Linear Algebra
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Lecture - 25
Pigeonhole principle in Linear Algebra

Come back to the second half of this lecture and I would be like to note the immediate consequence of the earlier theorem that I have stated. So, let me write it as a theorem.

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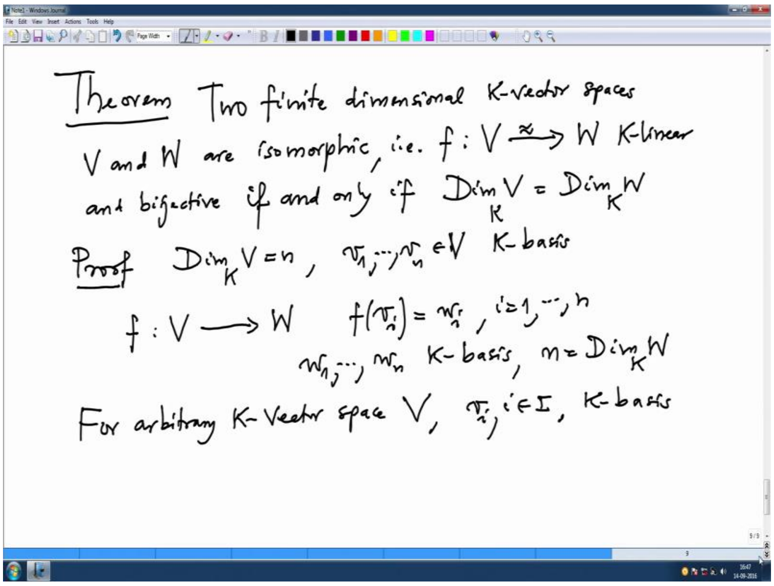

So, this theorem tells how do we test 2 finite dimensional vector spaces are isomorphic or not. Let me recall 2 vector spaces V and W; we say there isomorphic over K if there exist a K linear map from V to W such that f is budget. So, this is the isomorphism and we want to test economically how do we test given K linear map is an isomorphism or not. So, this is the contained of this theorem. So, 2 finite dimensional K vector spaces or isomorphic V and W, let us call it V and W are isomorphic; that means, there exist K linear map f from V to W which is K linear and bijective this also, I shortly derive like this is this means is an isomorphism K linear and bijective. If and only if V and W have the same dimensional dimension of V equal to dimension of W, we call that dimension means the number of elements in a basis and because we are assuming finite dimensional the dimension is finite; that means, dimension is natural number if these 2 natural numbers are same then there is an isomorphism from V to W conversely.

If there isomorphism from V to W then the dimensions are equal proof is very easy proof let us call 1 of the dimension; dimension of K V to be n and let us choose the basis v 1 to v n v K basis then we have seen to define a map f linear map from V to W all that we need to know is where should this v 1 v 2 v n should go. So, that mean f of v i if I know the values of f 1 v i if you call them w i is from 1 to n when can this f v n isomorphism that is if and only if earlier theorem, we have is check if and only if the image of a basis is a basis. So, in particular w n to w n is a basis K basis, but if it is a basis, this is the number of element there is the dimension of w. So, that is the n is the dimension of w. So, the map f bijective K linear map exists from V to W if and only if the dimensions are equal it is going to your proof. So, we have to proof this theorem. So, more for arbitrary for arbitrary vector spaces K vector spaces v choose always a basis we know basis exist v i i in I K basis then we have proto type of a vector space K power round bracket I.

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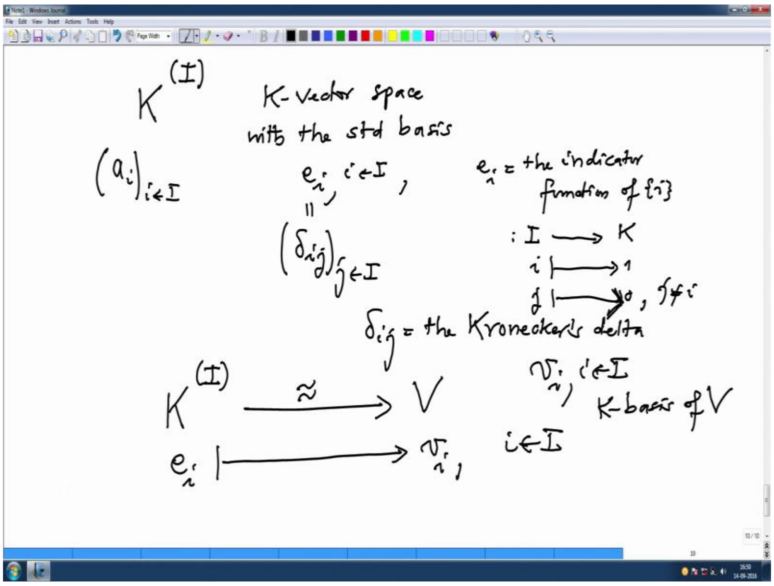

This is a K vector space with the standard basis; with the standard basis e i i is in I, what is e i? Either you think of e i with the indicator function of function of singleton i so; that means, e i is the map think of v i the map from I to K where i goes to 1 and all other (Refer Time: 05:50) j goes to 0 j not equal to f or you can think it is a tuple delta i j where J is varying in I. So, at the I th position it is 1 where delta i j is kronecker delta. So, this is we have seen that this e i equal K basis of this vector space and now we can define a map K linear map from K power I to V e i the I th basis element go to v i for all i in I.

Remember this v i is the basis of V, but this basis go to this basis therefore, this is a K linear map and this is an isomorphism. So, every vectors space is isomorphic to this vector space K power round bracket I, but remember this vectors space K round bracket I is a coordinated basis this e i is a very very special basis and when we wrote this K power round bracket I with that this e i came automatically. So, when we deal when we work in this vector space this basis is fixed and coordinates here elements here are the a i finally, mini i as are only can be nonzero. So, this comes with the given. So, any vector here is a tuple like this and when you want to calculate something you have to do the calculations with that only, but we would like to have a possibility to change the basis when you change the basis, you may simplify the coordinates and that might help drastically for complicated calculations this is what I will show you in the many many lectures which will come later.

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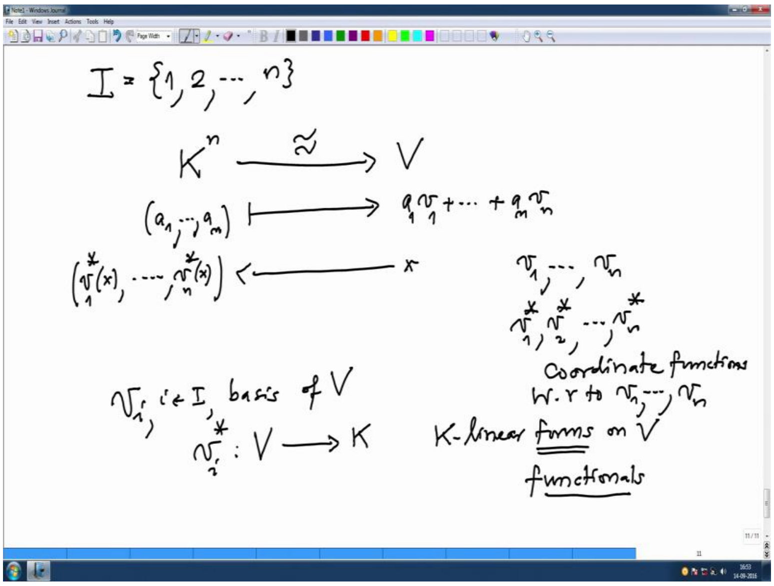

So, in particular case when I th finite when v finite dimensional; that means, when this I is the finite set if I is the set 1 2 n then K power n and v this is isomorphic. So, this is a tuple a 1 to a n actually going to the sum a 1 b 1 plus, plus, plus, plus, plus a n v n this is an isomorphism because a basis e 1 here goes to v 1 v 2 goes to v 2 and so on.

So, another example I want to do is actually in this case we can also write down the inverse map what is the inverse map inverse map is now your V should x; x should go somewhere x should go to the tuple here which tuple i will remain we have introduce this notation v 1 star x etcetera, etcetera v n star x we called that given a basis v 1 to v n we have define v 1 star v 2 star and v n star and this we called them coordinate functions with respect to the given basis v 1 to v n. So, that is the isomorphism or inverse map and this is not special about a finite basis in general we called at if v i is a basis of v then we have define this coordinate function v i star there are linear map from v to k. So, there K linear forms on v i use the word form because the values are inside this scalar flied some people also use the word functional there K linear functional on v.

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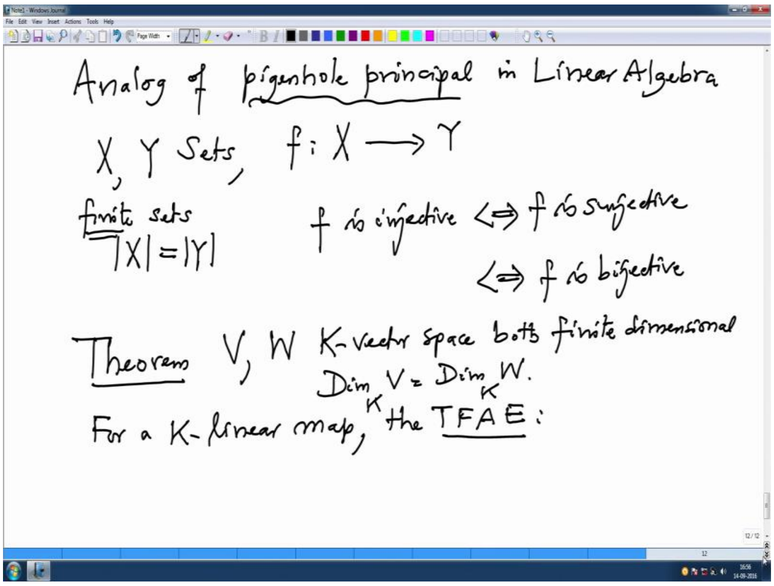

Now, the next one is very very important which we will use it many times. So, this is the next theorem is an analog of pigeonhole principle in linear algebra. So, let me just recall quickly what is the pigeonhole principle in set theory. So, in set theory if you have set x and y x and y 2 sets and suppose f is a map from x to y and assume that both are finite sets and they have the same number of elements cardinality x equal to cardinality y then any map f from x to y f is injective if and only if f is surjective, if and only if f is bijective this is the most concise we have state pigeonhole principle. So, this is we can put this in towards that if they are drawers and letters and the number of drawers etcetera the all this one can write it, but the most important is a map is injective if and only if it is surjective, if and only if it is bijective. So, the corresponding theorem for linear algebra is the following.

So, V and W K vector spaces both finite dimensional. So, this finite in set theory get analog is finite dimensional both finite dimensional. In fact, equal dimensional dimension v equal to dimension w that is given then for a map for a K linear map the following are equivalent this is the short form for the following are equivalent. So, what are the statements?

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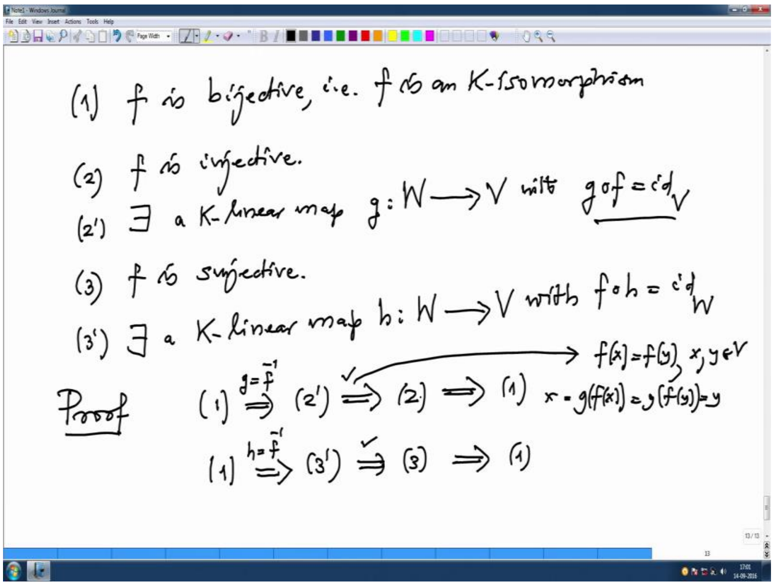

So, the similar to this, so, let us write them 1 f is bijective that is means f is an isomorphism K isomorphism 2 f is injective 3 f is surjective this 3 are equivalent this is the analog of the pigeonhole principle, but i want to write it 2 prime and 3 prime. So, injective, there exist a K linear map g from the other direction w to v with g o f is identity on V. So, this means take first f, f is a map from V to W this is f and then follow it by g this map is same as identity map on v. So, let me rub this part here what is 3 prime 3 prime is similar to this there exist a K linear map h from w to v with f compose g f compose h is identity on W. So, first h then follow it by f this composition is same as identity on w that is the assertion 3 prime again let me rub this part. So, now, we want to prove these statements are equivalent.

So, the arrangement in the proof is the following 1 implies 2 prime implies 2 implies 1, these equivalents is really prove 1 to 2 prime are equivalent and similarly 1 implies 3 prime implies 3 implies 1 if we proof this implications then that will prove 1 3 and 3 prime are equivalent. So, 1 is common in both. So, therefore, all this statements will be equivalent now let us see which implications among this implications are really very trivial to prove for example, 1 implies 2 prime we are given a bijective map.

So, if a map is bijective they will inverse and we have check that if inverse of a K linear map is also K linear. So, to prove 1 implies 2 prime, I will simply take g equal to f inverse which exist by 1. So, that will give map g and this with this condition similarly to prove 1 implies 3 prime, I will take g equal to f inverse h equal to f inverse of a K linear map is K linear therefore, h is K linear and this is identity. So, 1 implies 2 prime 1 implies 3 prime are trivial and 2 prime implies 2 we have given a map g of f is identity map and from here I want to check that f is actually injective, but that is to check this what we do is suppose f of x equal to f of y for some x and y in v then we wanted to check x equal to y, but you apply g to this equation g is given into. So, applying g we will get g of f of x equal to g of f of y, but g of f of y is y and this is x because of this. So, therefore, x is y. So, this implication is also trivial similarly 3 imply 3 prime is the same rate is trivial.

So, the only non trivial implications are to imply 1 and 3 implies 1 which we will prove. So, proof of 2 implies 1.

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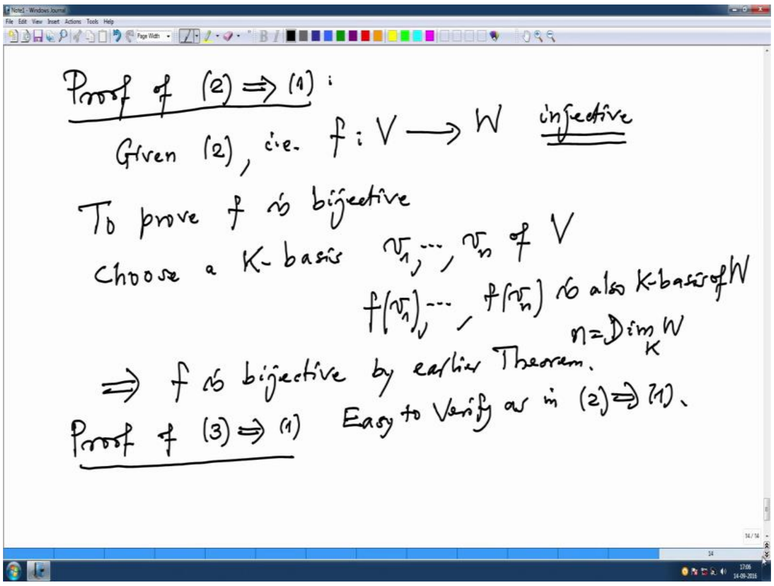

So, what do you want to prove? 2 we have given that there exist K linear map g from W to V with g o f equal to i d on v this is what we have given; oh no, no, no, sorry, what we have given? We have given that f is injective this is not given we have. So, sorry, I want to rub this. So, what do i given is 2 means given 2 that is f is the map from v to K linear map f is injective that is given and we want to prove to prove f is bijective this what we want to prove; that means, you also want to prove it is surjective you want to prove it is surjective. So, what we do we take a basis choose a basis v and w are equal dimensional finite.

So, choose a K basis v 1 to v n of v now because f is injective f is injective given then i then we have to noted earlier that the images of this basis f of v 1 2 f of v n this is also basis of w why that first of they are they are linearly independent earlier thing noted that if we have linearly independent the element and the map is injective then the images are also injective images are also K linearly independent. So, its K linearly independent element and n is correct n is the number of elements in a basis of W because V and W are equal dimensional. So, n is dimension of w therefore, this must be a basis of W because we have check that if a linearly independent set has a number of elements equal to the dimensional vector space then it must a basis. So, therefore, this map f maps this basis of v to this basis of w, but earlier statements say that to checks K linear map is bijective we need to check that the basis of V is map on to the basis of W. So, that f is bijective by earlier theorem.

So, prove of 3 implies 1 is similar. So, I will skip it instead of saying instead of checking when to check it is a basis I will check that it is a generating system. So, if it is generating system as a same number of a element as a basis then it must be linearly independent. So, this is, so I will skip this proof easy to verify easy to verify as in 2 implies 1. So, now, I want to consider 1 example. So, coming back again let. So, this is an example later on I will specialise to so on.

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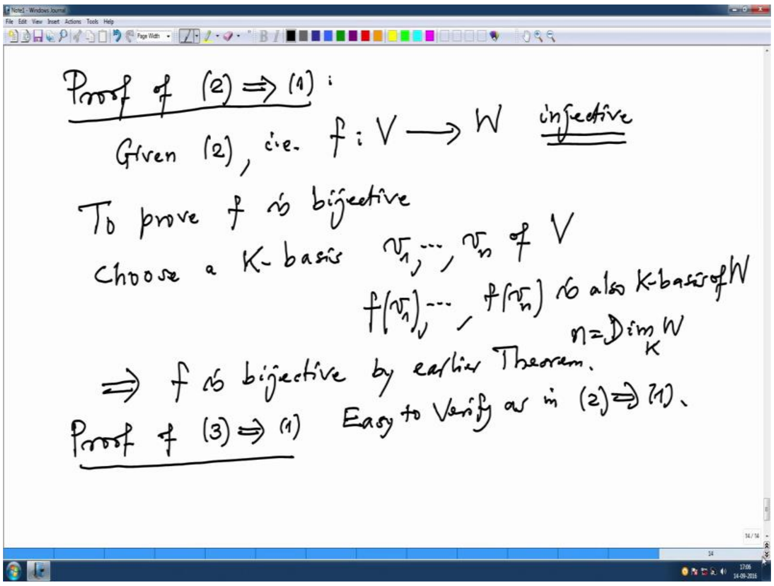

So, I have 2 vector space v and w v w K vector spaces and I have a linear map f from V to W and 2 vector spaces finite and equal dimensional. So, we will call the dimension of V and which is also dimension of W and suppose we have a K linear map f and what we want we want to solve is a j equation. So, given y in y, y in w; this is given then we look for all those x in v such that f of x equal to y this is a subscript of v this is also usually denoted by fibre or y f inverse y all those x which goes to y under f this is called the fibre of f over y or in short we are saying f of x equal to y this equation the equation. So, y is given and we are looking for this equation. So, this x is a solution. So, as a solution x, x in V, this simply means writing this simply means. So, that is the fibre over y is non inter set. So, there is at least 1 x so that than that will be a solution of this equation. In fact, all elements of the fibres are the solutions. So, ideally we would like to we would like to compute what kind of a space is this what kind of a subset if this fibre and if possible when it is a subspace we want to compute its dimension that is nothing.

So, for example, if you take y equal to 0 then fibre over y fibre over 0 f inverse 0 this is nothing, but the kernel of f all those elements of v which goes to 0 that is precisely the kernel of V, this is precisely all those x in V such that f of x is 0. So, this is a special fibre at y equal to 0 and this is not arbitrary this is not only a subset, but it is a K subspace of v and this fibre definitely has an element 0 in that. So, it is non empty. So, such as linear system as always a solution namely the trivial solution this is 0, solution is called a trivial solution.

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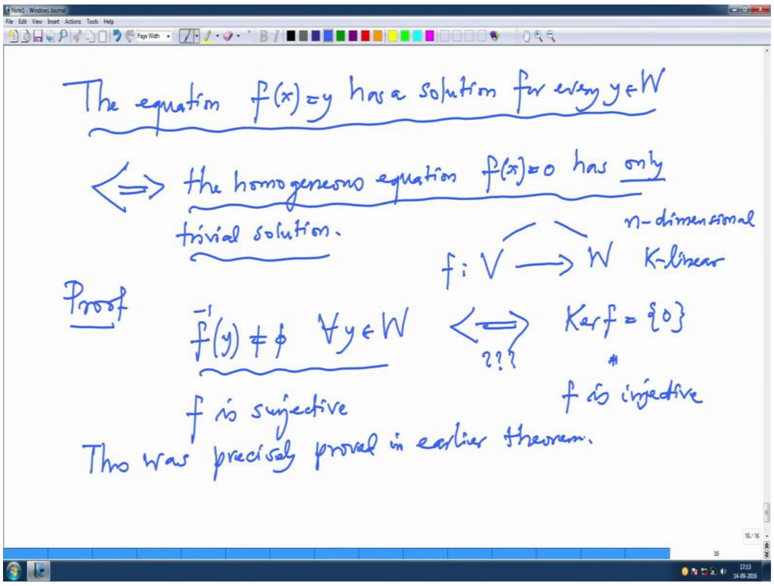

So, what the statement I want to make was the following, let me write that statement if statement I wanted to write was the equation f of x equal to y has a solution for every y in W if and only if the homogeneous equation f of x equal to 0 has only trivial solution the statement clear, this equation as a solution for every y this equivalent (Refer Time: 28:50) equation has only trivial solution. So, let us write the proof. So, let us translate this statement in our linear map language. So, f is a map from V to W finite dimensional both are n dimensional that is what given to us. So, what does this condition say the equation as a solution; that means, f inverse y if non empty for every y in g y in w this is what we have given one side the other side what is given the homogeneous equation f x equal to 0 has the only trivial solution.

So, that condition is simply kernel of a f is 0. So, we want to check this equivalence, but you see this is; what is this condition? This means f is surjective and this means this means f is injective and what are you asking of finite dimensional both are equal dimensional f is m linear map between them and one is surjectivity and injectivity saying this is what we are asking, but this is precisely the pigeonhole principle for linear algebra. So, this is this was precisely proved in earlier theorem. So, let us write still explicitly little bit more.

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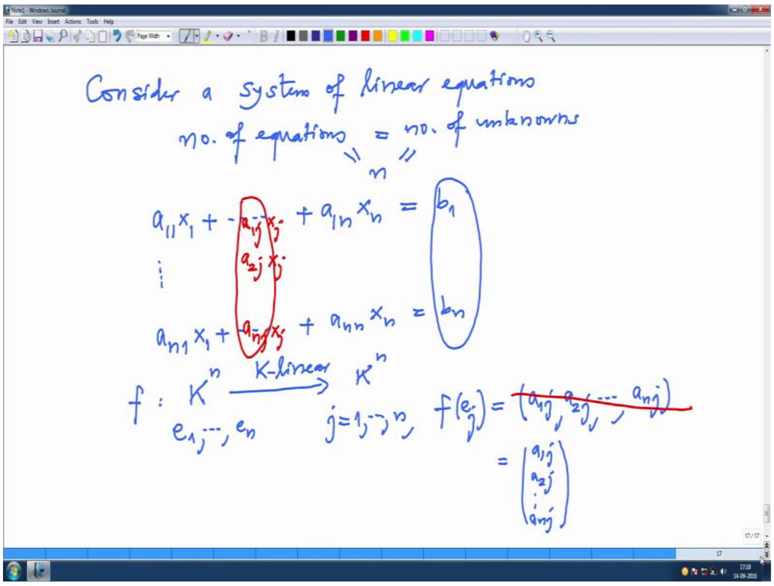

So, now consider a system of linear equations now I wanted to get a linear map from some vector space to the other vector space of the equal dimension. So, I will assume in this system of linear equations number of equations equal to number of unknowns. So, we are considering a system of linear equations where there are n equations and also n unknowns. So, that will typically look like a 1 1 x 1 dot, dot, dot plus a 1 n x n equal to b 1 and so on, this is a first equation is index by the first index n nth equation is index by n x 1 first index is n a n n x n equal to b n, now what are we saying earlier things says. So, this system you can think it is a linear map from f is a map from K power n to K power n, what is the definition of this f?

So, I want a K linear map. So, I will only, I have to give values on the basis e 1 to e n e 1 to e n is a standard basis of K power n. So, I have to note that for any j 1 to n where do f of v j goes. So, f of v j is another f of v j is a tuple n tuple. So, I have to tell, what are the elements there? So, this is by definition a 1 j comma a 2 j comma, comma, comma a n j this these are the coefficient here. So, these are it is easier to think this as not row I do not want to write it as row, but I want to write it as a column a 1 j, a 2 j, etcetera, etcetera a n j. So, I would prefer not to it this notation and I would prefer this simply because it is discussion here the coefficient of the jth variable see their coefficient of jth variable this is the term here will be a 1 j x j, next will be a 2 j x j and so on.

This will be a n j x j. So, this is a linear map, you can think, you should think this linear system as a linear map and what are we asking this as we our problem is to say that when this as a solution. So, for each for each b what we have approved in a earlier for each this tuple there is a solution here if and only if when I take the whole image system it as only trivial solution, but whole image system has only trivial solution means the map is injective and that is equivalent to using the map is surjective.

So, that is what the contained of this so; that means, what we i will just note it for every b 1 to b n in K power n.

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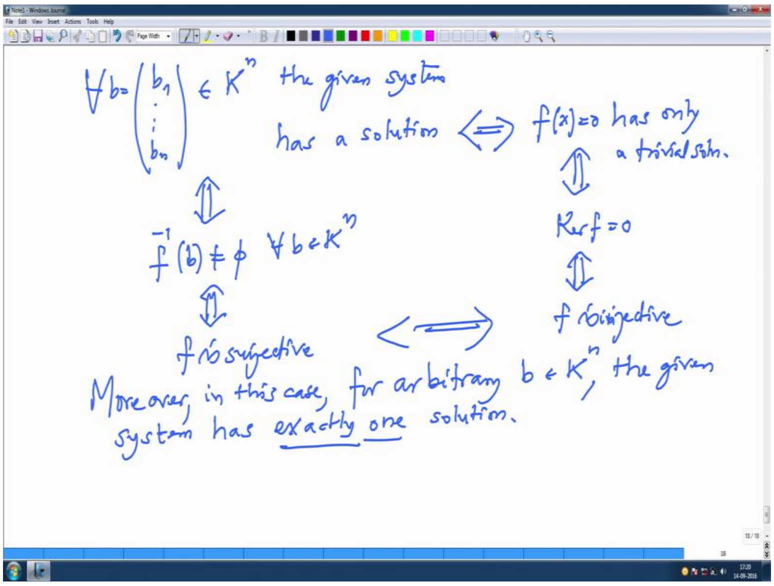

The system the given system has a solution if and only if f x equal to 0 has a trivial solutions has only a trivial solutions and this is equivalent to saying kernel f is 0 that is equivalent to saying f is injective and this condition is equivalent to saying f inverse of let call it b, b is non empty for every b if and only if f is surjective and this 2 are equivalent by the earlier theorem. So, therefore, we can conclude if these conditions in this case more over in this case what is the case either you say if the kernel is 0 or you say it is surjective in this case for every b for arbitrary b in K power n the given system has exactly 1 solution that is because we know these 2 are further equivalent to saying f is bijective. So, in that case fibre is non empty and not only non empty it has only one element. So, there is exactly one solution.

So, I will continue this system of linear equations to continue to think in terms of the linear maps. So, we will stop here we will continue next class.

Thank you.