Linear Algebra
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Lecture – 30
Direct sum decomposition of a vector space

So, welcome to these lectures on linear algebra, last time we have seen direct decomposition, direct space decomposition, direct sum decomposition of a vector space and projection maps. I want to do it today for many projection together more for arbitrary direct sum decomposition of a vector space.

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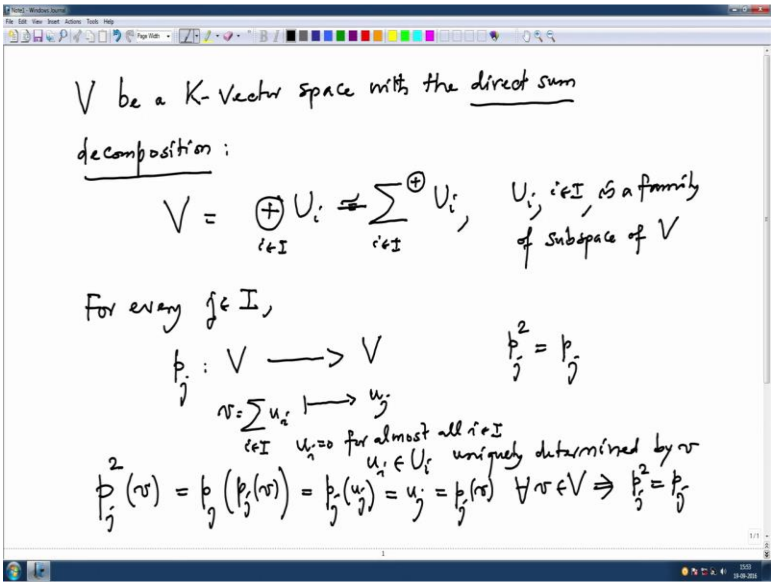

So, let us start with let as usual V be a K vector space, which as the direct sum decomposition with the direct sum decomposition. V is direct sum of U is. Strictly speaking I should be writing instead of this I should be writing some i in I U i, where U i is a family of subspace of V.

From this direct sum this from this direct sum decomposition I will define now many projection maps as many as indices by this set I and conversely if I have a family of projections on V, which are indexed by some arbitrary set then I will get a direct sum decomposition. These precisely these we have done it for 2 subspace in last lecture. So, now, I want to define for each. So, with this is given to us. So, for every j in j for every j in i, I want to define p j and that will be projection on V projection of V. So, it should be a map from V to V and also with the property that p j square should be p j and; obviously, this j should be used somewhere.

So, the definition of p j is take any element V in V, because V is a direct sum V as unique expression like this summation U j, j in not U j I want to use a different indices u i, i in I where only finitely many u i's are can be non zero with U i equal to 0 for almost all i in I. If you would have identified this we have check that this means this map is an isomorphism, we have identified this then you do not have to keep writing this and that for almost and that is understood.

So, map this, these and this u i's are unique, u i belong to U i they are uniquely determined by v, that is the meaning of that V i the direct sum decomposition. So, map this V to the j th component here u j this is; obviously, make sense and it is a map from V to V because this components are uniquely determined by V. Now we have to check several things we have to check that p j is a projection; that means, we have to check that p j square is pj.

So, let us check that first. So, I want to check that p j square on v start on any v. So, this is by definition of these p j, this is p j of p j of v, but this is p j of u j, but u j is already u j. So, this is nothing, but u j which is p j of v. So, on every V they are p j square and p j are equal. So, that this is true for every V in v. So, that proofs p j square equal to p j.

Now, we need to check now we also need to check what is a I want to say that p j is along whom so; that means, the I need to find out what is a image of p j. So, p j will be on to that and it will be along the kernel. So, let us find out what is we need to find out what is image of p j and also what is kernel of p j.

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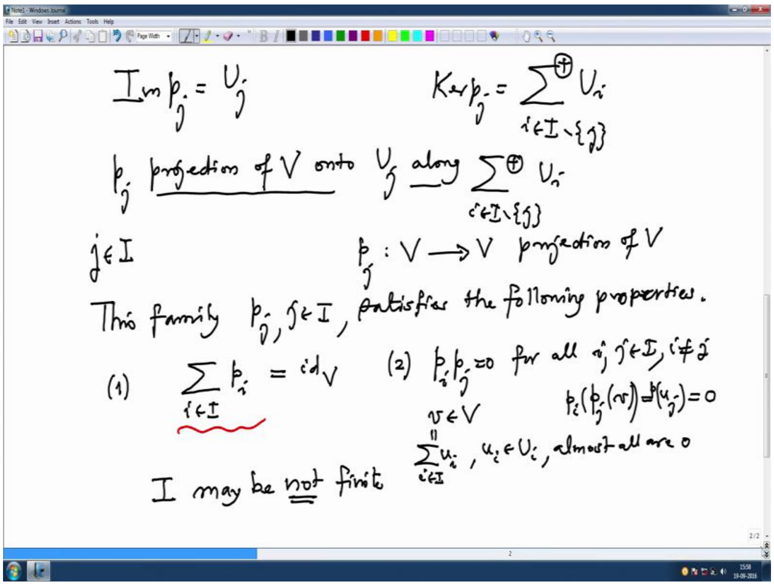

Then this is this is precisely the U j and this is the remaining sum this is the remaining sum i in I, but not j and this is U i this is also direct sum. So, then we will say that p j is p j is projection of V on to U J along is remaining direct sum that is the language you have we have been using for projection. What so each what we have defined for each j in I we have defined a projection p j projection of V on to U j along the remaining sum.

More over this family satisfies a more properties. So, these family of projections p j satisfies following properties; the first one is the summation I have to be little carefully a summation i in I p i, this is i d v. 2 p i compose p j is 0 for all indices I and j which are different I not equal to j and third one. So, this 2 is easy because you see the p j as image u j and p i as image u i, but p i will be 0 on the remaining part. So, it is 0 that is clear.

If I take p, so to check this, I have to check that on any v in V there is 0, but write V as summation u i, i in I U i's are in capital U i's and almost all are 0 almost all are 0 once I have written V like that what is p j of v that is u j by definition. When you apply once more p i that will be p i of this, but p i of u j is 0 because I different from j. So, therefore, this is this composition of 0 is clear now here little bit tricky here, first of all I have to justify what do I mean by sum because I may be infinite set, I could be infinite I may be not finite and therefore, we have to justify what do we mean by this sum.

So, that I do it in the next page. So, first I will note that this I will need this definition and again and again.

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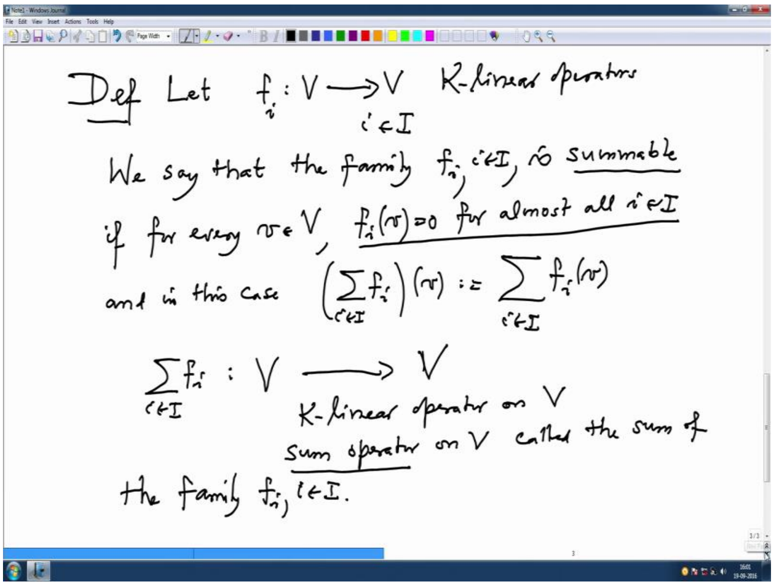

So, let us say take. So, this is a general definition which we will need again definition, let f i are linear operators on vectors space V K linear operators indexed by I or I is in I this I could be infinite shape, then we say that the family f i, i is I is summable if 0 for every v in V, f i x, f i V is 0 for almost all I only finitely we could be non zero and in this case we put summation f i, i in I on v to be the sum, because of our assumption that it is 0 for almost all I this sum make sense.

So, therefore, this symbol summation f i, i in I it is an operator again this is K linear operator again, this is map this is a map from V to V and it is very easy to check that this is k linear operator on V and it will be called a sum operator on V. So, the most important part is f i should be 0 for almost all I when the sum makes sense and the sum operator makes sense and that is called a sum of called the sum of the family f i.

So, in order to make the earlier part I will just show you we want to. So, this sum to make sense we need to verify that p i on any V is 0 for almost all i, but that is because V is every V as a decomposition like this summation U i where almost all U i are 0. So, only p i only whichever component of we are coming only those will survive, so therefore, that make sense and now we need to verify that this sum is identity, but that is very easy. So, we need to verify that the sum p i, i in I is nothing, but the identity operator on V.

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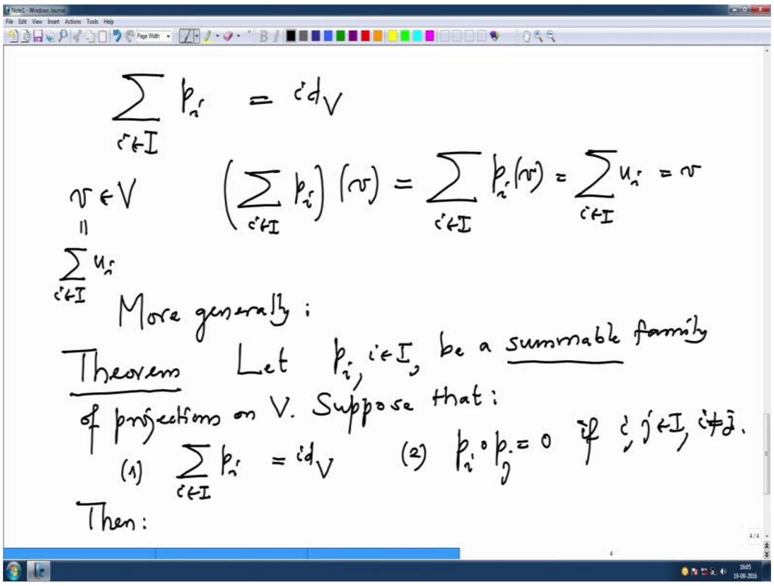

So, we should check that for any V in V both the sides are equal. So, so evaluate these p i i in I on V and this should be by definition it is summation p i V and what is p i V? P i V will only be when you write V as a finite sum, only those components are coming there. So, this is nothing, but summation U i, i in I but this is V because everything is unique or V is uniquely written as summation U i therefore, this make sense.

So, from the direct sum decomposition we got the family of projections with these 2 properties; there it is a summable family their sum is the identity operator, and they are p i times p j are 0 where I is different from j. So, these are also called orthogonal family I do not want to use the term because it is not really necessary to use. Now I want to do the converse conversely if I start with the projections on a vector space, family of projections which satisfies these 2 properties then I get a direct sum decomposition of v. So, that I want to write I t as a theorem.

So, more generally I want to state this as a theorem. So, theorem, let p i, i in I be a summable family of projections on V. Suppose you have 2 properties this satisfies this family should have 2 properties, suppose that first is first is the summation p i, i in I this operator sum operator should be the identity operator this summable therefore, this sum makes sense and we are demanding that the sum of this family is nothing, but the identity operator, and second condition is p i compose p j is 0 if i j are different in different indices.

Then what is a conclusion; then we want to decompose V into direct sum decomposition in such a way that the projections which will come from the direct sum decomposition as above they will coincide with this given family p i ok.

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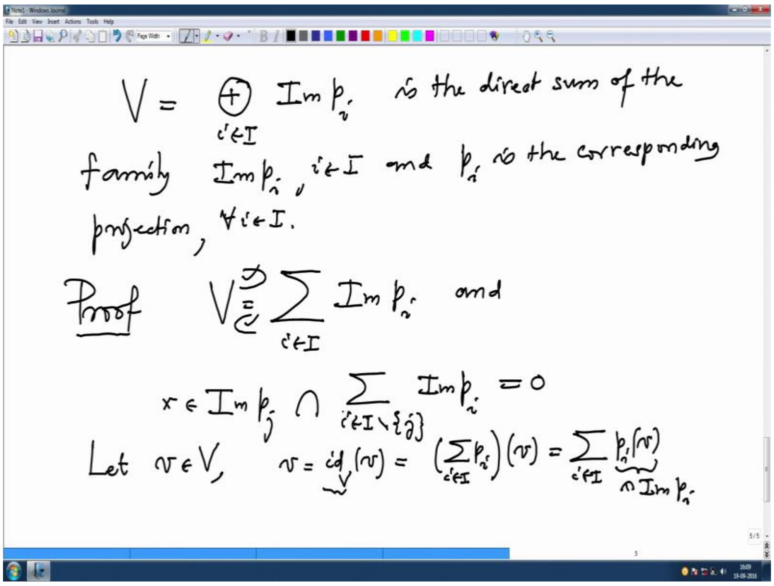

Then V is direct sum i in I of image p i is the direct sum of the family image p i, these are my U i's and the given p i is the corresponding projection for every i in I, this means using a direct sum we have defined a projection the same projections.

So, proof is very easy. So, let us write a proof. So, when we want to check V is a direct sum of this we have to check 2 things; first is we have to check that V is a some of this family; that means, V is a smallest subspace which contain all this case that was the definition of a sum space, and also we have to check that this sum is a direct; that means, if you fix one of them and intersect with the remaining sum that should be 0 space so; that means, image of p i or p j intersection with the remaining sum i in I minus j image of p i this should be 0, once we check this 2 then it will check the equality here, so let us check that.

So, to check the first equality I will have to check that first what I am checking; first I am checking that this equality first of all this is obvious, because this is in any case a subspace. So, I have to check that every V belongs here so; that means, I want to check that every v. So, let us check that every V should be written as sum of elements on image p i and only the finite sum. So, start with any v in V. So, v equal to V is same as identity operator operating on v.

But we have given the condition one what the summation of p i is I d. So, therefore, this I d I will replace by summation p i; we have given that this family p i is summable. So, therefore, this sum makes sense and that sum is I d is the first property. This evaluated at v, but the way it is defined this is summation i in I of p i v and only finitely many of these are non zero and by definition by evidently these belong to image of p i.

So, that check that every V is an element here I have written every V as a sum of this guess. So, therefore, we have check this, this was obvious. Now we want to check that they do not intersect other than 0. So, we should check that if somebody is in intersection then it is actually 0. So, let us take x here on this side and I will check it is 0. So, x you have taken on the left hand side.

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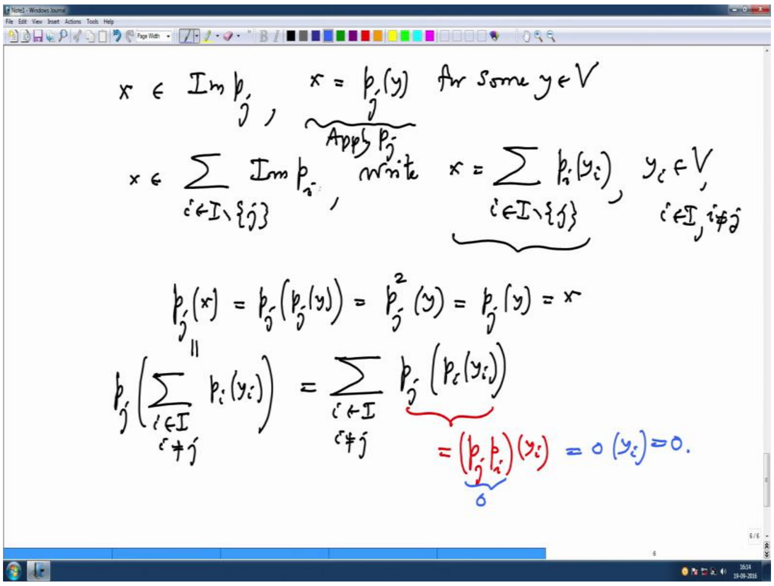

So, therefore, x we can write it as because it is in the image p j, we can write this x as x we have written as p j of y for sum y in V.

On the other hand it is also in summation i in I, i different from j image of p i; when x belong here that means, x is a finite sum coming from each image p i. So, write x equal to summation, summation is (Refer Time: 22:52) over i, but different from J and this belongs to image p i means it will be p i of p i of somebody p i of y i where y i s are elements in V and i is in I, but i cannot be j and this means this sum make sense means only finitely many of them are non zero almost all of them are 0.

So, we have written x like that and we want to conclude from here x is 0. So, apply p j to this equation apply p j to this equation apply p j. So, what we will get we will get p j of y no p j of x, p j of x which is p j of p j of y which is p j square y , but p j is projection. So, p j is p j square is p j this p j of y which is x. So, we got x equal to this, now I will compute p j of x by using this sum. So, apply p j to this. So, p j of x this is this. So, p j to the sum p j to apply to the sum, sum is over i and i is in I, but i is not equal to j and p i of y i, but this is a finite sum any and this is a linear operator. So, I will put that inside.

So, this is same as summation i in I, i different from J, p j of p i of y, but this one this here is nothing, but I will this is p j by definition p j compose p i, and then you evaluate at y i the definition of the composition, but we have given that this because i and j are different this composition is 0. So, this is the map 0 you evaluated y i which is 0. So, we got x equal to 0. So, that proofs the sum is direct and we have already mentioned that it is the whole v.

So, that proves the theorem. So, what we have proved is given a summable family of projection is equivalent to giving a direct sum decomposition of a vector space; given a direct sum decomposition we have the p j. Now here we I left little small thing to check that if you take in p j that the corresponding projection, but that is clear because in this case the subspace are image p i and therefore, it is clear that the corresponding projections from this sum which we get are the given p j s what I mean of this. So, that proves a theorem.

Now, in the after the break I will start dual spaces or let me start little bit of dual spaces, so that we can save some time. Now the next sub section I want to start is dual spaces.

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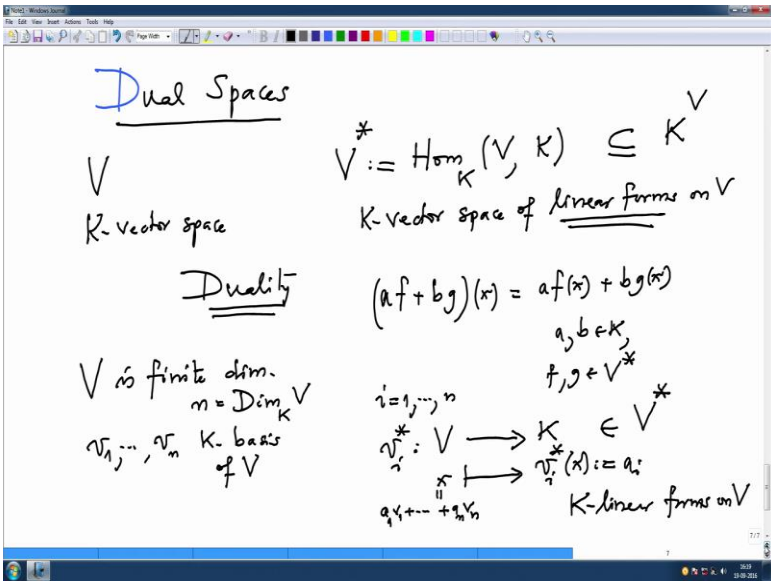

So, remember when we have a vector space V, V is a K vector space then we have this V star. V star is by definition K linear maps from V to the field K, V to K. So, they are called also these also vector space we have checked, this is a K vector space of linear forms on V, elements of this vector space are called linear forms and now we want to study V and V star together for example, we want to check what about the dimensions dimension V, dimension V star also given a basis of how do you get basis of V star or the other way and these study together we will sort a lead to duality.

Duality is the word used because the dual spaces in non dimensional. So, and it will also be helpful when you study for example, in a plane lines that is one dimensional of spaces where do they corresponding V star and so on. So, this interplay between V and V star is called duality. So, V star to start with this is a subspace of K power V. K power V is a very big vector space the, it is you cannot even write down the basis explicitly. So, here on the vector space structure on V star is a f plus b g evaluated at any x is by defined a f x plus b g x, this is for all scalars a b f g are in linear forms and this multiplication and this addition is happening in K now.

So, that is a vector space structure and remembers the most important thing we have done in earlier lecture is if I have let us take a finite dimensional case first. So, assume V is finite dimensional and then we can choose a basis. So, suppose n is the dimension of V then we have a basis V 1 to V n, K basis of V and we have from these basis we have defined coordinates in V. So, for each i in i from 1 to n we have defined this v i star which are called coordinate function; that means, they are maps from V to K and how are they define for any x in V, V i star is by definition.

So, take this x and write x in terms of this V i s. So, basis, so x will have unique expression a 1 v 1 plus plus plus plus a n V n and you map V i star is nothing, but the a i i is from 1 to n I is fixed here. So, I do not have to write this here. So, this definition makes sense because it is a basis, note that we cannot do this with definition does not make sense when it is a generating system or its linearly independent alone we are using a fact that it is a basis. Also one might think that V i start depends only on this V i here, but it does not depend alone on that it depends on the whole basis. So, these are called coordinate functions and we have checked that term their linear forms on V, that means, all together elements of the V star vector space, and now it is very easy to check that we will check that this V i is a basis of V i star and that will be called a dual basis. So, this we will do after the break.