# Linear Algebra

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## Lecture – 28
## Direct sums of vector spaces

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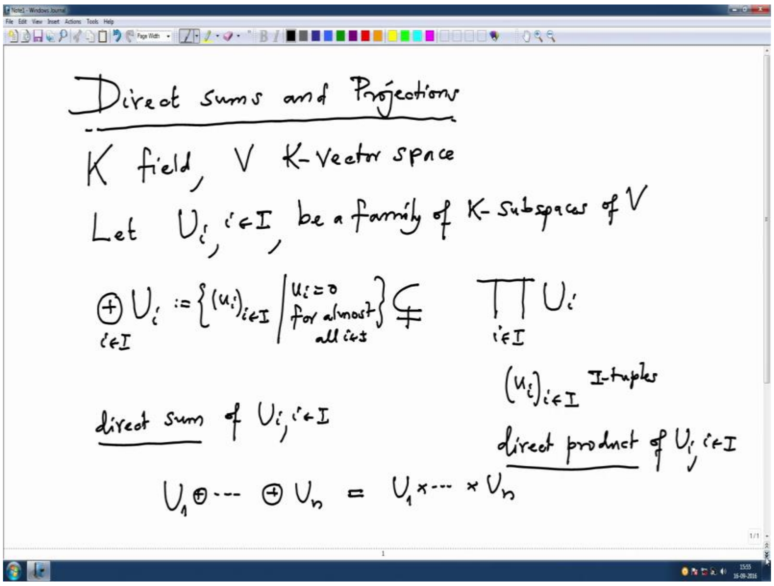


So, in today's lecture I will start the new section on direct sums and projections. Before we star formally I would like to motivate little bit direct sums easy concept which will be which will play very important role in the investigation of linear maps especially and it is also generalization of concept of basis that we will see in a minute and projections are special kinds of linear operators and vector space. So, when I have enough vocabulary that I will say more about it.

So, as usual we will say K is our field scalar field and V is K-vector space and what we are looking for is we want to break this vector space into smaller pieces, so that linear operators on V can be restricted to those subspaces and that study will become easier. So, this is the (Refer Time: 01:55) of the matter. So, I will recall first what is we have done earlier and then supported by examples how useful it is. So, let U i be a family of K subspaces of V could be finitely many, could be infinitely many and so on.

Now, remember we have defined product on the product set Cartesian product set we have a vector space structure here which is component by the addition which is coming from here and component wise scalar multiplication and the elements here we are denoting by I-tuples U i, i in I these are I-tuples. So, and we have consider subspace of this that we have called it direct sum, direct sum U i this is a K subspace of this bigger one precisely all those I-tuples for which U i equal to 0 for all most all i in I and this we have called direct product and this we have called direct sum of the family U i.

This is also of U i in general they may not be equal especially when I is infinite then they are definitely not equal if all of them are nonzero and infinite case they will always equal. So, if we have finitely many U 1 to U n subspaces then if you write like this direction sum that is same as the direct product because they are no condition on the tuples. So, in this case also we know there dimensions, dimension of this will be precisely the sum of the dimension not only dimension you will also know the basis if you know basis of U 1, basis of U 2, basis of U n.

Then you can put into together and the adding 0 to the other components will get a basis of this. So, this is what we have seen earlier and another thing another construction we have seen.

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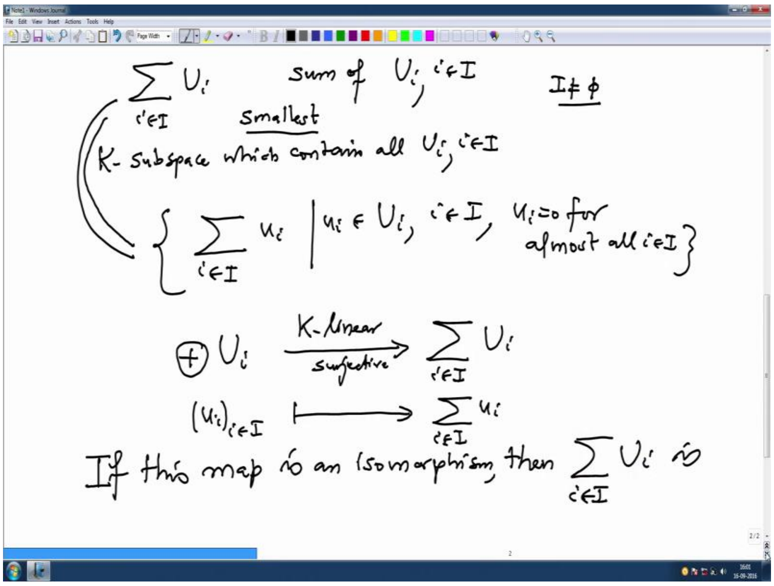


Another construction we have seen means for a family of subspaces U i we have also constructed a new subspace which is which we called it a sum, sum of the family U i and by definition this was a K-subspace smallest, smallest K-subspace which contain all U I’s and such a smallest subspace exist if I non empty of course, and that is denoted by this sum and it is denoted by this sum notation because we can actually described the elements of this, this is precisely the set of all linear combinations U i where U i where is in U i for each I and also this tuple U i actually belong to let me not write tuple because if the condition U i equal to 0 for almost all i. This condition is ensure that this sum make sense in general and it is very easy to say that this actually a subspace each contains all the U i's and it is the smallest one. So, therefore, this a quality here.

Now it would be very nice it would be very nice if we to compute a dimension of this sum for example, even for sum of two subspaces we had to use dimension formula because that is a like inclusion exclusion principle in linear algebra. So, it is not quite easy for many subspaces how do we compute the dimension, but they definitely there is one nice linear map from a direct sum to a sum, what is the map? Any element here is a tuple U i, i in I with a condition that almost all U i's are 0.

So, you can map this to the sum this make sense and these map is; obviously, K linear map, not only K linear also it is evidently it is surjective because any element of this sum just now we noted any element of this sum is a sum like this and that sum has only finitely many components could be nonzero and so it is coming from the corresponding I-tuple. So, if we can manage to compute the kernel then we could use rank theorem to decide what is the dimension of this and it would be also nice if this is an isomorphism because if this is an isomorphism then this dimension will be equal to this dimension, this dimension we know it is sum of the dimensions of the U i and therefore, this dimension we would know.

So, it is interesting to note when I will characterize soon characterization where this map is an isomorphism. So, that is aim first two minutes I will characterize the situation where such a map is an isomorphism. If it is an isomorphism, so let me also use the terminology if this map, if this map it is no obviously, it is not always an isomorphism especially when the intersection. So, even for two subspaces where the intersection is nonzero then this map cannot be an isomorphism, if this map is an isomorphism then the sum space this is called then the sum space is called the direct sum.

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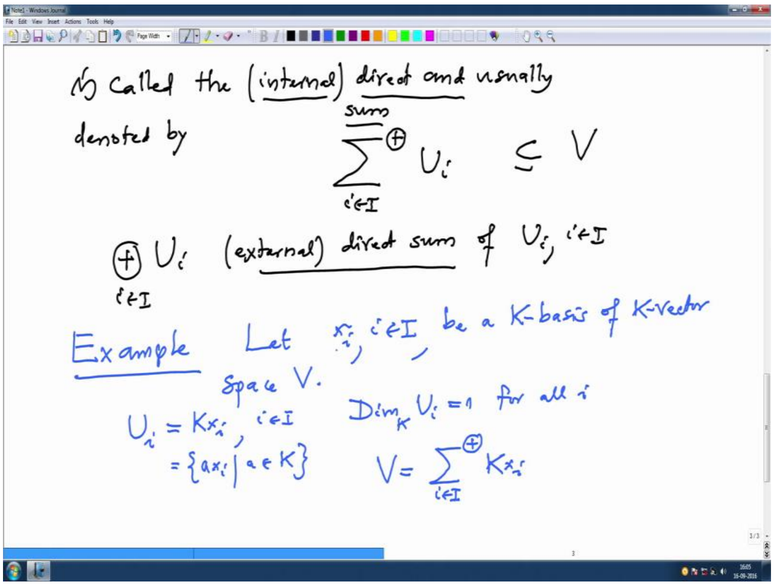

Now, to distinguish I will write here internal direct sum and we will write and usually denoted by sum and I will put that plus or on circle that is like this U i.

We will analyze when the sum is a direct sum soon. Why do we call it internal because see all this is happening inside a vector space V, this is this sum with a subspace in inside V with additional property that that above map is a isomorphism that is a reason why one call as it a internal direct sum, internal direct sum. In the other this notation, so for the same reason it is called external, similar reason external direct sum of the family U i, but if one is carefully it will it is visible from the notation because we are using different notations right. It is called external because this is not happening inside way it is happening inside a product phase. So, there a vector space structures are different. So, that is a reason it is called external direct sum.

So, let us see one example to illustrate this, this example will also illustrate that this concept is a generalize, see internal direct sum is an extension of the concept basis. So, that is what I want to illustrate in this example. So, V is a vector space. So, let x i, i in I be a basis be a K-basis of a K-vector space V then you look at the family U i, U i is the subspace generated by x i for each i.

So, this U i nothing, but scalar multiples of this fixed x i. So, they are a x i, a varies in the field K. This is one dimensional if dimension of U i is exactly one for all i and V in fact, whole V is a direct sum of sum of direct internal direct sum of all these K x i's this is clear because first of all V is a sum because every element of V is a K-linear combination of this x is because this is a basis and not only K-linear combination such combination is also unique the coefficients are uniquely determined and that will make that map to be an isomorphism.

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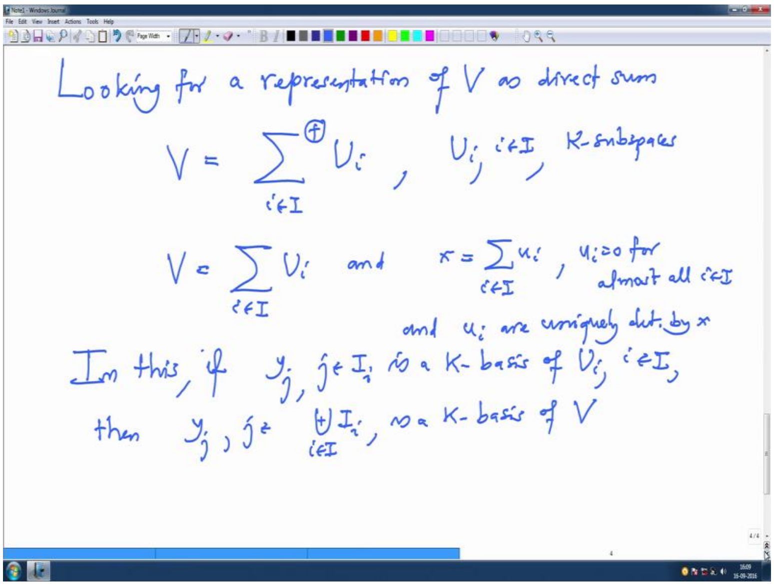


So, therefore, what we are looking for is we are looking for decomposition of V, we are looking for are representation of V as direct sum like this direct, not like this at least for not getting confuse we should use this notation i in I U i, V equal to this where U i a is family of K-subspaces. U i is may not be one dimensional U i may be more dimensional may not be one finite dimensional, but we want to break this if V like this and how we choose U i it will depend on the linear operator that we want to study on V and what property we would like to have.

So, that we can restrict to when we restrict to this subspaces U i we can induct on the dimension and many such problems we will become easier, so that is the idea. So, when the as we go on more and more of such examples I will keep saying. So, in this case what, what are this, this direct sum decomposition means; this means V first of all that two things V should be the sum and every element of V because of this equality every element x in V is can be written as summation U I, i in I with the property that this U i is are almost all 0 for almost all indices i. But such a expression in general may not be any and we want to insist now this U i are (Refer Time: 16:55) determined by x and U i are uniquely determined by x.

This is precisely the condition of the map the natural map we have from the direct sum to the internal direct sum that map is injective this is precisely the meaning of that. So, this will make this sum to be internal direct sum. So, that is; and in this case if I know in this case, so you can do more in this case if I choose a basis for each U i if y what do I call it, y j; y j as j varies in sum index inside I i is a basis a K basis of U i this is for each i in I for each subspace which is the basis and put it together; that means, what you can see then y j as j varies in the union, union of all these I i is and disjoint union i in I this is a K basis of V. Because they do not, they will not intersect. So, this is the main thing, and let me you will let us first analyze what happens. So, in the first thing I want to notice the following.

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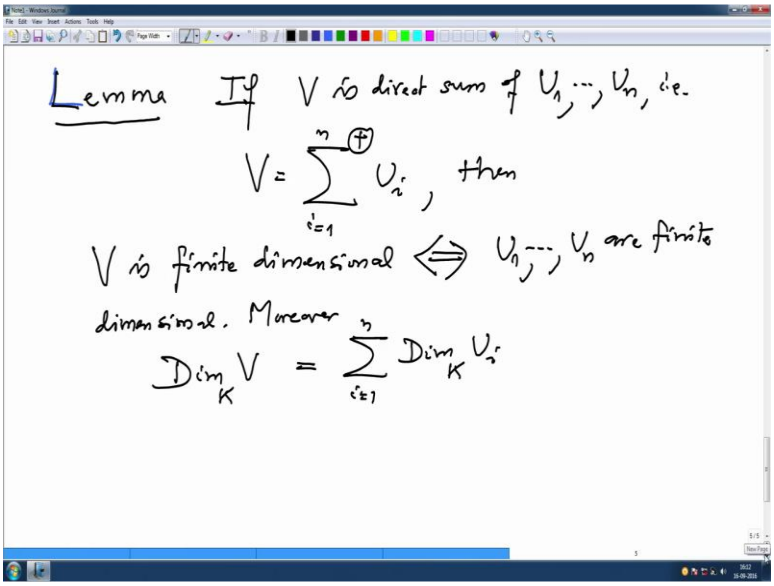


So, let us write it as a lemma. Lemma, if given vector space V is a direct sum of the subspaces finitely many U 1 to U n and now I am going to drop internal external it will be clear form a writing what I am talking, that is V equal to this, i is from 1 to n this means V is a sum and that is direct sum then V is finite dimensional if and only if U 1 to U n are finite dimensional and more over dimension of V will be the sum of the dimensions.

Just above that I said. So, the proof is just simple verification that you take the basis of each U i and put it together and that will be basis of V. So, this is precisely what we did in a when we proved the existence we proved that you in fact, you either one dimensional, but sometimes there may not we would like to have a decomposition of V in a direct sum, but with special other properties and that time one dimensional may not exist for example, when you are looking for Eigen vectors, Eigen values that time it may not be one dimensional and then we will have to, so this is where.

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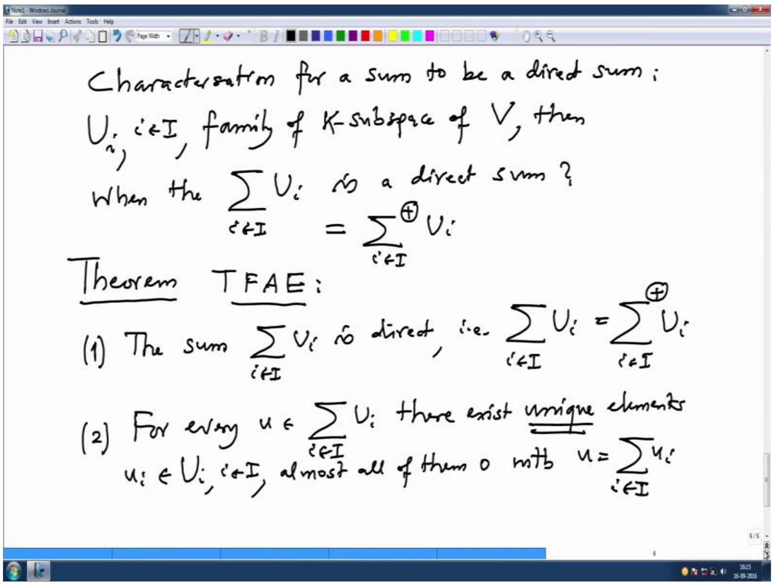


Now, I am going to characterize when the sum is a direct sum; that means, when that above the first map is isomorphism and especially the kernel injective, see more importantly the injective surjectivity is just definition. So, a characterization for a sum to be direct sum this is what we have done; that means, if I give any family U i, i in I family of K-subspaces of V then what we what I want to say is when the sum is a direct sum how does one test it easily because to check sum map is injective we have to find economical way to test when the map is injective tha, that the given map. So, let us write this in the form of theorem that is we are asking when it this equal to this.

So, with the same notation as above the following are equivalent. Number one the sum is direct, is direct. So, that is in the writing the sum equal to this to for every u, element u in the sum there exist unique elements u i in U i almost all of them 0 with u equal to summation u I; see every element as such expression that is no big deal, but what is more important is unique.

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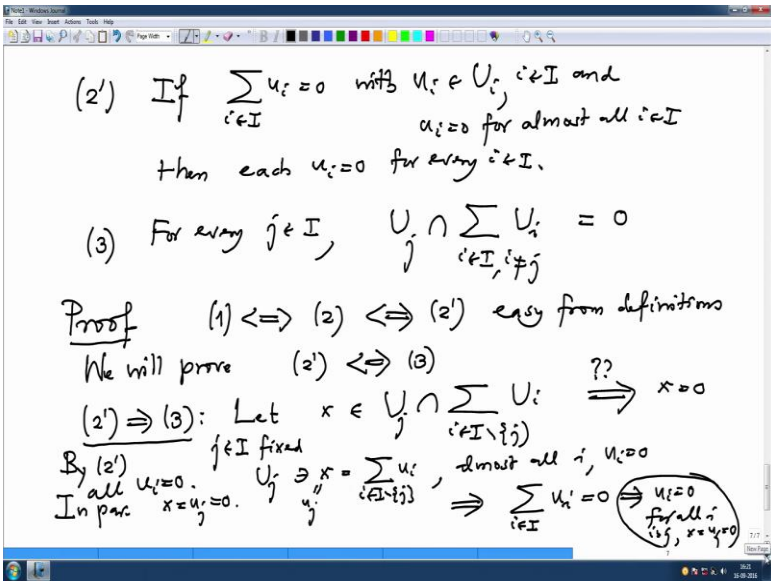

Third one or let me call it two prime. If summation u i is 0 where with u i's elements in U i for each I and U i is 0 for almost all then each U i 0 similar to linear independence this is two prime and third for every j in I, U j intersection the remaining sum i in I and i is not j, u is U i this has to be 0. Remember for two subspaces we have defined sum u plus w is a direct if intersection u intersection w is 0 this is similar to that for two it is matching with this.

So, let us prove this. So, proof 1 if and only if 1 if and only if 1 prime this is easy from definitions. I will indicate the proof 2 prime, so we will prove 2 prime if and only if 3 and that will make equivalence complete. So, first let us prove 2 prime implies 3; that means, we have given that if sum is 0 with each U i in U i and U i 0 for almost all then each U i 0 then I want to prove that this intersection is 0. So, take somebody common elements. So, let x belong to U intersection i in I, i minus j let me write it U i and j is fixed j in I, j in I is fixed then we want to prove, this is what we want to prove that x is 0 by assuming 2 prime.

Well, x is here means what x is here means first of all x is in this sum. So, x will I will expression summation U i, i in I minus j and almost all i's also got here j, almost all i, for almost all i U i's are 0, so that this sum make sense.

On the other hand this is also in U j because it is in the intersection so; that means, I can call it u j, call it u j and shift this to the other side, then what do you get? We will get summation U i now I can generate i in I because this is j added there with the minus if you like, but if u j is in u j then minus u j is also in u j and then this sum is 0 with the property that almost all of them are 0 because almost I have added these u j almost all are 0 and the sum the now 2 prime say that if the sum is 0 then all of them are 0.

So, that will imply U i equal to 0 for all i, in particular for i equal to j, for i equal to j x is u j and that is also 0 so that means, we approved x is 0. So, this is, I will write here if more this part is slightly invisible. So, I led with the sum U i 0 by 2 prime all U i's are 0, in particular x which was using that will also be 0 that is what we wanted to prove. So, that proves 2 prime implies 3. Now let us prove the converse, 3 implies 2 prime.

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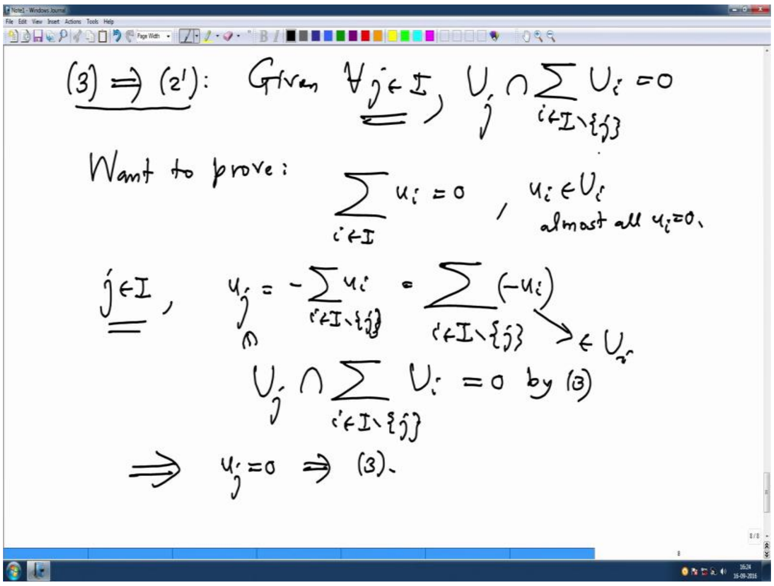


Now, we have given 3, that is what is given is given for all j in I, U j intersection the remaining sum this is 0 that is given for each j and we want to prove that the sums and what we want to prove? If the sum is 0, sum of U i is 0; obviously, we have a property that U i belong to U i and almost all of them are U i's are 0.

Then I want to prove each one of them is 0. So, fixed j in I then I want to prove u j here is 0. So, I will keep that u j on one side and remaining as I shift to the other side. So, what we will get then we will get u j equal to minus u i, i in I minus j this, this equality because I shifted, but this is same as summation i in I minus j minus u i and this is guy if u i's are new then this is get also in U i, but; that means, on one side this belong to U j on the other side this sum belong to the remaining once. So, this element is in the intersection, but we have given that for each j this is 0 this is by 2 prime by 3 summation.

So, u j minus be 0. So, that implies u j is 0 and that is true for all. So, that proves 3. So, that implies 3. So, we have proved that to test some sum is direct or not we have to test for each subspace intersection with the remaining sum should be a 0 subspace. So, we will stop and continue after the break.