Linear Algebra
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Lecture - 26
Interpolation and the rank theorem

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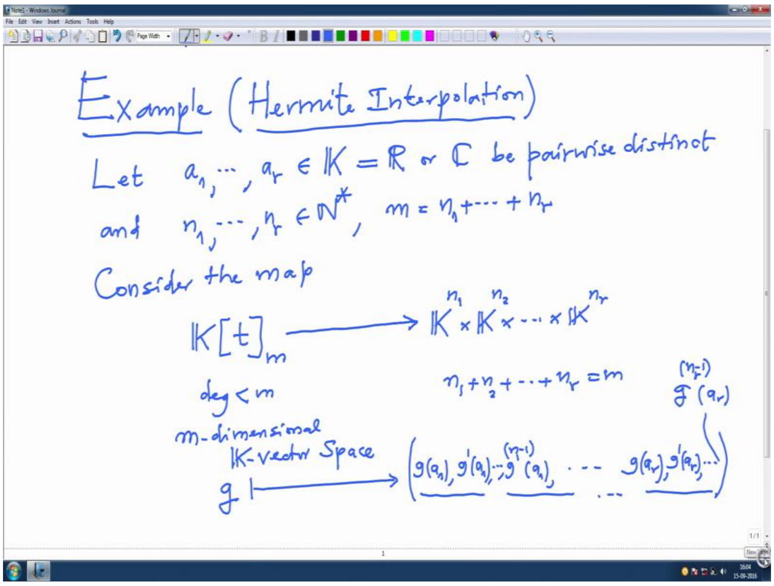

We were studying Linear Maps together with basis and I want to give the last example of this section which is known as Hermite Interpolation. Interpolation problems are usually it is about finding polynomials with some conditions and also it involves construction of such polynomials. So, one of such problem was Lagrangian interpolation and Newton interpolation with both their methods should concerned polynomials where at given distinct points it has a given prescribed values and we wanted to construct polynomial unique polynomials with some bond on the diesel. So, this is more finer than that.

So, let me just state the problem first. So, the problem is if you have R distinct elements in the field K, K double line is remember our notation this is either R or C - complex numbers or real numbers and I assume that this be pair wise distinct numbers.

And also let us take corresponding to a 1 is the n 1 into n r these are nonzero natural numbers and let us call into this summation of this n 1 to n r. Now what do you want to do? We want to construct the polynomial of degree is smaller than n such that the values of this polynomial at a 1, a 1 is 0 of this polynomial of multiplicity n 1, a 2 is 0 of multiplicity n 2 and a r is 0 of multiplicity n r. So, we are looking for polynomials and the degree should be bonded by this m.

So; that means, we consider the map, map is some the set of polynomials with coefficient system in K degree at most time, at most time. So, these are all polynomials of degree, degree of this polynomials is strictly less than m. So, this is we know this is m dimensional K vectors space. On the other hand because m is a m 1 plus m 2 plus n r, we have also this K power n 1 this is a vector space of dimension n 1, if I take a product with that K power n to etcetera etcetera K power n r this is vector space of dimension n 1 this is of n 2, this is of n r altogether this is go vector space of dimension n 1 plus n 2 plus plus plus plus n r which we are calling it m.

So, both these are vector spaces of the same dimension and there is a natural map here the map is I will write the map here take any polynomial g, g is of degree smaller than m and map this 2 now we have to map it to big tuple that will be blocks n 1 block, n 2 block, so on, n r block. Here that first block you write evaluate g at a 1 then the next we will evaluate g prime at n 1 a 1, g prime is the derivative of g and so on, go until g power n 1 minus 1 evaluated at a 1 and the last block will be g evaluated at a r, g prime evaluate at a r and so on. The last here will be g power n r minus 1 evaluated at a r.

These are the derivatives of g and then we are evaluating derivatives at the points a 1 to a r. So, I am going to use this map. So, we have checked that and both are vectors spaces of dimension m.

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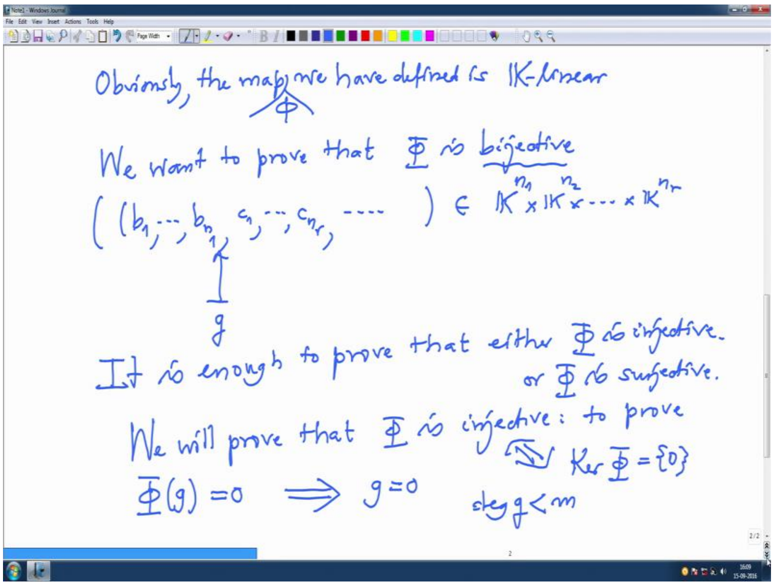

The map where defined is clearly obviously, the map we have defined is K-linear that is obvious because derivatives, second derivatives etcetera this is K-linear.

Now, I want to prove that this map is bijective. So, did I give I want to give some name to this map, let us give a name to this map capital phi; so obviously the maps phi is we have defined its K-linear. Now we want to prove that, we want to prove that phi is bijective; that means, given any a tuple of blocks it will come from the polynomial; that means, so this will serve this is what this will answer over question namely given then be so.

If this n if I given b 1, b n 1 and then c 1, this is 1, c 1, c n 2 and etcetera given such a tuple in K n 1 cross K n 2 cross K n r then this will come from some polynomial; that means what? That means, this polynomial g when I evaluated at a 1 I get b 1, when I evaluate its derivative at a 1 I will get b 2 etcetera then I will evaluate this derivative at n 1 minus 1 at a 1 I will get b n 1 and similarly for the next blocks and the not only that the polynomial will be unique because this map is bijective.

So, therefore, what we need to prove that is the map is bijective. To prove the map is bijective because both the vectors space is K-linear and both the vectors spaces have the same dimension it is enough, it is enough to prove that either phi is injective or a phi is surjective.

Obviously in this case I will choose to prove that phi is injective because it is easier than proving phi is surjective, surjective of phi in fact the problem of the interpolation. So, we will prove that phi is injective; that means, what do have to prove? If phi of some polynomial is g 0 then g should be 0 polynomial. So, to prove this injectivity is equivalent proving the kernel is, kernel of phi is 0 subspace; that means, if somebody in the kernel it should be 0 and remember here degree g is strictly smaller than m.

So, let us prove this. So, what we have given? We have given that phi of g is 0, but by definition of phi of g this is g at a 1, g prime at a 1, so on till g n minus n 1 minus 1 th derivative evaluate at a 1 and so on.

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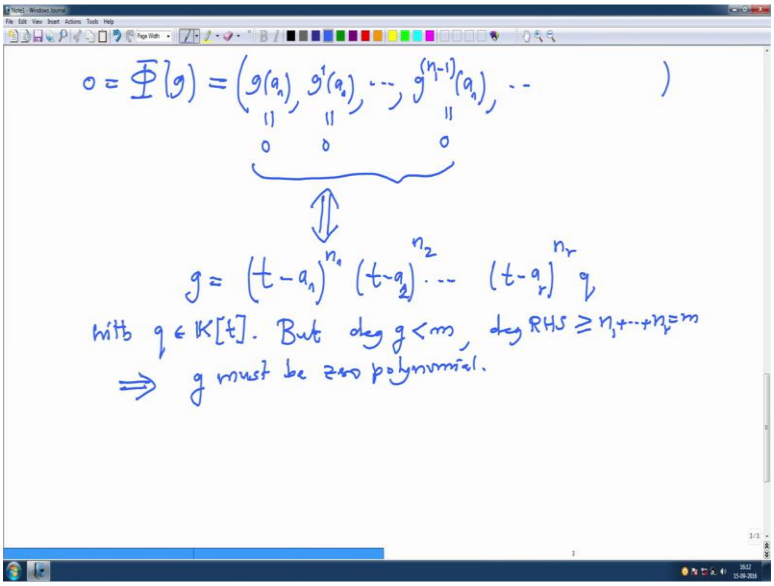

This is 0, this is 0, this are all 0 because the tuple is 0 then each component is 0, but this block full block is 0 means derivative of g, g derivative of n 1 minus 1th derivative of a 1 is 0 earlier derivatives also 0 that is equivalent to saying that t minus a 1, t minus a 1 is a factor of g that is clear from the first component because g validities at a 1 0, but the derivative is 0 will mean the square will also effective, n 1 minus nth derivative 0 means this power n 1 is the factor of g, factor I mean write from more. The second block is 0 means same argument t minus a 1 power n 2 is the factor of g because these are different a not a 2 these are different a 1 and a 2 different, so this factors are co prime. So, if this divides, this divides, then product also derived and so on.

So, this t minus a r power n r this is also factor of this and this factor is co prime to the other factors so; that means, the product divides that and may be something still have let us call it as a polynomial q with q some polynomial in t. But because the degree of g is strictly smaller than m and this is degree of the left hand side, but on other hand degree of the right hand side is degree of the right side RHS is definitely bigger equal to n 1 plus plus plus plus n r which is m.

So, the only possibility is g is 0 polynomial, so that implies g must be 0. So, that proves that proves of fact that this map is injective and therefore, bijective and therefore, we get what we want.

So, this proof is an existential proof; however, if one want to explicitly construct g in terms of this given numbers a 1 to a r then it requires a different argument which I will not give here because this example was to illustrate how do we use so called (Refer Time: 13:53) or principle in linear algebra.

So, now, with this I also want to, now you continue investigation of linear maps between 2 finite dimensional vector spaces and for this very easy or very trivial, but very important theorem which one keep using again and again is so called rank theorem, the rank theorem.

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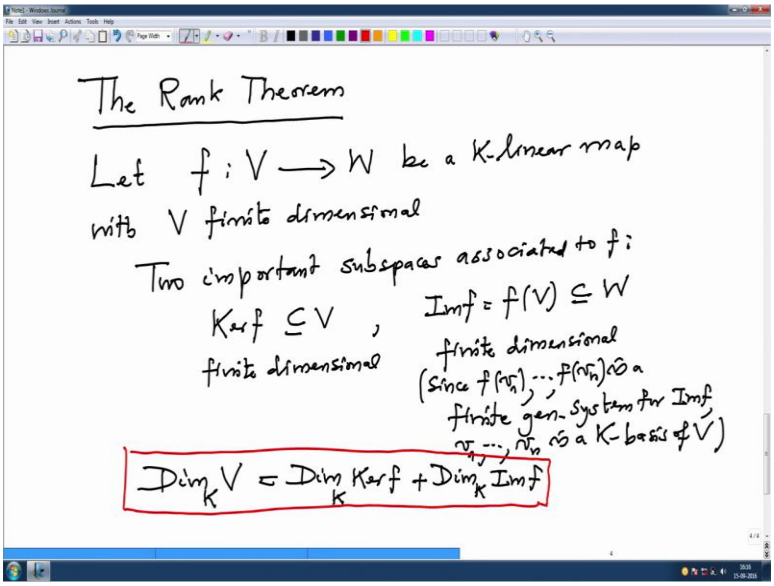

This is reality well theorem as we will see from the proof, but on the other hand it is also very very important theorem. So, this is in fact the beginning of the investigation for the linear maps between the 2 finite dimensional vector spaces.

So, let start with the linear map let f from V to V be a K-linear map and I only need to assume V is finite dimensional, with V finite dimensional and as we have seen in earlier lecture that the 2 important subspaces we have attached to this linear map - one is the kernel and the other is image. So, 2 important subspaces associated to f are one is the kernel of f which is the subspace of V and the other which is image of f which is by definition f of V and we have seen this is the subspace of W. And note that if V is finite dimensional that should we have assuming then kernel f is also finite dimensional because it is subspace of a finite dimensional and image is generated by a basis of f, a basis of V and V as a finite basis because we finite dimensional, so image f is also finite dimensional because in may a as if finite generating system. I will write it in the bracket since f of v 1, f of v n is a finite generating system for image where v 1 to v n is a K-basis of V. We have checked it in earlier lectures is somebody is generating system then image of that generating system is a generating system of image space. So, both this are finite dimensional.

Now, the rank theorem concerns the relation between the dimension of kernel, dimensional of the image and dimensional of V. So, then what we want to prove is dimension of V equal to dimension of the kernel plus dimension of the image this is what the rank theorem is; very nice important formula which is used very often.

So, let us prove this, proof is I will see it also trivial. So, proof we know kernel is a finite dimensional subspace. So, I can always choose a kernel, I have basis for the kernel.

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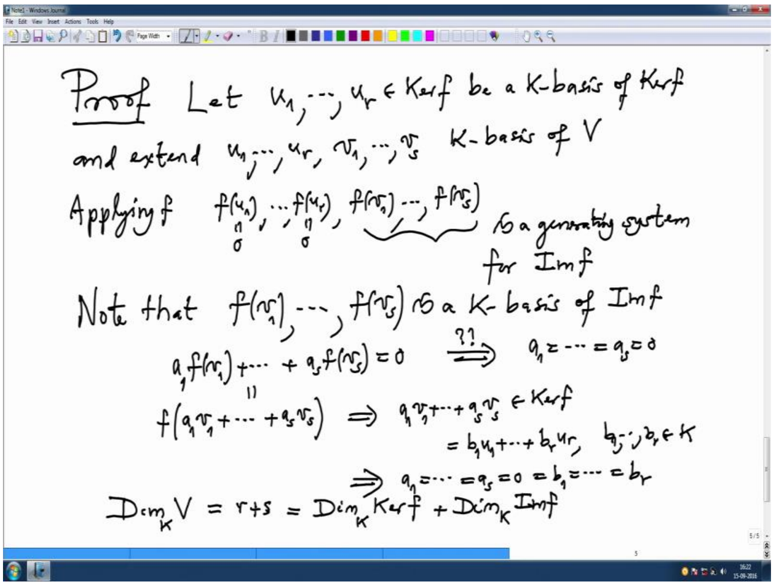

So, let before, let you want to u r in the kernel f be a basis, be a K-basis of kernel f and we have leart that whenever we have a basis of a subspace we can always extent to a basis of bigger space. So, and extend u 1 to u r by adding some more elements and let me call the extra elements to be v 1 to v s I have added extra vectors, I have extend this, this is the K-basis of V.

Now, what is the images under f when I apply f to u 1? So, applying f, f to this basis of v, f of u 1 is 0 this will happened to f of u r these are all 0 and then f of v 1, f of v s; obviously, then this is a generating is a generating system for the image because this is basis of the image and this is guys are 0. So, only this will contribute. So, this is a basis of the image.

Now, if I have prove that, if I know this is a generating system if I prove that this is a basis then I would know what is the dimension of the image space I also know dimension of the kernel space and then this so, so first note that f of v 1 to f of v s is a K-basis of the image. So, it check that you will have to, one will have to check that these are linearly independent because its already a generating system and if I check that it is linearly independent then by definition it will be a basis.

So, if a 1 f of v 1 plus plus plus plus a s f of v s this is 0 then we want to conclude that all this a’s are 0, but because f is linear this is nothing, but f of a 1 v 1 plus plus plus plus f of a s v s this f of this is 0, but that will mean that this combination a 1 v 1 plus plus plus

plus a s v s belong to the kernel, but kernel is as a basis we want to u r that mean this we can write it as combinations from the u 1 to u r. So, that is b 1 u 1 plus plus plus plus b r u r, for b 1 to b r a scalars.

But then this is this will give us if they should be imply that all area all b are 0 otherwise it will be a contradiction to the fact that e 1 to u r together with v 1 to v s is the basis. So, that should imply all the a's are 0 and all the b's are also 0. So, we have proved that they are linearly independent. So, we have proved that this image spaces is the basis, but then look at the formula what do you wanted to prove dimension of V because you want to u r we want to v s at basis and any 2 basis we have same number of cardinality therefore, this dimension is r plus s, but r is the dimension of the kernel and a is the dimension of the image.

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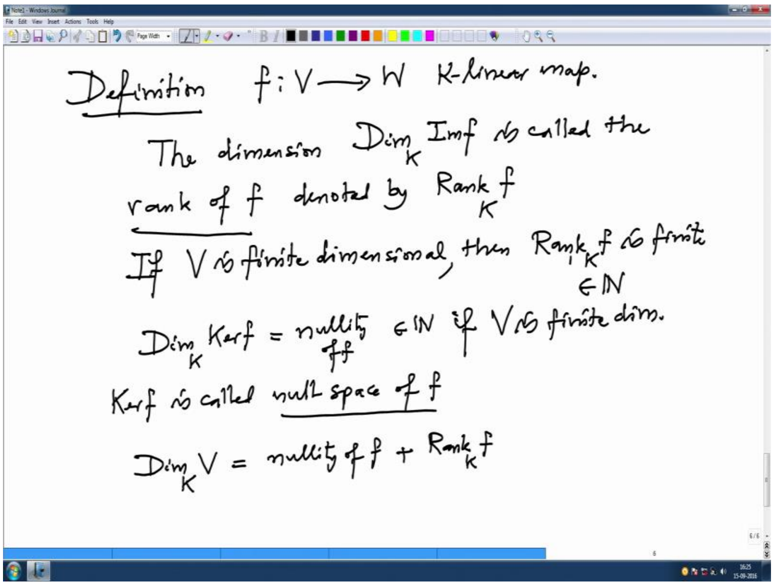

So, that proves our theorem. So, it is really very very easy theorem, but as soon I will show the example this is very important to know this theorem. So, before I go on I just want to introduce your definition. If f is a linear map from V to W, K-linear map then the dimension Dim K image f this is called the rank of f denoted by rank K f. Note that this dimension could be infinity also, but definitely it is finite when V finite dimensional.

So, if V is finite dimensional then rank of f is finite; that means, it is a natural number. Similarly some people also called the dimension of the kernel, kernel is also called null space, null space of f and with that also some people called dimension of the kernel as nullity and that is also finite, it is a national number n if V is finite dimensional.

So, the rank theorems simply says dimension of if V is dimensional then dimension of V equal to nullity of f, is nullity of f plus rank of f. So, it will also some times is known as rank nullity theorem. I will not use this terminology I will strict to what rank theorem, fine.

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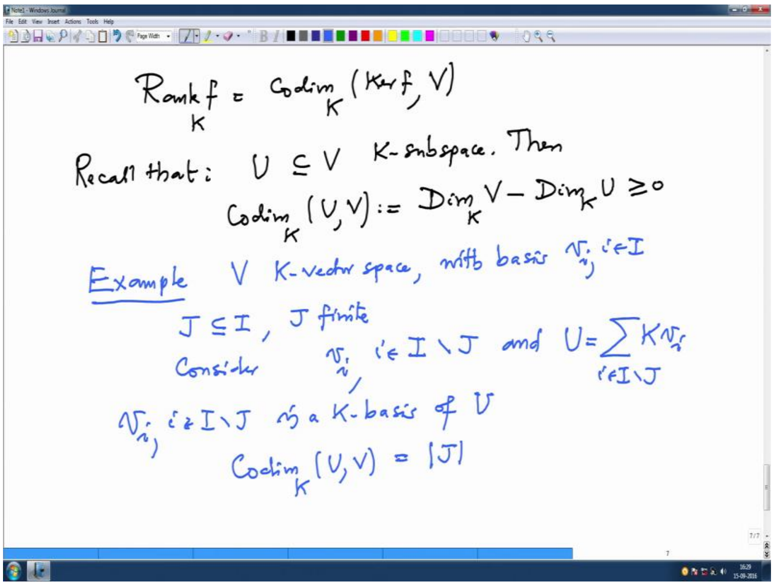

So, in other word also. So, the same statement, one can also write it as rank f is also co dimension of the null space in V. Recall that remember if U is subspace of V, K subspace then co dimension of U and V this is by definition dimension V minus dimension U definitely this is bigger equal to 0 because U is a subspace, co dimensional is bigger is 0. Sometime it may happen that V may not be finite dimensional, but co dimension U could be finite dimensional.

Lets you get the example. So, example for example, if you take V any K-vector space and you take basis, choose a basis with basis v i, i is the arbitrary index set. So, now you fix a subset, fix J subset of I, J finite subset and only J from the omit J from I so that means, you consider the family v i were is (Refer Time: 29:58) in i, but not in j. So, only if finitely many basis elements are not here in this family, namely those which are index by J and let U we have subspace generated by this v i's, K v i as i varies in I minus J.

Then obviously, this family v i, i varies in I minus J is; obviously, linearly independent and in its generating system for U, so this family is a K-basis of U. So, if we look at dimension V minus dimension W it is extra elements needed are J. So, therefore, in this case co dim K U V this is nothing, but the cardinality of J, is nothing but the cardinality J. So, this is one way to produce arbitrary co dimensional subspaces. This I will come back to this studies systematically in the section called quotient spaces.

So, I want to consider some examples which will illustrate the use of rank theorem.

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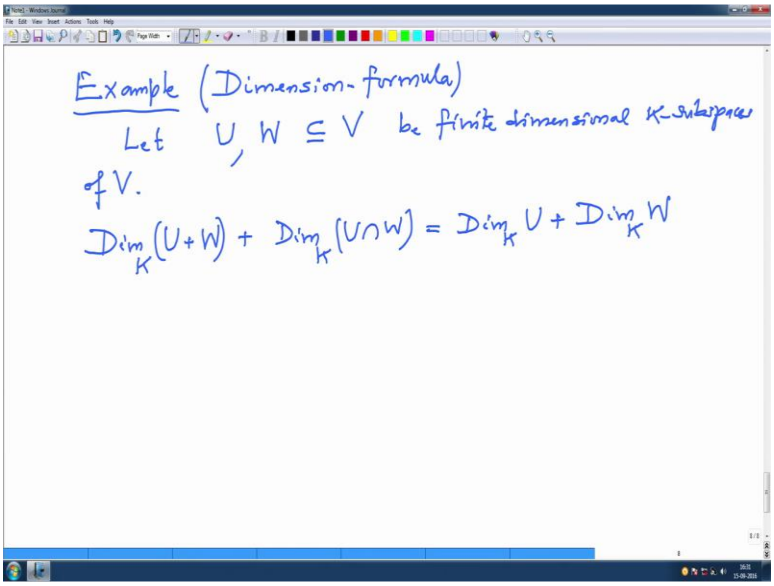

So, first example this is who have approved earlier also, but once again we prove by using rank theorem, this is dimension formula which say that if let U and W be 2 finite dimensional subspaces of V, K subspaces of V.

Then the dimension formula say that dimension of the some space dimension K of U plus W plus dimension of U intersection W equal to dimension of U plus dimension of W. we have proved this much earlier, but this time I want to give approve by using rank theorem and before I give a proof I also want to say that this is the analog of principle of inclusion exclusion from set theory. No, principle of inclusion exclusion is the cardinality dimension is replaced by the cardinality, but before I give a formal proof I would like to take a break and then we will continue to after the break.