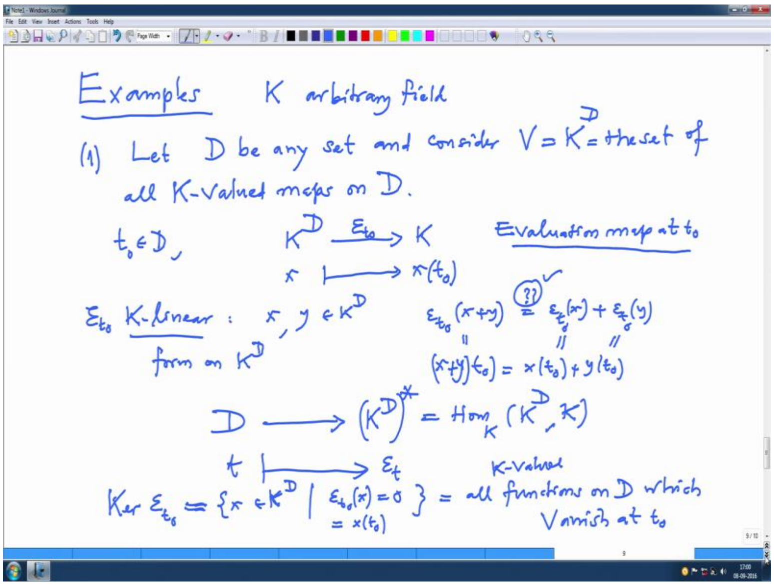

So, first one let us fix K be arbitrary field when you specific field is considered then I will mention it that time. So, D let us take a D be any subset any set and consider the vector space V equal to K power D. Remember K power D is all K valued functions on the set D; the set of all K value maps on D and our addition is component wise scalar multiplication is also component wise and let us fix some t naught in D and let us look at the map K power D to K.

So, given any function x, I will call the functions to be x now x is a function from D to K and I evaluate this function at this given t naught. So, this goes to x of t naught x of t naught is an element in K because x are K value. So, this is a evaluation map at t naught that is a even I denote it by epsilon suffix t naught. This is evaluation map at t naught want to check that this map is K linear map it is epsilon t is K linear. So, why do I have to check? That means, we have to check that if you are given 2 functions x and y in K D epsilon t 0 of x plus y equal to epsilon t 0 x plus epsilon t 0 y we have to check this, but what is this? This is the function x plus y is evaluated at t naught by definition of epsilon t naught x plus y evaluated t naught.

But the definition of addition of functions is point wise which is same as component wise this is same as x at t naught plus y at t naught, but x at t naught is by definition epsilon t naught at x this is a epsilon t naught at y. So, we have checked this. So, we have check that this is K linear and if because the values are in K it is actually K linear form on K power d. So, they are many K linear forms in this case and when you vary t naught in D at each t naught will get linear form. So, actually you have a map from D to K linear forms on D K D that is if you remember our notation K power D star this is by definition hom K; K power D K.

So, t going to epsilon t this is a map. So, there is a big flow of linear forms on K power D this is very very useful for studying a vector space when they are many many linear forms we can use this linear forms to study a vector space better. So, next time i want to compute what is a kernel of epsilon t naught. So, this means this is all those functions x in K power D which go to 0 under epsilon t naught this is precisely the kernel of this map and what is this condition this is precisely means x at t naught is 0.

So, the kernel is precisely all functions K valued functions on D which vanish at t naught usually such considerations are done in analysis course to study functions and so on. So, this language of linear algebra will help tremendously studying analysis also the next example we will we will tell you this more.

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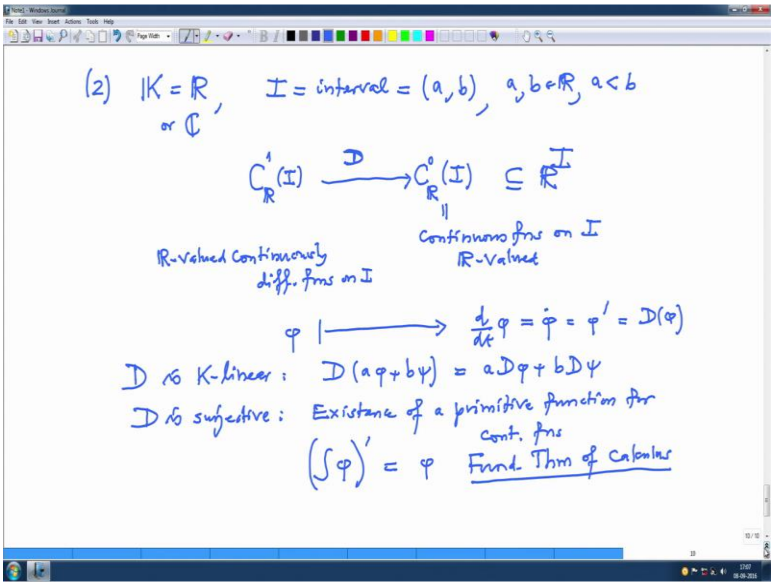

So, for example, now let us take the field K to be the field of real numbers R and let us take I to be the interval I is a interval that could mean open interval or close interval or half open half close and so on, but this typically interval means a comma b where a and b are real numbers and a is less than b.

So, these intervals as enough points then what vector spaces are we considering one vector space we consider is C R 0 and I. So, what is this? This is the space of continuous functions, functions real value on the interval I R value and nothing special about field of real numbers where ever the continuity make sense. So, for in sense I could have taken C complex number C, but for illustration and fixing it the real numbers strictly speaking I should have written this capital this standard notation which is reduce earlier double line K is either R or even C, but let me strict to real value, alright.

So, this is we have seen that this is a vector space this is just because one learn when you learn continuity one learns that if you have 2 continuous functions then the sum which is point wise sum which is also component wise sum this is again continuous functions and a scalar multiple of a continuous function is again continuous function all this one learns in a first course analysis this is a subspace of R power I.

So, big vector spaces R power I and this is a subspace their another subspace here is if you remember when I did on examples of subspace now continues more than continuous C 1 functions; C 1 R I; that means, continuously differentiable functions on I; real value

R valued again, this is a subspace because 2 continuously differentiable functions this sum is continuously differentiable this scalar multiplication is continuously differentiable and derivative of such function is done by sum wise. So, I will just say it is unique and continuously means that differentiable and derivatives also continuous.

So, we have a natural map here namely let me write it capital D; capital D is a differentiation. So, what is the map D given any function phi map it to D D t of phi which also when denotes by different disciplines in science or engineer they denote in a different way for example, I will see in the notation like this phi upper dot t upper dot this is a differentiation of phi or phi dash or of course, D D t and I am going to use it D of phi.

So, this is a map and; obviously, if phi is continuously differentiable then the derivatives are continuous function. So, therefore, this D is indeed a map from C 1 R I to C 0 R i and I want to say that this map is linear map D is K linear what is; that means, that means we need to check that D of a phi plus b psi equal to a D phi plus b D psi, but these are the usual standard rules of differentiation that one learns in a first course and differential calculates.

So, another its K linear another thing is D surjective what a; that means, that means give an any continuous function they are the continuously differentiable function derivatives is the given function. So, this is precisely the existence of a primitive function for continuous functions this means given an any continuous function there is a big there is a function which is called a primitive function namely in some people also call it integral. So, that the fundamental theorem says that integral of phi exist and derivatives of this integrated functions is precisely the given function this is also known as fundamental theorem of calculus.

If you see the history between the derivative and integration who came first is actually the integration came first because of the need derivative come later and the beauty of this theorem is those concept came much different time, but they are finally, connected with this nice equality which is therefore, it is called fundamental theorem of calculus a surjectives existence of primitive.

Now, going continuing the next example continuing now I want to give a map in the other direction the map in the other direction is that is C is from the continuous functions on an open and a interval i real value to C 1 functions on I.

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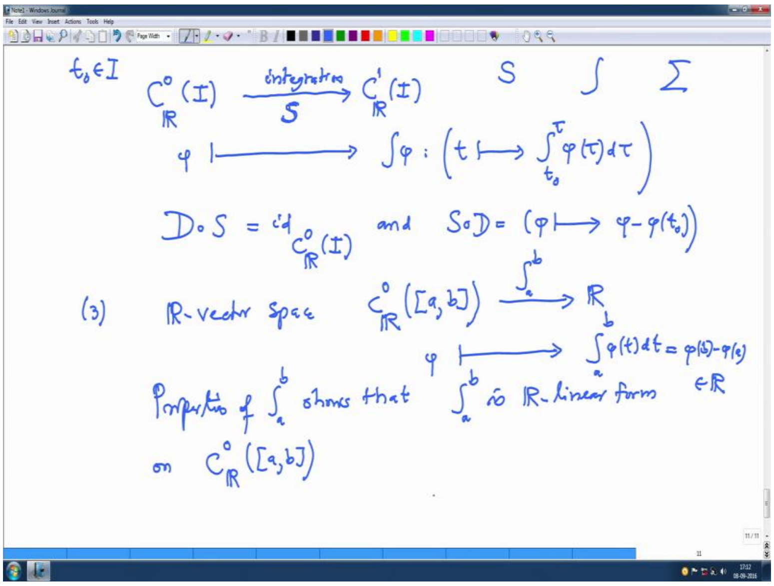

This is also natural map here namely the integration any continuous function phi you want to map it to each integral. So, that operator this integral operator I will denote by S; S is evolved from integration and the integration evolved from the sum. So, we sum when you become smooth that become an integration and then for our English purpose for our operator purpose it is denoted by S.

So, this is the function now it should be a function which function is it that function let me get also this is integral phi this is this should you should think of the map from t I to R. So, this is the map t going to integration from t naught to t phi t D t actually strictly (Refer Time: 14:15) I should use the different letter, but it is; so this is goes to let me view more precise. So, this is phi to tau tau D t.

After sometime I will also explain you why this D tau is written here, this is a map and now what is the just now want the fundament theorem I said was if you look at D compose S see the D operator was here and then compose with S that is nothing, but the identity operator on C 0 R K and what is the operator either composition also make sense S compose D is precisely the operator which send the map f map phi to the map phi minus phi evaluated t naught where t naught is fixed t naught is fixed in the interval I, this is because then function does not determine the integral uniquely, but its determine up to a concern and that concern is uniquely determined if we evaluated at a given point.

So, that is why this phi at t naught. So, also from this equality it is clear this D is surjective see because D composes identity. So, D is surjective is clear from here D surjective is clear also S is injective is clear, but they are not isomorphisms another another extension or let me write the number three again now let us take R vector space of real valued continuous function let say on a close interval a b and if you look at the integral definite integral. So, this is my vector space from here to R then this operator integral a to b any function phi goes to the integral a to b phi t D t this is actually a constant.

This is a real number integration definite integral is a real number which real number that is phi b minus phi a this is a real number and the properties of integral that we study that say that this operator is a linear operator. So, properties of integral transferred to this of definite integral shows that these operator is R linear for on this vector space of continuously related function on the close interval.

So, their many linear forms this integration is one of them and that is why analysis studying analysis is important and I think special about the close interval one could have taken arbitrary topological space actually and could have done this in a more generally. So, that is enough for this, now let me take up another very important examples which we will which we is the art of this course.

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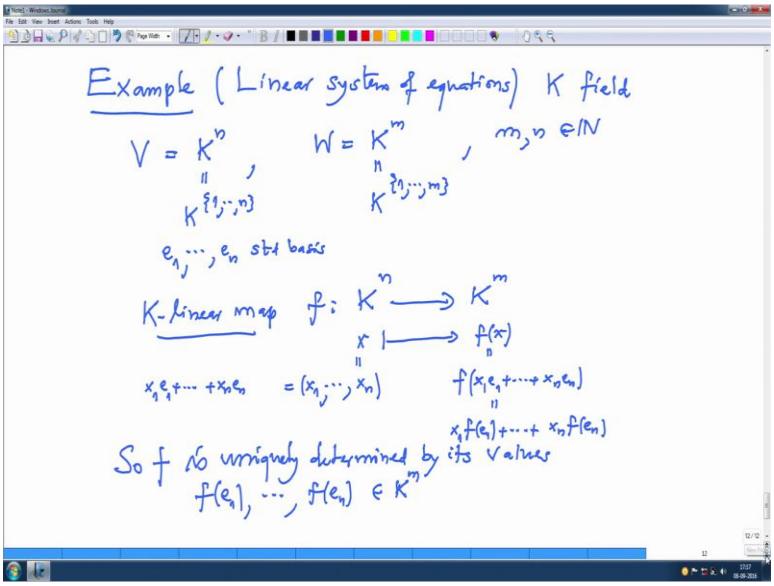

So, another example is what these called linear system of equations, what I want to show you in this example is studying a linear system of equations is equivalent to studying linear maps and it is better to study linear maps then linear system of equations and that is much more general and much more easier.

So, let us fix a K as a field; K as a field and then my vector space is now K power n and W is K power m and n are 2 natural numbers remember these also is same as either you can think the elements of v as n tuples or you can also think them as functions from 1 to n to K this also a functions from 1 to m to K or m tuples whichever this is also still these vector spaces are given with a basis the basis also fixed here when we have written v equal to K power n the standard basis also fix.

So, the standard basis here is e 1 to e n, but here the standard basis here now I cannot need not e 1 to e n because there will be confusion right these are n tuples and these are n tuples. So, whenever I want to use standard basis of w i we careful and not use e 1 to e n otherwise will be confusion and let us look at the map f K linear map f from K power n to K n.

I will show you such a linear map will give rise to a linear system and then we will analyze what does one mean by solutions of linear system and what how do you decide in terms of this linear maps there consistent and so on and so forth. So, this linear map is given; that means, given any x here it goes to f x remember a if x is a tuple x 1 to x n and these x 1 to x n, they are component of this x they are precisely the coordinates of these vector x with respect to the standard basis.

So, this is also written as x 1 e 1 plus, plus, plus, plus, plus, x n e n. So, these coordinates are uniquely determined by this basis that is what we have been noting earlier. So, now, where will this go? So, f of x is therefore, f of x 1 e 1 plus, plus, plus, plus, plus, plus, x n e n, but f of this because f is linear this is same as x 1 is a scalar. So, x 1 will come out f of x 1 e 1 plus, plus, plus, plus, plus, x n f of e n. So, this means this map f is uniquely determined. So, f is uniquely determined by its values f of e 1 f of e n which are vectors in K power m.

So, they are m tuples. So, let us analyze what are. So, therefore, I want to write this f of e 1 f of e n their n m tuples. So, I want to write them.

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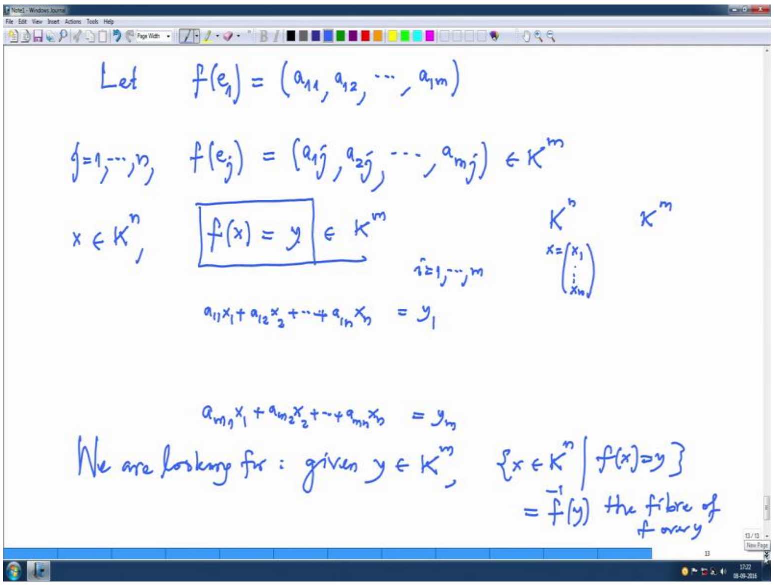

So, let f of e 1 is a vector in K power m. So, it as m components, so, therefore, I can write this as a 1; a 1 to a 1 m. So, this again this representation we have use a standard basis of K power m to write this and these a 1 1 a 1 2 a n; a 1 n are the coordinates of f of e 1 with respect to the standard basis of K power m.

So, in general if for any j in 1 to n, we have f of e j equal to a 1 j a 2 j a n no, a m j their m tuples they should be in K power m and they should depend on the j (Refer Time: 24:02) plus this as m n entries this as m components because there in K power m. So, now, if we look at the equation, f of x for any x in K power n f of x is some y; y in K power m.

So, now instead of denoting the standard basis as role I want to denote as a column. So, this 1 will be; I want to compare the elements of K power n and K power m instead of writing row vectors, I am writing it as a column vectors. So, f of x as a column what are the entries we know that is what I have to do I have to take this x which is x 1 to x n and then i take x 1 and then e 1 and so on.

So, when I compare the components for each I from 1 to n, what do I get here? Here I will get y 1 on one side that is a first component first coordinate of y with respect to the standard basis of K power n on the other hand what will it be it will be a 1 1, first component of e 1 times x 1 that is a 1 1 times x 1 plus a 1 2 times x 2 plus, plus, plus, plus, plus a 1 n times x n that is a first component of f of x remember because this f of v j we are thinking as a column vector.

So, the first component will be a 1 j times x 1 a 2 j times x 2 and so on. So, the in general the m th component will be y m on 1 side, the other side it will be a m 1 x 1 plus a m 2 x 2 plus, plus, plus, plus, plus a m n x n. So, which is in did looks like a system of linear equations there m equations in n n known. So, the problem is to find given y you want to find x so; that means, we want to solve this equation given y we want to solve how many x are there with this condition.

So, that is precisely. So, what are we looking for we are looking for given y fix y in K power m we are looking for all those x in K power m such that f of x equal to y. So, this is nothing, but f inverse of y the fiber of f over y and the this system being can system is equivalent to saying the fiber being non empty and our fiber is non empty there will be some x in that and that x will be a solution of that system.

Student: It will present with the index (Refer Time: 27:59) when you please taking column vectors.

We are taking column vectors all right. So, let us do it little bit more. So, that it will become more and more clear. So, therefore, the problem is a following.

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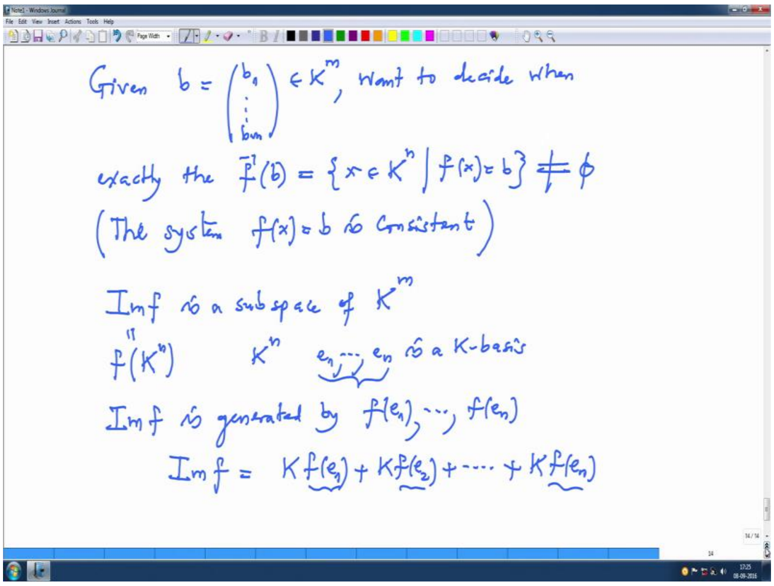

So, let us to be consistent with the earlier notation given b and think of b as a column b 1 to b n this is given in K power m we want to know want to know want to decide when exactly the fiber f inverse b which is by definition all those vectors x in K power n such that f of x equal to b is non empty.

In the language of system of linear equation language this means the system the system f x equal to b is consistent that is equivalent to save the fiber is non empty you see also it is very neat and we have to write much less the writing is less and the confusion is also less. So, we want to decide that. So, now, note what is a image f remember image f is a subspace of K power m which subspace this is a f of v f of K power n, but K power n K power n we know e 1 to e n is a basis is a K basis that is K power n is generated by this in particular therefore, the image will be generated by the images of this.

So, image of f is generated by the vectors f of e 1 f of e n in our earlier writing this means we were writing like this image of f equal to K f e 1 plus K f e 2 plus, plus, plus, plus K f e n these all precisely the meaning of subspace being generated by these vectors this is the smallest subspace which contains this all this vectors.

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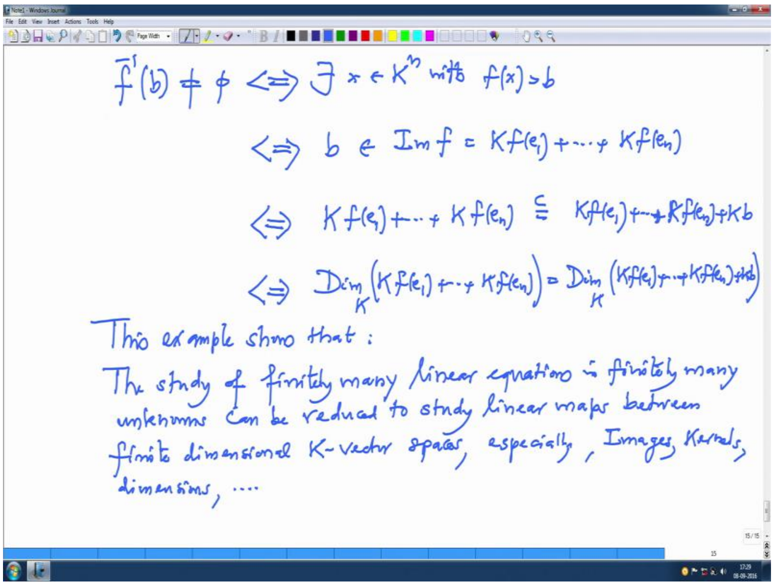

So, now the question becomes therefore so f inverse b is non empty if and only if their exist x in K power n with f x f of x equal to b because such a x will be in the fiber of b, but that is if and only if these b belong to the image of f, but image of f. We know it is generated by phi 1 plus, plus, plus, plus, plus f e n, if this b belongs here that is equivalent to saying the 2 subspaces generated by f of e 1 to e n and f of e 1 plus, plus, plus, plus, f of K f e 1 e n plus K f K b if I add this b to the generating system it will still generate.

So, that is a meaning of this is equal this inclusion is the obvious what is more important, this b belong to this set, but how do you check this just by using one click that is equivalent to saying the dimensions are equal. So, dimension of the subspace K f e 1 plus, plus, plus, plus, plus K f e n this dimension of this subspace equal to dimension of the subspace generated by f e 1 plus, plus, plus, plus K f e n plus K b.

So, this is this is very good we just have to check the dimensions are equal. So, I will formally note these example shows that the study of finitely many linear equations in finitely many unknowns is or can be reduced to study linear maps between finite dimensional vector spaces K vector spaces especially their image space images kernels their dimensions and more in variants which we will introduce in the due course of this lectures. So, with this I will stop this lecture and we will continue next time.

Thank you.