Linear Algebra
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Lecture - 27
Examples

So, I will continue this lecture with the proof of this dimension formula. We have already proved this is formula once before, but this time, I will give a proof using rank theorem.

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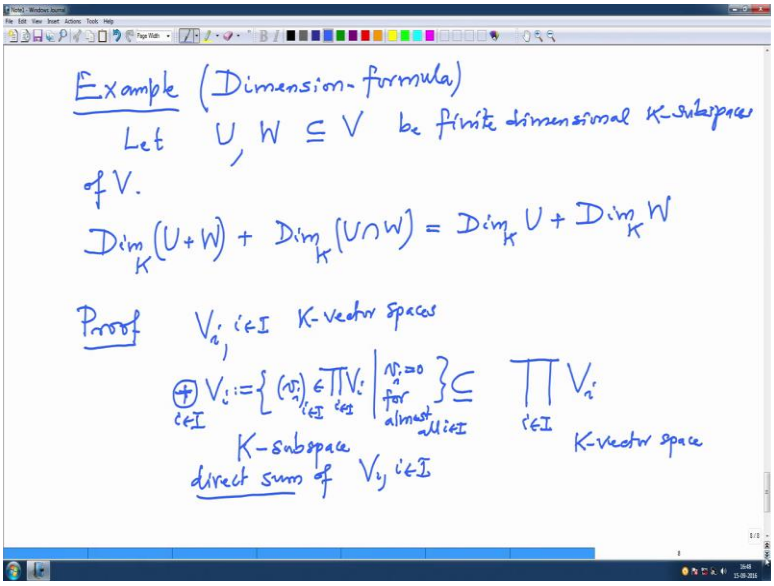

So, proof; so, let me recall first when we have 2 or many vectors spaces then we have to consider their direct sum of vector spaces. So, first for arbitrary family if V i where vector spaces K vector spaces then the product set on the product set Cartesian product of V i, we have a natural structure of addition namely the component wise and also component wise scalar multiplication. So, that make this Cartesian product as a K vector space and I can consider a subset of this which is denoted by this direct plus circle plus this is a subset of those i tuples V i with the condition that almost all v i's are 0, v i equal to 0 for almost all I, this subset is called and this is subset is also clearly subspace of this product space.

Because if you take 2 such tuples with this property that V i are almost all 0 then the sum will also that property scalar multiplication also we will have that property. So, this is K subspace of this product subspace and this is called the direct sum of the family V i, but in case of finite if I were a finite index set then; obviously, this subspace equal to the whole space.

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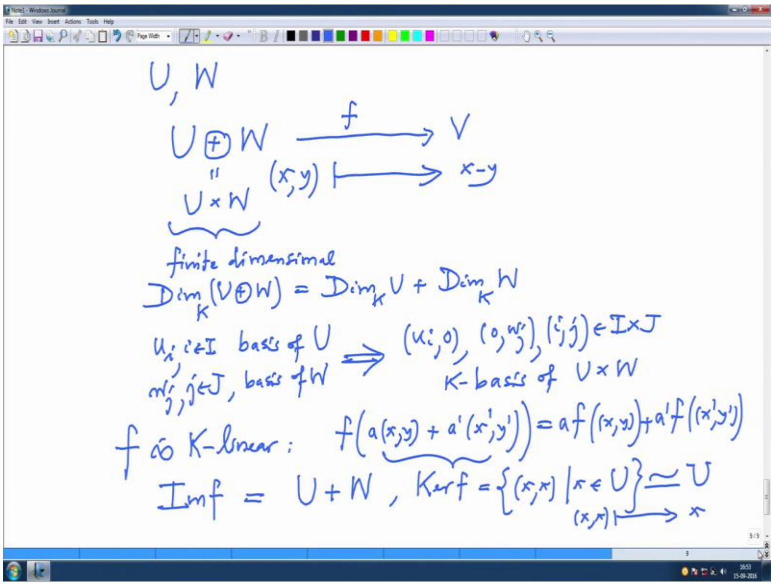

So, we are going to use it for 2s vector spaces namely U and W. So, the direct sum U direct sum W is simply the direct product with the component wise structure and; obviously, we have a naturally map from U direct sum W to V is a very natural map also note that if U and W are finite dimensional then this also finite dimensional and dimension of this will be equal to sum of the dimensions this is also very simple to prove what one does is take a basis of U and take a basis of W and look at the new set put first component the basis of the element of U and the second component 0 and the other way.

So, what I am saying is the following if u i i in I basis of U, w j j in J basis of W then this is very easy to see that u i comma 0 and 0 comma w j i j varying I is varying or I comma J varying in I cross J, this is the K basis of U cross W. This is very simple checking in particular dimension of the direction sum equal to sum of the dimensions and now look at the map which is defined from U direct sum W to V, there is a natural maps. So, the pair x comma y goes to x minus y that is the definition of these maps.

So, let us call this map as a f. So, what is easy to note is f is linear f is K linear so; that means, you have to check formula like f of a x y plus a prime x prime y prime this should be equal to a f x y plus a prime f x prime y prime, but this is very easy to check because this definition here is component wise addition component multiplication and the f is simply taking the difference. So, this is just a re agreement of vectors and scalars in the vector space. So, that show that if is scale linear.

Secondly, I want to say that the image of f is precisely the sum of u plus w that is; that means, we need to check that image of f is the smallest subspace which contain U and it contain W also, but look here if I take y equal to 0 then where this x is nu y is in W. So, if i take y equal to 0 that is in x is in U and that that goes to x. So, therefore, clearly use in the image U is the subset of the image similarly if W is the subset of image and it is the smallest one is apparently very easy to check. So, I will leave it to you. So, image is U plus W.

Secondly, I want to compute the kernel. So, kernel f is all those pairs x comma y where it goes to 0; that means, x minus y is 0; that means, y is x, but; that means, kernel is precisely the diagonal x comma x where x varies in U that is the kernel; obviously, and also note that this kernel isomorphic to u as a vector space simply you can take x comma x map to x this map is clearly linear and clearly bijective therefore, kernel of f is the isomorphic to. So, K vector space u in particular the dimensions are equal.

So, therefore, when we apply a rank theorem what do we get? Rank theorem says this dimension equal to dimension of the image plus dimension of the kernel.

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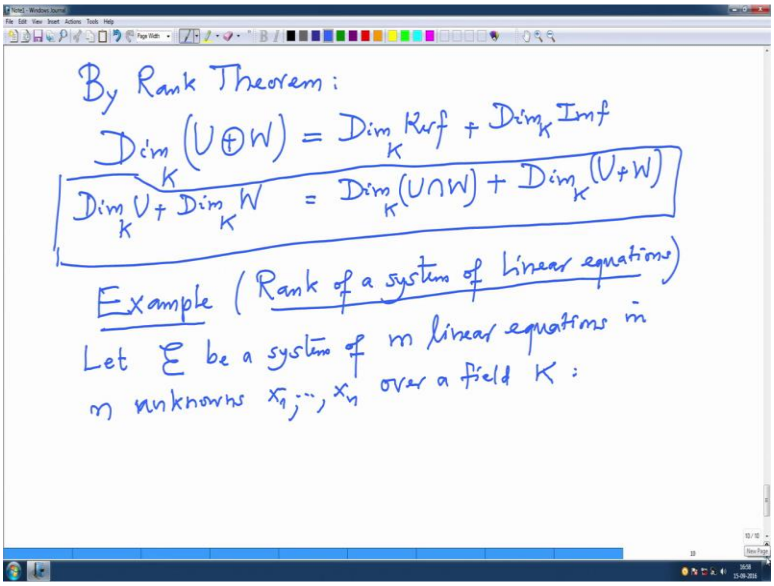

So, let us write down the rank theorem by rank theorem dimension of the direct sum equal to dimension of the kernel plus dimension of the image, but we have just seen dimension of the kernel is precisely dimension of U; oh! I made a little mistake.

So, let me go back to the earlier page. So, kernel f this is not isomorphic to U, but isomorphic to U intersection W because when we write x comma x that is x is should be nu as well as x should be in w. So, this is really u intersection w the map goes inside that and which is clearly K linear and K isomorphism. So, going back, this is kernel is the dimension of the intersection and image we have checked that that is nothing, but the sum u plus w and the other side we have just noted this is nothing, but the sum of the dimensions. So, it proves the formula what we want this is the dimensions formula it is written this side is written this side and this side is written that is the only difference.

So, one can also use this rank theorem to prove the one which theorem earlier namely the map f on a finite dimension vector space to finite dimension vector space is an bijective map if and only if it is injective for example, injective is equivalent to say in kernel f is 0, but then that will mean that the image f will have the full dimension, but that will mean that it is surjective and therefore, all together bijective. So, I will not write this explicitly here, but I would like to now discuss rank of a linear system.

So, the next example; rank of a system of linear equations, so, once again, start with the system of linear equations let, I will write E be a system of m linear equations in n unknowns which we are calling it x 1 to x n over a field K so; that means, when we did the Gaussian relation that is how we were writing.

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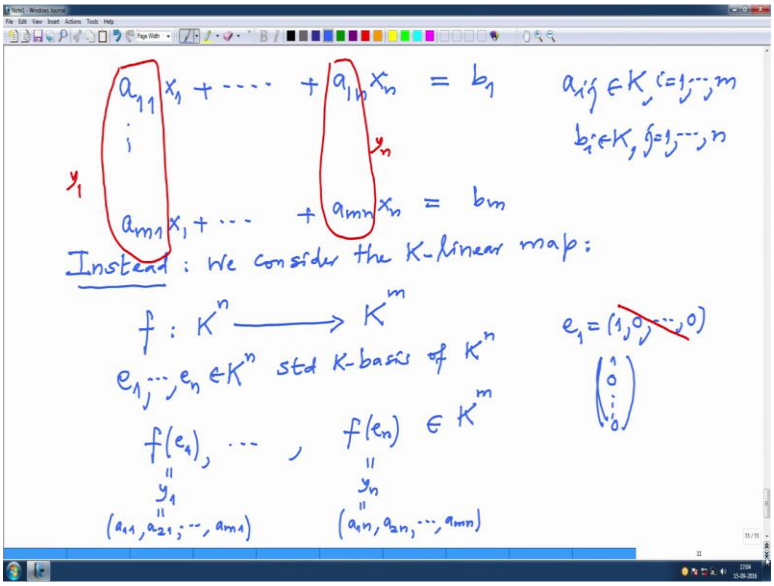

So, how we are writing we were writing the first equation the first index is for the equation first equation number 1 1 x 1 plus, plus, plus, plus, plus a 1 n x n equal to b 1, this is the first equation the first index of the coefficients a i s; a i j s is the equation number second index is the variable number or unknown number so and so on.

So, m th equation m 1 x 1 plus, plus, plus, plus a m n x n equal to b m where all these a i j there in K and b i are also in K where I index 1 to n and j is the variable number 1 to m. So, that is the data given to us and we are looking for the solution space that is x 1 to x n in K power n. So, this is how we were doing, but instead of thinking like this I want to think consider. So, instead make that notation as also very compact and easier instead we consider the K linear map f let us call it f from the vector space K power n to the vector space K power m.

So, note this vector space is this space of all n tuples with entries in K this is a space of m tuples with entries in K this as a basis the standard basis is e 1 to e n this also have standard basis, but because we will get confused with which e where we are using I will try to avoid using e's instead of I will write n tuples and m tuples. So, but I will need to use one of them at least. So, I will stay that e 1 to e n belong to K n, this is standard K basis of K n, I would also think them rather than rows, I would also better think as columns instead of writing e 1 for example, instead of writing like this n tuple like this first entry is one and remaining into the 0 that if first basis element here instead of writing this I would write it as 1 0 0.

So, this is instead of this, this is preferred so; that means, I give what is the linear map and just remember that you give a linear map from a vectors space 1 vector space to the other vector space I just have to give its values on the basis. So, I just have to give you what is the f of e 1 f of e n and they should be elements they should be vectors in K power m; that means, they should be m tuples. So, this one is what let us write down this 1 I want to call it y 1 this one, I want to call it y n and what is y 1? Y 1 is a 1 j a 2 j. So, on a n j m tuple either you write like this or like it as a column and for the references write it as a column similarly y n is a n 1, no, no, y m it should be it should be the other way sorry, this is correct, this is also the other way I have wrote it this is there is no j here, sorry.

So, this let us correct this y 1 is y 1 is y 1 is a 1 1 a 2 1 a m 1 y n is a 1 n a 2 n a m n, this should be m tuples. So, these are this 1. So, these are this is the first column this is you see here this one; this vector y 1 is nothing, but this one, this is y 1, this is y n, this is y 1 and this is y n,. So, that is the how the linear map is defined then what do then what do we do with this then what are we looking for.

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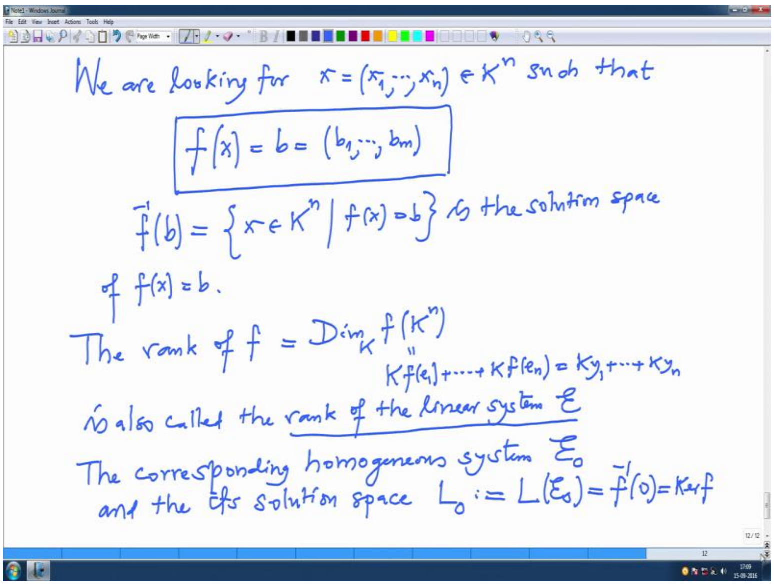

And let us write down what are we looking for and we are looking for solutions we are looking for tuples which tuples x let us call it x which is x 1 to x n in K power n such that f of x equal to b, b is b 1 to b n b 1 to b m we are looking for this. So, given b we are looking for all x with that this is equation is satisfied. So, one can write; one can say that this is linear system of equations. So, just f x is equal to b for a given b and now how do you get a x; obviously, then the fiber over b is the solution fiber over b which is by definition although x in K power n such that f of x is equal to b this is the solution space of f x equal to b.

So, now I want to compute the rank of this linear map f. So, the rank of f we this is the dimension of the image space f of K power n, but you know the image space is generated by this is generated by this is K f of e 1 plus, plus, plus, plus, plus K f of e n, but this is f of e 1 we have call it by y 1. So, this is K y 1 plus, plus, plus, plus K y n. So, you want to compute this rank. So, this is by this rank is also called the rank of the linear system e and what is what is the solution what is the corresponding homogeneous system in the corresponding homogeneous system remember that we have denoted by e 0; that means, in e in the equation seen e, we are putting b 0. So, that is and the solution space the corresponding homogeneous system denoted by this and the solution space and its solution space we have denoted by L 0 which is L 0 by definition L of e 0.

But what is this? This is nothing, but now f inverse of 0 because we are taking b equal to 0 the 5 over 0, but this is nothing, but the kernel of f.

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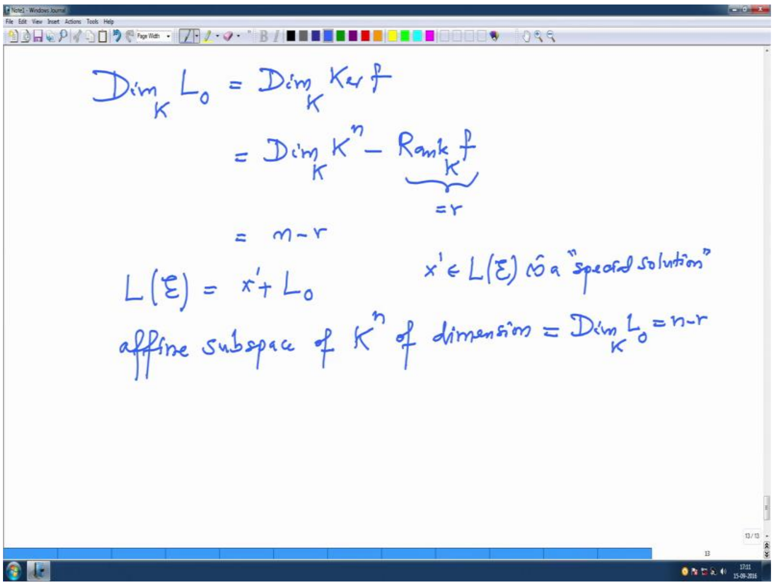

So, therefore, dimension of the L 0 is nothing, but the dimension of the kernel f which is by rank theorem this is nothing, but dimension of v v in this case is K power n minus dimension with the image rank minus rank of f if you call this rank to be r then this is and this dimension we know K power n dimension n minus r. So, we know dimension of L 0 is a n minus r where this r is a rank f of the rank of the linear system that is how we are defined rank of a linear system and then we have we have note we have proved that as a solution space of the e is a nothing, but this a fine space x prime plus L L 0 where x prime is a special solution in L E is a special solution defined some of the special solution and once you know one solution then all other solutions are precisely the elements of f L 0 add it to that special solution.

So, therefore, this L of E is an affine subspace of K power n of dimension of dimension equal to dimension of dimension of L 0 which is n minus r where r is a rank of the system. So, this language affine spaces etcetera, etcetera, we shall we are planning to do it a second course on geometrical linear algebra. So, with this I want to end the section and I would like to start now studying more precisely.

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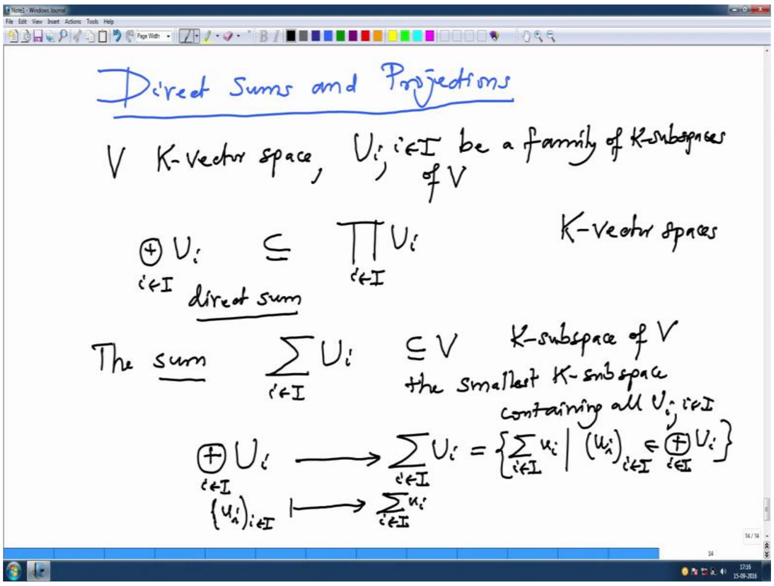

I want to start a new section called direct sums and projections direction are already we have done something, but I will do it together with a projections. So, I will as I said that V K vector space and let u i i in I be a family of K subspaces of V then we have this product space u i and also the direct sum which is a subspace of subspace is default. So, note that though they are subspaces of V. This product is not a subset of V and therefore, this direction is not a subspace of V. So, they are some K vector spaces.

But I still want to because these are this is this is not an arbitrary familiar of vector spaces this family of subspaces which came actually from v. So, there I have to be some connection with the given vector space. So, this is what I want to do it more precisely for example, we did already some similar thing in the example when I proved that dimension formula that I did it only for 2 subspaces, but now I want to carry on that study more precisely for arbitrary number of subspaces of e. So, as for the vector space is concerned I can consider a sum of u i s, this is the subspace of V K subspace of V, you know it is the smallest K subspace containing all u i s and I would to like relate these 2 this is the direct sum and this is the sum.

So, this is the sum the sum space and this one is the direct sum. So, apriori they are different the; they are even as a sets are different, but i would like to relate them the linear map and; obviously, one can guess from the earlier remark that there is a natural map from the direct sum u i i in I to a sum remember the elements of the sum space is precisely the finite sums of elements which are coming from each u i.

So, this is we have seen earlier this is sums like summation u i i in I and only finitely many components are nonzero that can be written simply has the tuple u i belongs to the direct sum. So, it is obvious the tuple u i in this direct sum as only finitely can be finitely many components are nonzero that is map to their sum and; obviously, we will study this map and what we have to do is we have to compute a kernel we have compute to image and so on. So, this I will continue from next lecture and then along with that projection maps sum some special endomorphism of a vector space. So, all this concepts I will try together and try together nicely so that these concepts will be very useful for the further use.

Thank you.