Linear Algebra
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Lecture – 03
Examples of Vector Spaces

We will recall what we have been defining in lecture one. Last we have ended the lecture one with the definition of a vector space.

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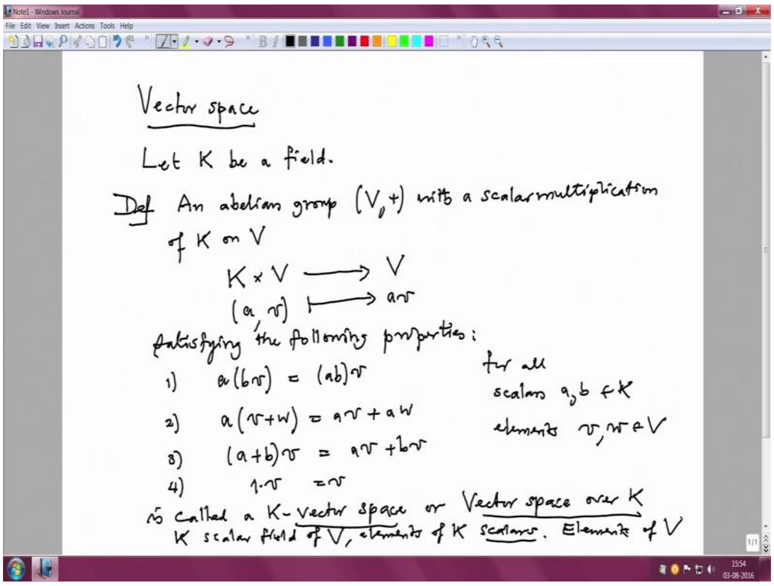

So, let us recall it definition once more to define a vector space we need a field. So, I will always start with let K be a field then definition an Abelian group V plus is with a scalar multiplication of K on V which, is a math from K cross V to V which, the image of the pair a comma v, I will denote a, v satisfying a proper the following properties; four properties if you recall these four properties are basically the properties about compatibility of the field structure on K and these addition on V and the scalar multiplication of V.

So, the four properties are; a, b tends v same as a, b, v where b, v is a scalar multiplication and is further scalar multiplication on the new element b, v. On the other hand multiply a and b in the field and then take scalar multiplication of them. Similarly, with the addition of first addition of the vector space that is a times v plus w equal to a, v plus a, w and third one is when you add elements in the field a plus b tends v equal to a, v plus b, v and last one is one times v equal to v. This property should be varied for all scalars a, b and k and all elements v, w in v.

So, even Abelian group as a scalar multiplication of K and it satisfy these property then in Abelian group (Refer Time: 03:51) is called a K vector space or all though sometimes I will say vector space over K. This field is called K, we will be then referred as scalar field of V and elements of K are called scalars, elements of V are called vectors.

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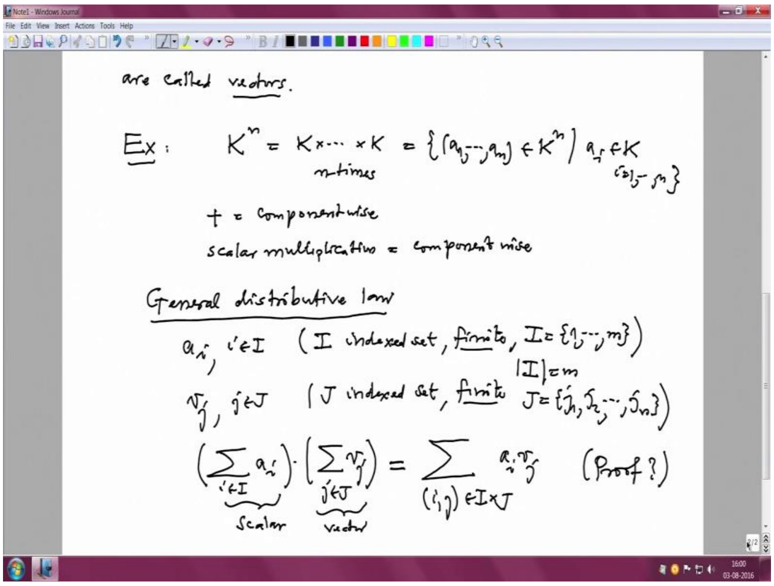

Later on we will see that how this concept is related to the older concept of the vectors in the geometry.

Today I want to give many examples, many different examples which we will constructed from the given once. So, recall that yesterday, the model example I give was K power n Cartesian product of K n times. So, elements of these are tuples, these are n tuples with coordinates in K and we saw it is a vector space in a very natural Abelian group structure namely the component wise addition which is coming from K, addition is component wise and scalar multiplication is also component wise.

So, the addition of the field K is extended to the set of tuples and multiplication in the field K is also extended as a scalar multiplication on this tuples. And later on we will see that every so called finite dimensional vector space will look like this, but before go on, I want to mention some of the properties that one need to check, I will just mention them without proofs. For example, general distributing law; this generalization of the property 2 and 3 sort of so that is if we have finitely mean elements a i, i in i, this I is indexed set and finite.

Typically one will take I equal to for example, 1 to n, 1 to m. So, (Refer Time: 07:37) elements, but sometimes it is useful to, who denote or enumerate the elements in I differently. So, for that use in we will not in general assume that I is 1 to n, but we will assume I is a finite set. It has only finitely mean elements and cardinality some natural number m. So, cardinality is some m and suppose we have finitely many vectors v j, j in J again J is indexed set, finite indexed set. So, for example, J could have n elements which one might denote j 1, j 2, j n. Finite is important because without finite, the statement will not to make sense.

So, on one hand we have this summation a, i. So, we are adding elements a is, the finitely mean element and remaining adding them in the field. So, you get again an element in the field. So, you get a scalar. So, this is scalar, we have seen that associatively. So, this is well defined does not matter we will put the bracket. So, this sum is well defined because of the associatively in the field. So, this and from the vectors v j we get will sum them up in by using the addition in the vector space. So, they will get a vector, so this is a vector and when I take the scalar multiplication and now what do we get? What I want to stress is, you can open the bracket.

So; that means, you are using the property 2 and 3 repeatedly. So, we can write this as summation over i comma j in the products at a, i, v, j. So, this needs the formal proof; when i as two elements, j as two elements then it is precisely either 2 or 3. So, I will leave it for you to prove this. So, this is known as general distributive law; that means, you can use the distributivity as many as times you want, what is more importants is finitely mean time because infinite sums do not make sense in generally in the vector spaces. Another sort of, this I call it sign rules when one makes a calculation with formulas, we will obey this.

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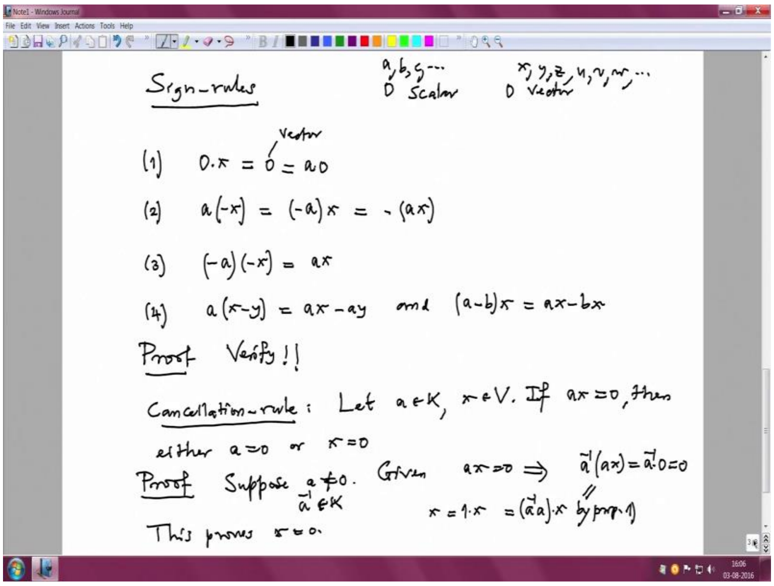

For example, you have 0 in the field, zero scalar also; we are denoting the same notation and zero vector also we are denoting by the same letter 0. So, it should be clear from our context what we mean for example, 1 I want to write, 0 times x as I told you earlier scalars we are denoting by the letters a, b, c etcetera. Vectors we are denoting by x, y, z, u, v, w etcetera. So, here 0 is a scalar, x is a vector because we cannot multiply vectors. So, this result should be 0 and the 0, this should be vector. Now on the other hand also I can take like that any scalar a, and multiply the zero vector then, I should get a 0.

So, this is also 0. So, I could write the equality here because this 0 as a vector. So, one need to proof this, so second one, if I take any scalar and multiply by negative of the vector, minus x. Negative of the vector simply means inverse in the Abelian group of x. So, they should be same as you take the scalar and take the negative of that; that means, inverse in the additive group of the field and multiply by x, this is result should be same or also you could also take this. You take the scalar multiplication of a on x and take the negative of that vector, all these three quantity it should be equal.

Another one, third one minus of the scalar, negative of the scalar times negative of the vector is same as scalar tend to vector. Fourth is, if I take scalar and take the scalar multiplication in the difference vector that should be same as a x minus a y. This is the vector subtraction and same thing with the subtraction of the scalar a minus b times x should be equal to a x minus b x. So, these rules are so natural that and they will also simplify our writing which will be exactly same as what we were doing with our number systems. So, proof I will leave the proof to their. So, I will simply say verify, this will give you a good practice about how does one write the equations in Abelian group or in a vector space and so on.

More important I want to say is, following cancellation rule and this term will proof it suppose. So, let a be a scalar and x be a vector then, if a times x is 0 then, either this scalar a 0 or the vector x is 0. So, let us write a proof. So, we want to prove either or statement. So, always in mathematics you assume one of them any are true and prove the other is true. So, suppose a is not 0 then, we must prove that x is 0. We have given a x 0 when, you multiply this equation by a scalar a inverse because a is a non zero element in the field, a inverse is also an element in K. That is why we have defined our fields.

So, this will imply a inverse time say a x equal to a inverse times the vector 0, which is 0. Just now we are wrote the sign rule 1 for example, on the other hand this a inverse times a x we can use the property 1, which will say that I can first multiply a inverse and a in the field and times the vector x. This is by property 1 only in the beginning I will write this, after sometime I will not write this, but this is same because a inverse is the inverse of a in the field, this is same as 1; 1 times x, but property forces 1 times x is x. So, we have proved x is 0. So, this proves x is 0. So these cancellation rules say that, if your scalar multiple of a vector is 0, either the scalar is 0 or the vector is 0.

This was the intuitively clear when we studied the vectors. Next now let me give lots of examples of vector spaces.

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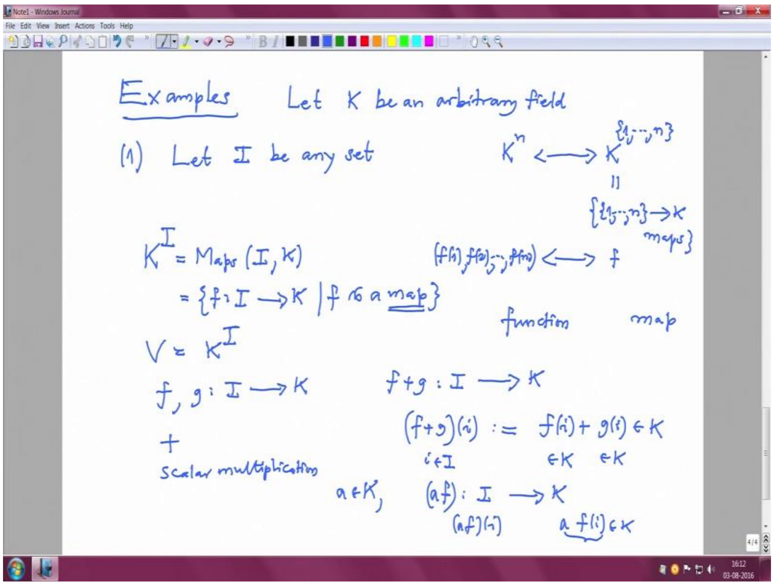

So examples first one; so in all these examples, when I do not mention anything about the letter K, we will always take K to be a field an arbitrary field. It could be finite, it could be infinite, it could be or field of rational numbers or field of real numbers or field of complex numbers or a finite field with P elements. So, the first example let I be any set. So, this example is generalization of the, our prototype example which K power n.

So, here I would like to mention K power n also you can think, either you can think the elements are tuples or you can think they are mapping so on. They said 1 to n to K, this is the set of all maps from 1 to n the field K maps. So, these correspondences individually either you think of function f or you think of the tuple f of 1, f of 2, f of n. Each tuple will give you a unique function and each function we will give a unique tuples. So, when a set is infinite or you do not know beforehand what they said, how many elements it has etcetera, it is better to think about functions from I to K. So, that is what I am considering here K power I, this is the set of all maps. Sometimes I will write these also maps from I to K or like this f from I to K; if f is a map.

Also I will like to take this opportunity about my saying sometimes. So, the difference between the function and a map, these then the very fine distinction between these two words and most of the people in general they do not do this, but I would like to do this for many reasons. Functions means where the values are that is our one of our number system. For example, it can be rational number, it can be real number, it can be complex numbers. When we do not know about the field then, we will call it a map when the values where it goes. If it is our known set like one of the number system then we will call it a function, otherwise we will call it a map.

So, that is a reason I am writing a map here. If K was set of real numbers I would have said function. So, our V is now K power I and I want to check that these V is a vector space, K vector space in a very naturally way. So, that means, I need to define a addition operation on this maps and get a map again and with that addition it should be an Abelian group and also I should tell you, what is the scalar multiplication and when one need to check all those four properties of the vector space. So, it is think they are very natural like we will see if we have f and g two maps from I to K then, I need to define, what is the plus f plus g? And the result should again be a map from I to K.

That means, I need to define, what is f plus g 1 arbitrary element of i in I and what can we do? Obviously, on one hand we can take f of I, which is an element in K, also you can take g of i also element in K and add them use the addition in K. So, we have added these values. So, result is again in element in K because K is a field. So, I define this f plus g evaluated or image of i under f plus g is take f i take g and add them in the field. So, you get a map from I to K. Now I need a scalar multiplication. So, if have a scalar a in K then, I need to define what should be a times f and this should again be a map from I to K. So, I need to define, what is a f evaluated on i?

Well what do I have? I have f of I, this is an element in K and I have a given element in a in K. So, I can multiply in the field. So, this is a result I will get a product of two scalars which is again a scalars; so which again an element in K. So, with these I would define plus 1 V and also I have defined a scalar multiplication.

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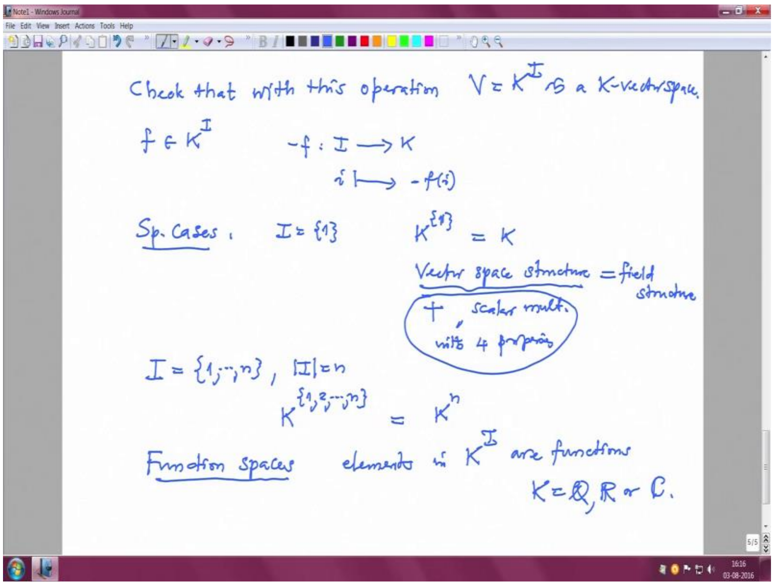

Now, see these two operations are so very natural and canonical that all the properties of the vector space are satisfied that again I will leave it for you to check. So, I will just mention. So, I will just first mention check that with this operations V equal to K power I is a K vector space.

Notice: that if I take f in K power I, the negative of that, if that maps I to K which maps I to f of i and minus of that, this is that map and it is obvious that this serve the purpose of the inverse. So, that with respect to plus it is indeed an Abelian group. So, for example, let us take special cases. For example, if I were a very single term then, what is K power single term, 1; this is nothing, but just to K, the copy of K because you are taking K cross K cross one times. So, it is only one copy and the vector space structure is nothing but the field structure.

So, vector space structure is exactly same as the field structure. So, when one says, the vector space structure; that means, two things one is that addition of the vectors and the scalar multiplication. With those four properties, all these together is called a vector space structure on V. The plus give an Abelian group structure, scalar multiplication gives the scalar multiplication and the four properties combined them to. So, I equal to say 1 to n or any set I where I is exactly n elements then, K power 1 to n is nothing, but K power n which was the prototype property, example that we have mentioned in the before the definition of the vector space.

So, these examples are also called function spaces. Function spaces because here vectors are functions. So, elements in K power I are functions. This strictly speaking I should use this word only when K is Q, R, or C. That is a reason why they are called function spaces and you will see this examples more often in your analysis courses.

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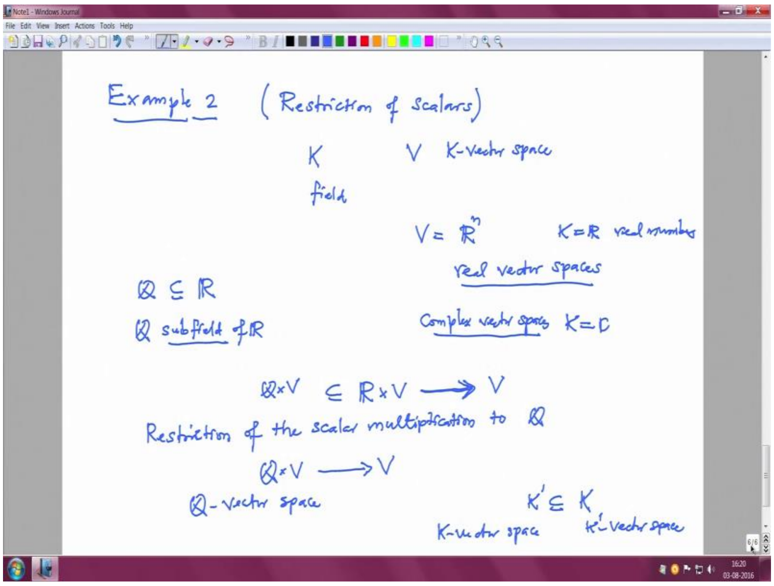

Now, example 2; this is what I am addressing you to, what is called restriction of scalars? So, typical situation will arise when we are given a field K and a vector space V, K vector space. For example, V equal to R n; R power n here K is R; field of real numbers.

Such vector spaces are also I will, people call them as real vector spaces. So, if the field is C, then one calls them as complex vector spaces. So, it is clear from the writing what field what scalar field one takes. Sometimes it is useful to take complex numbers as scalars. Sometimes it is useful to take real numbers. So, this is a field. So, this is a real when we are given a real vector space for example, that is a V equal to R n and, but this field of real numbers contain the field Q and Q is actually sub field of R. Sub field simply means that when you restrict the field structure of R to this subset Q, it should give the Q vector spaces.

It should get the original field operations in the Q, that is because we have extended the plus and multiplication from Q to R in a very natural way. So, it is a sub field. So, one would like consider this real vector space as a Q vector space. So, what do we have to do? Already we have given an Abelian group structure that will not change; only we have to give a scalar multiplication of Q on V in a very natural way. So, scalar multiplication of R is given. So, R cross V to V we have given a scalar multiplication, in multiplication. I simply restrict and there is a subset here Q cross V. So, I simply restrict this scalar multiplication to Q.

So, restriction I will just say restriction of the scalar multiplication to Q simply means you take this scalar multiplication, given scalar multiplication map and restrict to this set. So, you get a map from Q cross V to V and with this already the properties are satisfied. So, that it becomes this V will become a Q vector space and then nothing special about Q and R. In general you could take arbitrary field K and a sub field. As I said sub field should mean that when you restrict the field operations of K to K prime, they should get the operations of the field K prime.

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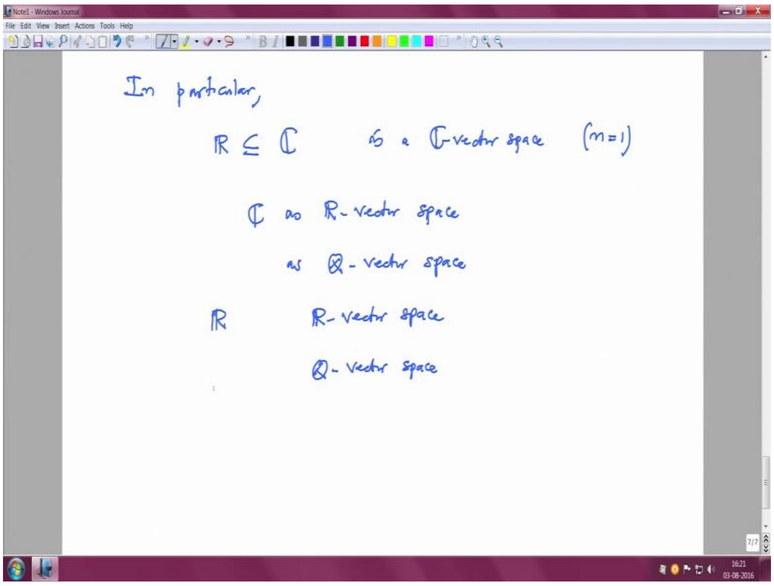

In that case every K vector space you can think K prime vector space. Before I stop for the break I would like to mention here in particular it is important case. So, I want to record it here in particular; obviously, C is a C vector space. If like in a earlier example n equal to 1 and R is a sub field of C. So, the same C, I can think as C as a R vector space where restriction or I can also thinks C has vector space. So, here the scalars allowed are natural numbers, here the scalars allowed are real numbers or even also R is a R vector space. Then it is also Q vector space by restrictions, but here I cannot say R is a complex vector space because to extend I need to do more. For restriction I do not they have to do anything, I have to just simply restrict.

So, I stop here and we will come back after few minutes.