Linear Algebra
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Lecture – 29
Projections

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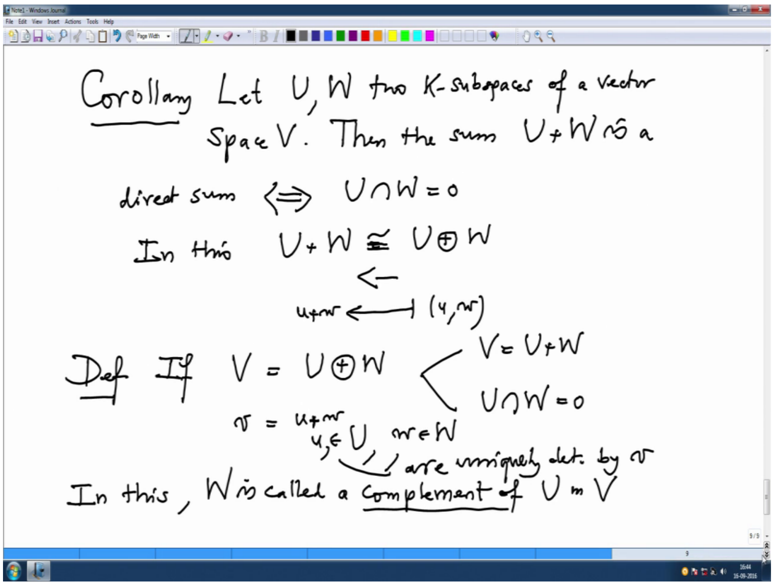

Come back to the second half of this lecture. We just now proved a theorem which characterizes when the sum of subspaces is a direct sum immediate corollary for this, let me note it down U, now it is only for 2 subspaces U and W 2 case subspaces of a vector space V. Then the sum U plus W is a direct sum if and only if U intersection W is 0 has only 1 condition now U intersection W in 0. And in this case, in this case U plus W is written is like this U plus W isomorphic actually, the map we have defined these direction, the map defined in these direction is U comma W is map to U plus W.

This map is clearly surjetcive and this condition means it is injective and this is because of this then the dimension of U plus W is dimension of U direct sum W which is then dimension U plus dimension W. So, that now leads to a definition we say that definition if V is a direct sum of U and W, what does this means? This means 2 things V equal to U plus W the smallest subspace which contain both U and W is V and U intersection W is 0. So, therefore, any element v in V has a expression u plus w where u and v, u is in U, w is in W and u and w are uniquely determined by v.

In this case we will say that W is called compliment of U in V. Now note that for any subspace there is a compliment.

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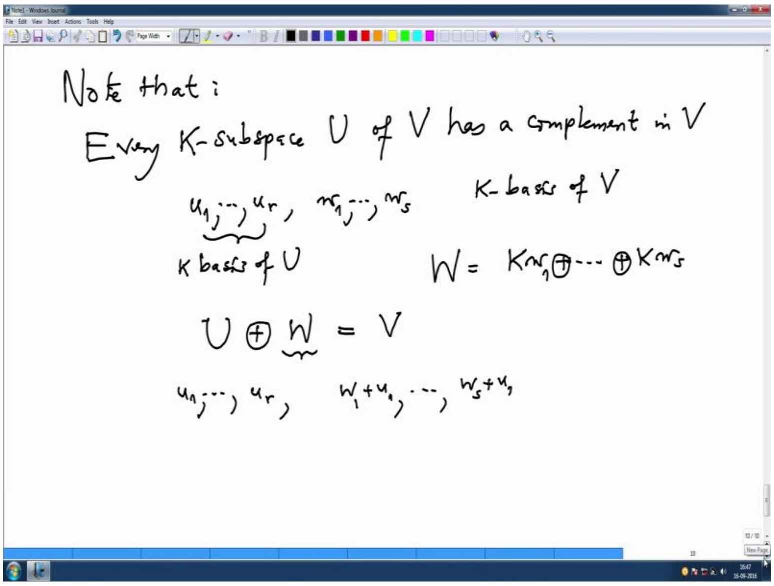

So, let me note that in the next page. So, note that every subspace every K subspace U of V has a compliment in V. Why? Proof is very simple what you do is you take a basis of u. So, u 1, u 2 are basis of this is a basis of U and extend by adding if you more vectors, extend it basis of V and extend w 1 to w s K-basis of V and take the W to be subspace generated by K w 1 to w n that is W is K w 1 plus plus plus plus K w s and because W is are linearly independent this sum is same as the direction.

And also it is clear that U direct sum W is V. First the sum is V that is clear because this is a generating system for V and it is direct because nobody is in common because this is a basis if there is a common element on one and it will have a K linear combination between u 1 to u r. On the other hand it will be K linear combination between w 1 to w s, but they cannot be equal unless their 0 because otherwise it will give a dependence relation among this basis.

So however, this complete W is not uniquely determined by U there for a given subspace U there may be many compliments because this is not the only way to extend it to a basis there is a many ways to extend it to basis. For example, you can also do it to this basis you can add u 1 to them, this is also will also be basis of V, this is a basis, this is also basis and you know the number is correct and generate is also correct. So, that is a basis.

So, there are many ways to get a compliment of a subspace in a bigger space V. So, now, let me define first let us start in a example and then that we will defined formally what projection is.

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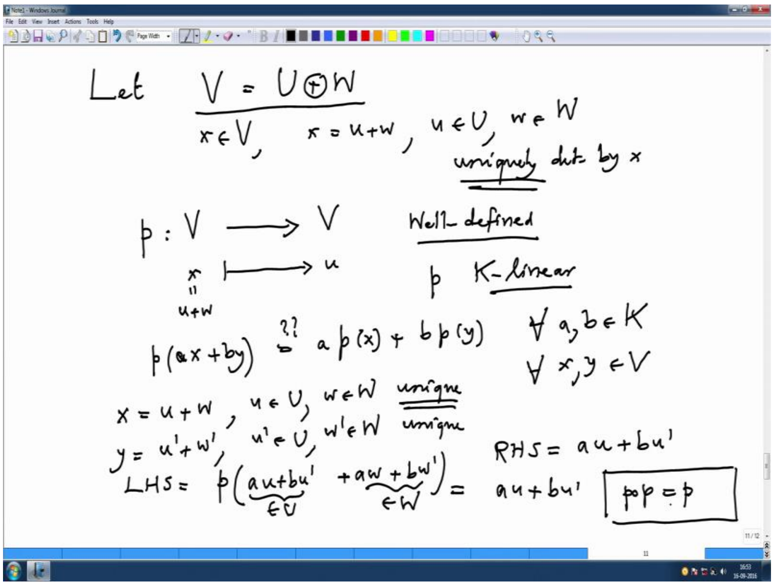

So, let us consider suppose to understand let us, let V is a direct sum of 2 subspaces remember this means V is a direct sum of U and W and U intersection W is 0, that simply means that any x in V has a unique expression x equal to u plus w where u in U, w in W and uniquely determined by x uniquely determined that is very important. So, I define a map now from p, p is a map from V to V. What is the map? Take any x and x write x as U plus W, u is in U, w is in W and they are unique because we are assuming this and map this to u. Look at this map first of all I want to say that this map is well defined.

Well defined simply means that if x as a another expression and whatever we were taking that one should be unique, but that is because it is a uniquely determined this map is well defined and also p is K linear that is what we have to check is if I have 2 elements x and y then p of a x plus b y equal to a p x plus b p y this is why what we need to check for all scalars a b and for all vectors x y in V. So, start with x y in V and write x has u plus w where u is in U, w is in W unique, unique is very important similarly y has u prime plus w prime, u prime is in U, w prime is in W and unique. So, therefore, by definition RHS is nothing, but a times u plus b times u prime that is RHS.

What is the left hand side? LHS is first I have to add this. So, when you add and collect the elements in U together, so p of a x is a time this. So, that is a u plus a w I have written it a part because I want to collect. So, here it is u prime and times b. So, plus b u prime plus b w prime. Now this is in u because u is a subspace this is in w because w is a subspace and because this sum is direct they will be uniquely determined by x and y because u is uniquely determined, u prime is uniquely determined, w, w prime are also uniquely determined. So, this are uniquely determined.

So, by definition of p this is nothing but first component that is a u plus b u prime, but that is also RHS. So, that checks the linearity of p. So, such a map is called projection.

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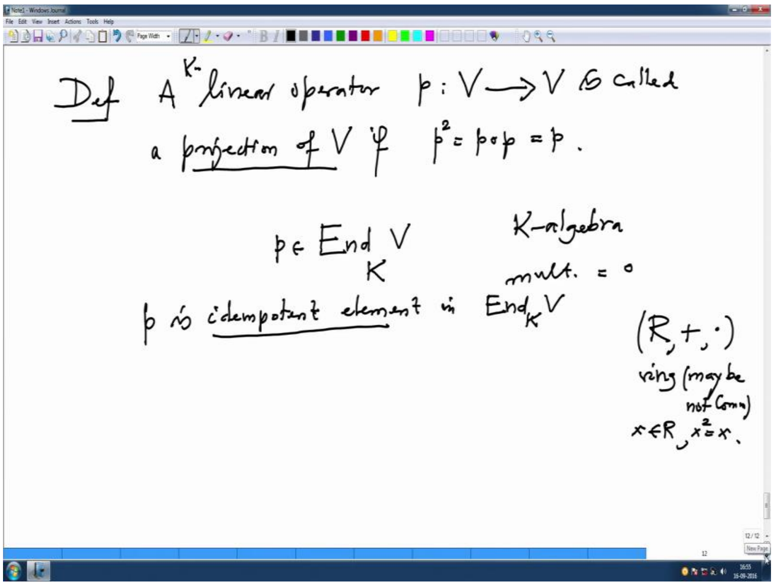

So, let us formally defined, so definition linear operator K linear operator if 1 1 2 is specific p from V to V is called projection of V if p square which is p compose p is p. I n the earlier map I just forgot to check here this map, please note that if I compose p with p I will write here p compose p is p. Let us check this orally in this example because I have to check that for every x both the sides is evaluated are equal.

So, p of x is by definition that u, what is once more if I apply p because u is already in U now. So, the unique decomposition of U is U equal to U plus 0. So, therefore, by definition of p, p of u will be u. So, that checks p compose p is p, is it clear. So, therefore, it is worth considering in general linear operate K linear operators on vector space which have this property p square is p, p square is by definition p compose p. So, what does this mean actually what does this operators, you remember we have we have been we want to study this K algebra and K V this is a K algebra multiplication in this K algebra is nothing but the composition, composition of maps.

So, this is a K algebra and in this K algebra what are, this are p, p is by definition p has to here and this condition means they are idempotent elements in this K algebra. So, p is idempotent element in the K algebra endomorphisms of V idempotent. More generally if I have a ring R, R plus dot if this is a ring and R may not be commutative may be not commutative. So, with respect to plus the ring is very good because addition in the ring is good because it is an abelian group. So, all you notable that all property are there, to study ring what is very interesting to study the multiplication what kind of property, the multiplication has whether it is commutative what are the units.

And these elements are special element they are idempotent elements, idempotent means x in the ring is called idempotent if x square is x.

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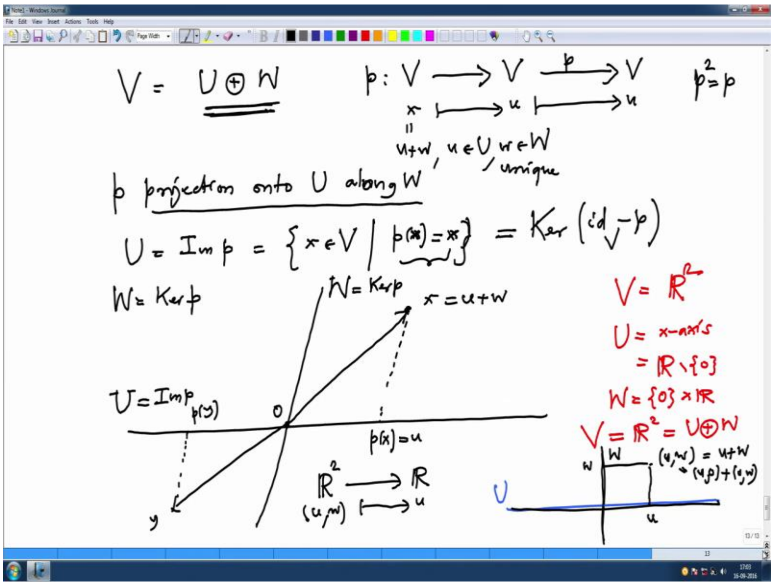

So, now let us try to do try to draw the pictures. So, now, I want to draw picture and show it is convenient. So, now, we are still considering V is U direct sum W and the projection here p is V to V and the map is clear x write x uniquely has u plus w and map it to this u, u is in U, w is in W and these are unique, uniquely determined by the given vector x and if you compose once more this u goes to u by definition of p. So, p square is p.

This is also called more specifically one can call it p is projection on to U along W, I want to use this expression U is a image and, so this I am calling because U is nothing but the image of p which is all those x in V such that p x equal to p, p x equal to x, but this is also same inside thing is also same as if you shift x to the other side this means let me write this, this is also same as the kernel of identify on V minus p see this is also linear map identity minus p and what is the kernel of this? All those x in v so that this i d v minus p on x is 0, but that is same as this equation because you shift (Refer Time: 17:01) to the other side.

So, how will how does on draw picture. So, this is my U, this is my U and this is W. So, this is U and this is W also note this W is what; W is a kernel, W is kernel p where will if x is only W then it goes to 0 because it has know U component and expression are unique that is very important. So, this direct sum, being a direct sum is very important we cannot to this for a arbitrary sum. So, if I have say x arbitrary x, x here which is u plus w then what will be p of x in the picture it will be this, this is p of x which is nothing, but u, this U is image p and W is kernel p.

This is 0 here the intersecting 0 until you see the picture is clear, this is if y is here this is x, y is here, this is y you can think of this is y. What is p of y? p of y will be somewhere here this is p of y. So, all the elements go to U from V and this is so, this is in the standard picture, so this is I can draw perpendicular for the reason because we have no concept of perpendicularity. But if you consider for example, V equal to R 2 and you take U equal to x axis; that means what? This U is; that means, V cross 0; R cross 0; R cross 0 that is x axis and W equal to 0 cross R and note that V is a direct sum V is which is R 2 which is a direct sum of U direct sum W.

That is because it is a sum first that is clear and they do not have anybody other then 0 in common. So, this is U and this is U and this is W, now its perpendicular because we have the dist we have inner product on R 2 which I talk about inner product we will see then we can talk about orthogonal d perpendicularity and so on, but in general in a vector space we cannot talk about perpendicularity there is no measurement for perpendicularity. So, when we do inner product spaces precisely we will introduce a tool which will major perpendicularity which will also major a distant and so on.

Right now in arbitrary vector space we do not have this tools and then if this was this is in standard notation it is U comma W, but then one could also write this as U plus W because when you write this a pair when you write it U comma 0 plus 0 comma W and are indentifying U with this 0 U comma 0 similarly W has 0 comma W and so, that justify why we are writing U plus W. And what is the projection on to U? That is precisely this U and on to W it is this one and if you have taken the first projection b 2 R 2 to R, u comma w going to u this is a first projection and the image is precisely the x axis and kernel is precisely the y axis right we could do the other way also, you can projected on to the y axis.

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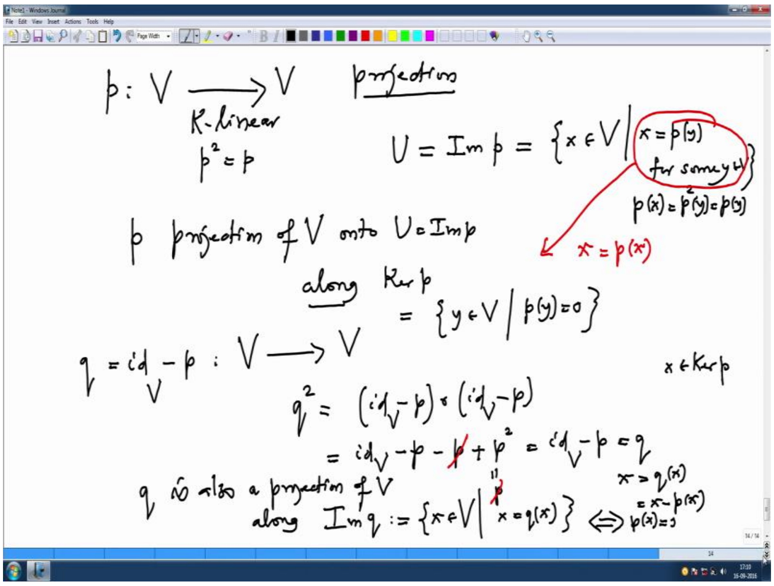

But these are complicate to each. So, I will now define, so let us continue now start with a projection, so p where projection from V to V projection. Remember I do not have a I do not have a decomposition of V as a direct sum, but I am going to get by from the projection. So, what is the projection? Projection is simply and an K linear map from U to V with the property p square is p, and we have seen if p square is p then I am calling U as image p and imitating the above concrete example. So, what is that? This means all those x in V such that x equal to say p y for sum y right that is the image, but you see do not have to write so complicated because when I apply once more p what happens? Then p x is equal to p square y, but p square y is p y because p square is p.

Therefore I could have simply written p y as p x. So, instead of thinking such this thing I would simply replace this by x equal to p x. So, that is the image p. So, therefore, then in our definition you remember we have said that p is a projection of V onto its image right so, but image is U on to U with this is image p. Now I wanted to write along whom along the kernel, kernel p. So, what is kernel p? Kernel p is precisely all those y in p y in V such that p of y is 0, but if p of y is 0 again I will do the same thing I will apply p once more.

So, what will I get; p square y which is p y which is 0. So, that will not give much, but I want to also note that look at the nu linear map which will come from p which is i d v minus p let us call it q, these also linear map from V to V because we know linear maps can be added set of linear operators is a K algebra actually. So, I can add multiply by scalar and composition if the multiplication. So, I want to test whether these q is projection or not. So, I should compute what is q square. So, q square is by definition i d v minus p compose with i d v minus p and we have to we have to be little carefully here because composition is not commutative.

So, do its really i d v, i d v compose i d v is i d v then i d v tends this minus p that is minus will come out in a minus i d v compose p that is p. The next one is minus p time i d v that is minus p again and minus p times minus p minus p compose minus p that is minus minus will become plus and p compose p that is p square, simply by this, this is i d v minus p and this p square is p again and that will get cancelled with one of them. So, you get this which is nothing, but q again. So, this q square is also projection. So, q square what we approved is q is also projection, q is also projection of V; and along whom? Along the image, so I will to find image of q image of q.

What is image of q? Let us find that. So that means, all those x in such that a limited this because you already said it is a projection the x is q x, but what is x is q x? x equal to q x means what? q x we know that is i d v x minus p. So, this q x is x i d evaluated at x is x minus p of x, so that means, this is precisely what do you have make mistake. So, this means no, there is some mistake somewhere x is q x. So, you cancel x from here. So, you will get this is equivalent to saying p of x is 0, but that is equivalent to saying x is in the kernel.

So, this image q is nothing, but the kernel p. So, therefore, we have two p and q are projection, but they are not independent they are related to each other.

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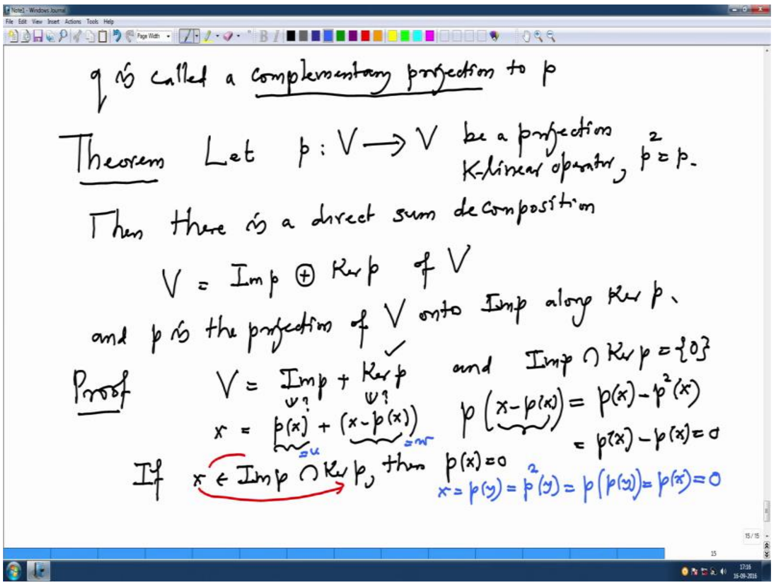

So, this q is called, so q is called a complimentary projection to p, complimentary and we have proved the following actually and this will actually this projection will give you a direct some decomposition of V. So, let me write in the theorem form. So, theorem, let p from V to V be a projection, projection again let me repeat each K linear, K linear map K linear operator and with the property that q square is p then there is direct sum decomposition for V. V equal to image p direct sum kernel p of V and p is the projection of V on to image p along kernel p.

So, what we have to prove? We have to prove that V is, so proof you need to proof 2 things - V should be the sum of image p plus kernel p and the intersection of image p and kernel p should be 0 that is a meaning that it will be a direct sum. So, first I will prove that every well, every vector in V is sum of somebody coming from image p and somebody coming from kernel p and also I will prove the kernel this is 0. So, let us prove that. So, start with any vector x. I want each clear that x is. So, I want some bodies in image p.

So, that is; obviously, I will take p x and I want somebody in the kernel. So, plus x minus p x; now we need to check that this is in image p this is what we need to check and also we need to check this is in kernel p that will give a said sum and we will vary about this the next part. So, visibly this p x is in the image p because its p of somebody. Now I want to check this is in the kernel p; that means, when I apply p to this it should become 0. So, apply p to this x minus p x p is p is K linear. So, it is p of x minus p compose p is p square of x, but p square is p because p is a projection. So, this is p x minus p x which is 0.

So, indeed this is an element in kernel. So, we have written every vector in V as a some of vector some of 2 vectors one is in image p the other is in kernel p. Now we want to check there is nobody in the intersection other then 0. So, suppose there is x, if there is x in the intersection image p intersection kernel p then what happens then because it is in the kernel p, p of that should be 0, p of x should be 0 because we are using the fact that x belong here. Now I want to use x belong here so; that means, what; that means, that means x is p of somebody x is p of y.

But p is p square which is also p is square of y because p is p square, but p square is p of p y, but p y is x, so this is p x, but x is in the kernel already we have check x is your assuming x in the kernel, so p x is 0 because x is in the kernel. So that, check that if it was in kernel as well as the image then x has to be 0. So, that proves that this is a direct sum once it is a direct sum then what is what is p equals p equals exactly to p is exactly the map we which sends x 2 p of x. So, that is, so this decomposition tells you this is my this is u and this is w and the map p is definitely x going to u. So, that is precisely the projection, along u and along u no; on to u and along w.

So, typical picture you see what we draw picture was in R 2 the picture of this was U and this is W. So, you are projecting on to this axis along this W. So, when we talk about projection both the subspaces are very important; one is the image going in to that and the other also the direction is also directed by the W. More, this will become more and more clear when the once we bring geometry into the picture that is one thing and also now the next time I want to do not one projection, but several projections and then we will get the decomposition of a vector space into the U i, now family U i. So, that is what we will do it next time.

Thank you.