Linear Algebra

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Lecture – 24
Linear Maps and Bases

Welcome to this lectures on linear algebra. So far in the course we have seen that every vectors space has a basis, any 2 basis have the same number of elements or the same cardinality and now we where started with last 2 lectures we started studying linear maps. I would like to continue this study of linear maps more and basically in this sections I want to use existence of basis to study linear maps in more systematically, in particular produce more linear maps on a given vector space.

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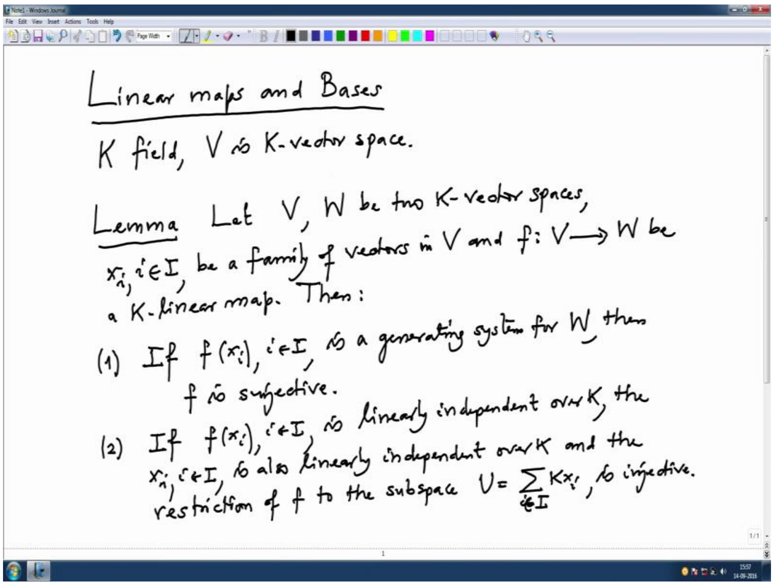

So, this section title is linear maps and basis. So, as usual K field are arbitrary field and V is a K vector space. In the first lemma I want to collect the facts be aware of generating systems be aware of linear independence and be aware of the basis with respect to a linear map, so above their images about their inverse images and so on. So, this I will write the lemma. So, lemma let V and W be 2 K vector spaces x i, i in I be a family of vectors in V and f V to W be a K linear map then many statements, but there will be very easy to prove we will see a first statement.

If f of x i the images of this given family x i is a generating system for W, then the linear map f is so subjective. 2 if f of x i, i in I is linearly independent over K then the original family x i, x i, i in I is also linearly independent over K of course, and the restriction of f of f to the subspace if U which is generated by the given family x i is injective, so a 2 statement.

Let us first use the proof of this 2 statement with theories, we want to proof that f is surjective.

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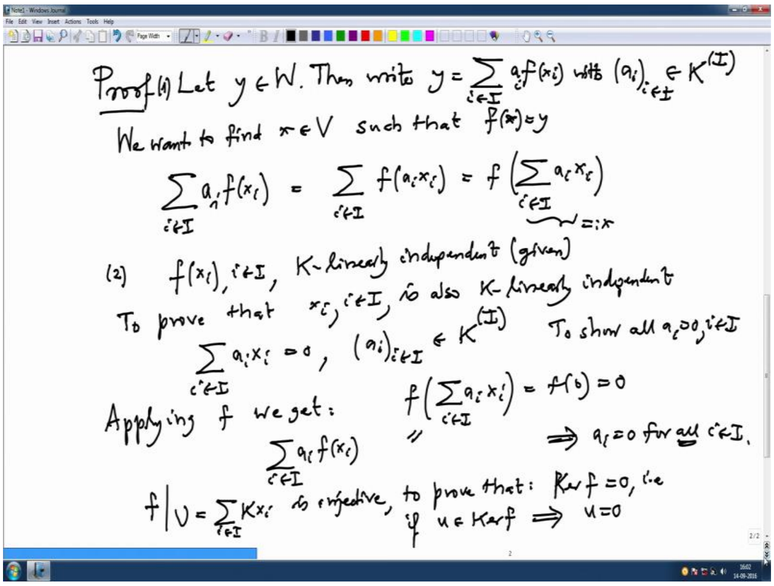

If the images of x i generate W then we want to proof that f is surjective. So, proof, start with W let y in y, y in W to proof f is surjective we want to write f y. So, what we want is we want to find to find x in V such that f of x is y that is the meaning that it is surjective every y in the image.

So, start with y because we have given that f of x i generates W. So, write then we can write y as a linear combination of the images of x i, a i, f x i, i in I with this coefficient of a i is in a K round bracket I. Remember K round bracket I is a notation for those tuples for which almost all components are 0. So, this one makes sense and this is the meaning that y is in the subspace generated by f of x i; because f is linear a i f x i is same as f of a i x. So, therefore, this sum equal to this sum and once again I can take a f out because f V also respect the addition, and now we can take this as x and then we got x so that f of x is y.

So, this means f if surjective this was a proof of 1, proof of 2 what we have given we have given that the family image family f of x are linearly independent f of x i, i in I this is K linearly independent that is given and we want to proof that to proof that x i, i in I is also K linearly independent. So, take the linear sum which is 0 dependence relation a i x i is 0 i in I where again this coefficient tuple is in K round bracket i. So, almost all are 0, but we want to proof that all I i where 0.

So, to show all a i are 0; well here this equation given. So, apply f to this applying f we get f of this sum is f of 0, but f is linear. So, f of 0 is 0, on the other hand this sum is because f is linear it is a i, f of x i, but; that means, this is dependence relation among f of x is, but f of x is are linearly independent given. So, that implies all a as 0. So, that proofs 1 and 2; know we still have to proof that f restricted to U where U is a subspace generated by x is this are the notation for subspace generated by x is, we want to show that this is injective, but; that means, we need to check that. So, it prove that f of. So, we should prove that kernel of f is 0, that is if U belongs to kernel then U should be 0.

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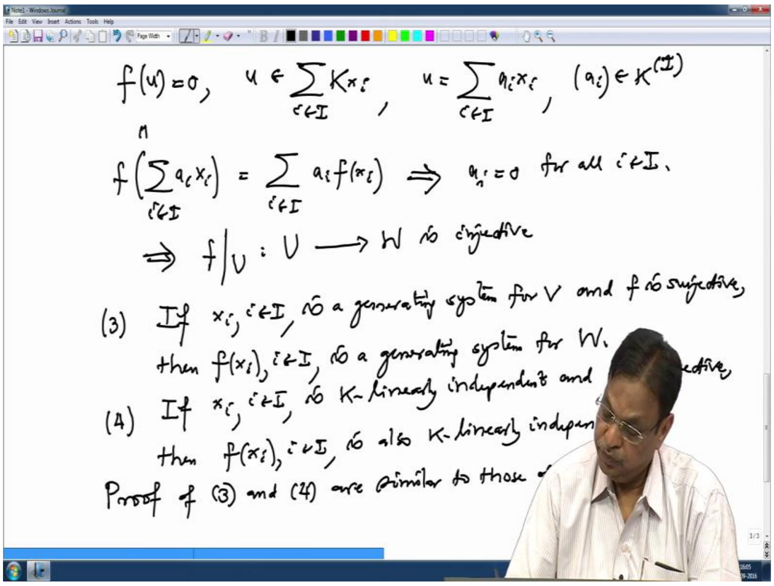

That U belong to kernel means f of U is 0, and U is in the subspace generated by x is; that means, U I can always write it as linear combination of x is, and again the coefficient a i is in K power round bracket I, but then when you plug it in here this is f of summation a i x i, but this is because f is linearly summation a i f of x i and this is 0.

But if f of x i where linearly independent, so a i are 0 for all I in I. So, that proofs that f restricted to you which is a map from U to W is injective; this was that extension third extension now so the other way if x i, i in I is a generating system for V and f is surjective then f of x i, i in I is a generating system for W.

Forth one again the similar see the earlier 2 statements where about the images of the family, these are these three and four will be the corresponding statements about the family. So, if x i, i in I is K-linearly independent family and f is injective then the images the image family f o f x i, i in I is also K linearly independent.

So, the proofs of this proof of three and four are similar to those of 1 and 2. So, I will skip the proof.

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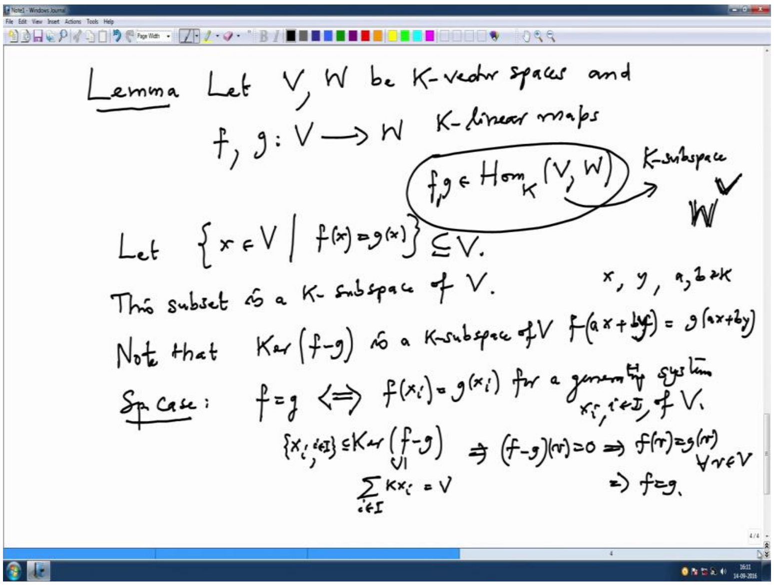

So, having done this, so, let us write the image here sum of the image at consequences of this. So, this one the next lemma for example, checks how do we check the given 2 linear maps are equal or not for example. So, let V and W be K vector spaces, and f and g are 2 linear maps from V to W K-linear maps.

But I want to also remind you remember we have introduce this notation Hom K V W for the set of linear maps from V to W. So, these writing will get shortly if you simply write f comma g belongs to this instead of writing this line, and we shall adopt this more often because that will say some sometime as well as space in writing; now consider. So, let look at all those vectors x in V. So, that f and g agree on x f x equal to g x this is a subset of V this, this, this is a, this subset is a K subspace of V well that mean that if x is their y their then a x plus b y is also there for all scalar a and b.

So, either one checks you can take x in that side y in that side and a b arbitrary scalars then we just have to look at a x plus b y, and you evaluate a x plus b y and you apply f to this as well as g to this, and check that both this are equal because f and g are linear because this side is a f x plus b f y, this side is a g x plus b g y and f x equal to g x and g y f y equal to g y therefore, this equal, but you can also make it little shorter by noting note that this one is precisely the kernel of the linear map f minus g.

Whenever you have 2 linear maps we can add them you can multiple by scalar. So, we have checked that this one is a K subspace of a bigger space which is W power V, W power V is a K vector space again by component by the addition and component wise scalar multiplication this V arbitrary set and the W is a vector space. So, this is the kernel and kernel is always the W subspace of V, K subspace of V.

So, that by the assertion in this lemma and the particular special case of the assertion is f equal to g if and only if f and g agree on a generating system for a generating system x i, i in I of V if we become an when generating system of vector space V where f and g agree then f equal to g, this is because in this case this kernel of f minus g this is a subspace and it contains a generating system x i.

The generating system is contained in this therefore, a subspace generated by this will be contained in this, but subspace generated by this is precisely the whole V this is equal to V and this is contained here and if kernel contains whole V; that means, it the map f minus g vanishes on every v; that means, f equal t o g, this means f minus g on any V is 0 we got this means f y equal to g V; that means, this is true for e very V in V. So, that is f equal to g.

So, this is one very quick way of testing quick way of testing whether 2 linear maps agree or not equal or not and that is enough to check that it is enough to check that they agree on a generating system. So, in case of a finite dimensional vector space we can always choice a finite generating system and then we checking f equal to g we involve only finitely many steps ok.

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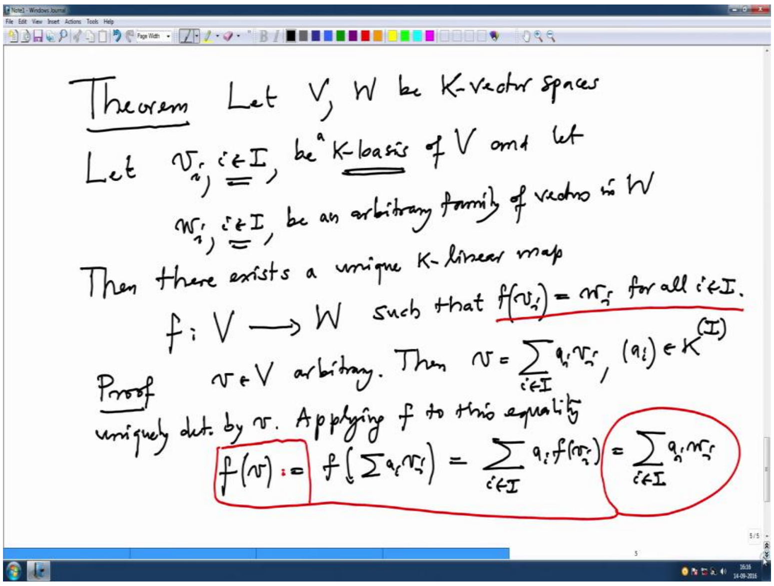

Now, let us write a theorem this is very very important statement. So, this shows that linear map we can define a linear map on a vector space if you can define only a basis. So, let me write in the form of theorem. So, theorem let V and W be K vector spaces and let V i, i in I be a K basis of V. We approved in first part of first 2 lectures that every vectors spaces as a basis may be finite may be infinite, but such a basis always exist and let W i, i in I be an arbitrary family of vectors in W, I have taken the arbitrary family, but index by the same index inside that is important.

Then there exist unique K-linear map f from V to W such that f of V i equal to W i for all I well; we this is we just have to show that on a arbitrary V how we can define f. So, I will so continue this. So, let me prove this first. So, what we need to do is if we have V in V arbitrary then because this V i is a basis of V this V we can write uniquely a i V i because it is a basis why this coefficient tuple a i this is in K power round bracket I and this are unique uniquely determined uniquely determined by V, that is because it is a basis, basis is a important.

So, when I apply f applying f, f to this equation this equality and what do you want to linearity. So, f of V should be then f of summation a i V i and this; obviously, we want summation i in I a i f of V i and; obviously, f of V i demand is it is W f. So, this is nothing, but a i W i and because this a i are uniquely determined by V that do not depend on the W is. So, directly we could have said that define f V f of V by this formula, it is a definition of f and it is it make sense also it is a uniquely determined by V.

So, that gives f and by very fact that we have check this 2 equality rows this equality between that shows is a this f is a unique map with this condition, the condition is this once you know this then the map is unique. So, I want to note what more property this unique map f as which as this which is this condition. So, let me write.

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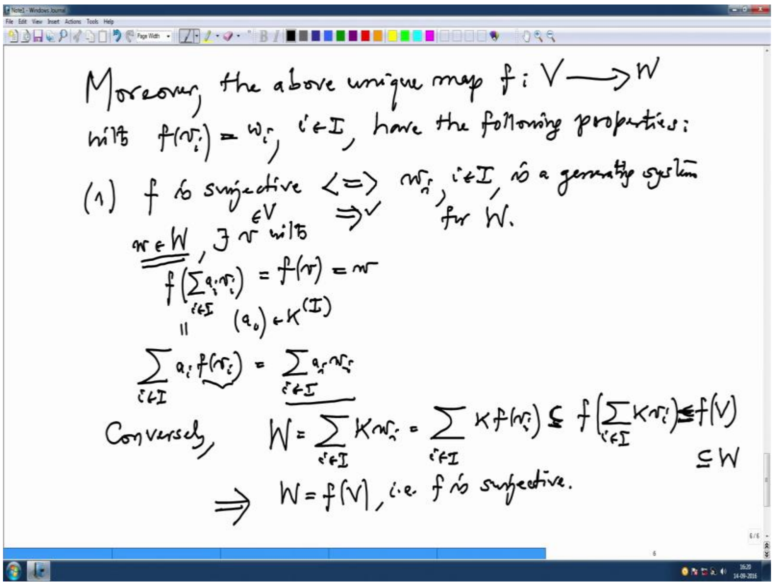

So, more over the above unique map f V to W with f of V i equal to W i for all i in I have the following properties ok.

One f is surjective how do we test f is this given this unique map is surjective or not. That is if and only if W i, i in I is a generating system for W, this is very easy if f is surjective of course, this as to generate because surjective means every W is coming from some V. So, this means w in W there exist V with f of V equal to w, but f of V is V is in V is in V. So, we can write as a unique linear combination of V i, a i, V i where with a i tuple is in K round bracket I that is because V i is a basis, and because f is linear this is nothing, but summation a i f of V i, i in I, but f of V i is W, W i.

So, this is nothing, but summation a i W i. So, this means every W in W we have wrote as a linear combination of this W i; that means, if W i is generate W. So, we have approved this assertion this way, where assuming it is surjective we approved it is a generating system for W. Conversely if I given that a W as a generating system for W then I want to proof is surjective, but if W as a generating system for W means W is a smallest subspace or W is a subspace generated by W i means this, but this is same as i in I, K f V i because W i if f of V i, but this is same as this is contained in f of I can take the the sum inside and V i to (Refer Time: 27:32) f is linear, every elements here is a combination and finite combination we can take f out, but this is contained in f of V this is equal to of V because V i is a basis, but f of V is contained in W we should W is contained in W with all these must be equal because W is all these equality that happening in a big space W i.

So, that implies that implies W equal to f of V, that is f is surjective that is all that was proof of this one I still have 2 more properties.

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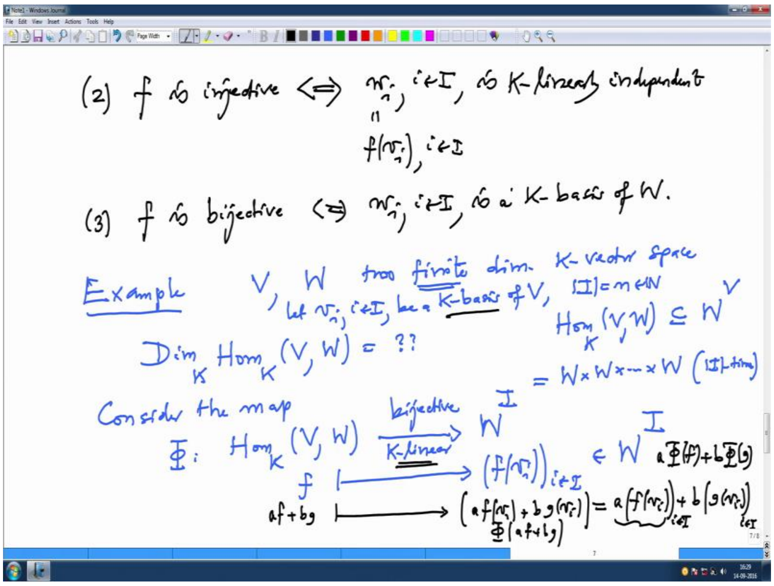

So, there will similar. So, I will skip the proof. So, now, f is surjective now f is injective f is injective equivalent to saying if family W i, i in I is K linearly independent remember this W i is nothing a image of basis V i. So, I will skip of this, this is similar to that of 1 and then combine 1 and 2, so f is bijective injective and surjective is bijective if and only if and linear independence and the generating system is a basis.

So, that is W i, i in I is a K basis of W so; that means, whenever we want to check the map is given linear map is bijective, we need to check that basis some basis of V the image of some basis of V is a basis of W then we can say that the map is bijective. I will soon apply these in a example. So, let me write example. So, example, so the problem I want to deal in this example is given V and W to finite dimensional K vector spaces, then we have this vector space Hom K V W. I want to compute the dimension of this is what I want to do.

So, remember this Hom K V W is a subspace of W power V this is to big subspace. So, this is not even finite dimensional. So, if you and whether this is finite dimensional not it is not clear from this inclusion. So, we want to find the better way to embed this in a smaller vector space so that we can compute the dimension. So, look at consider the map the map this map is f, this map I want to call it phi this is a map from Hom K V W to W power I what is I? I first choose a basis of V.

So, let V i, i in I be a basis be a K basis of V; and now because we are assuming V is finite dimensional the cardinality of I as to be some natural number n; that means, the I is a set which as exactly an elements we may also take 1 to n or you can call it I 1 to I n the elements of I. This map capital phi is a map which so take any linear map from V to W and where should I map, I should map it to the I tuple which as entries in W.

So, that I tuple is nothing, but the values of V i values of f at V i. So, f of V i this is; obviously, an elements in W power I, and we do not need to write a round bracket here because I is a finite set. So, this is this is set of all I tuples with coordinates in W. So, this is a tuple now I want to check that this map is bijective, if I check the map is bijective then the dimension of these vector space its bijective and also want to check K linear; if I check that then the dimension of this will be equal to dimension of this, but W power I is nothing, but W cross W cross cross cross this is a product mod I times because functions from I to W we can think a tuples and how many coordinates as many as the set as I has elements.

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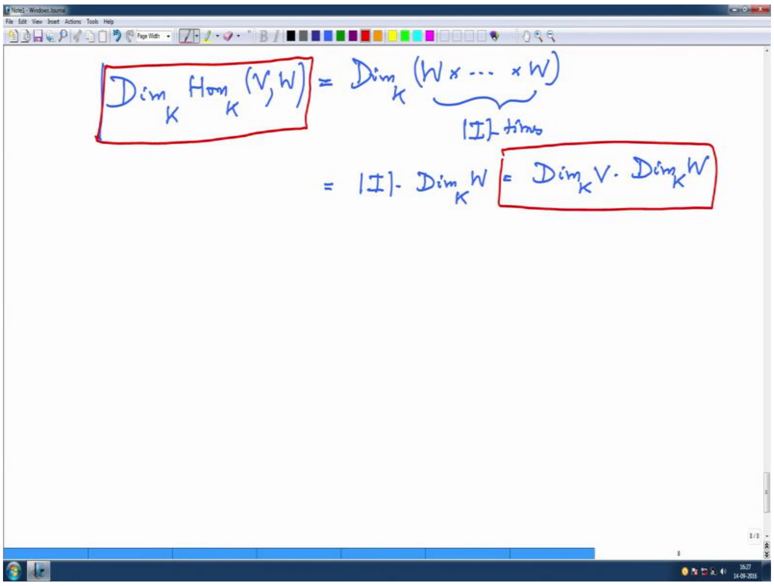

So, this is this. So, therefore, the dimension of this what we will get is dimension of the vector space Hom K V W will be equal to dimension of the product V W cross W cross W how many times mod I times. But dimension of the product is the sum dimension of the product is sum how many times I times. So, this is nothing, but mod I times the dimension of K dimension of W, which is and this mod I the dimension of W. So, this is dimension V times dimension of W. So, we get a nice formula dimension of the Hom, dimension of the Hom this dimension equal to this dimension ok.

So, what this left now let us let me just go back and show you what is left is not difficult to check, we want to check this map is K linear and this map is bijective. So, all that we have to do is you take any combination like this a f plus b g and where does this go under this map this goes to a f of V, f of V i plus b g of V i or this is same as the image of a f and image of. So, this is same as a f of V i, plus b g of V i and this is this one is same as a phi of f, and this one is same as b phi of g and this is this is phi of a f plus b g.

So, that checks that this map phi is linear; now to check bijectivity we have to show that every elements here is coming from a linear map that is surjectivity, and the injectivity that if f goes f and g goes to the same tuple then f equal to g, but that is precisely what we have approved in earlier 2 statements f equal to g if and only if they agree on a basis, and V i is a basis and similarly that give an any W i we can always find linear map f unique. So, that f of V i is precisely the given family W i. So, that completes the proof of this example completely and we will take a break and then continue after the break.

Thank you.