Linear Algebra
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Lecture – 31
Dimension equality and examples

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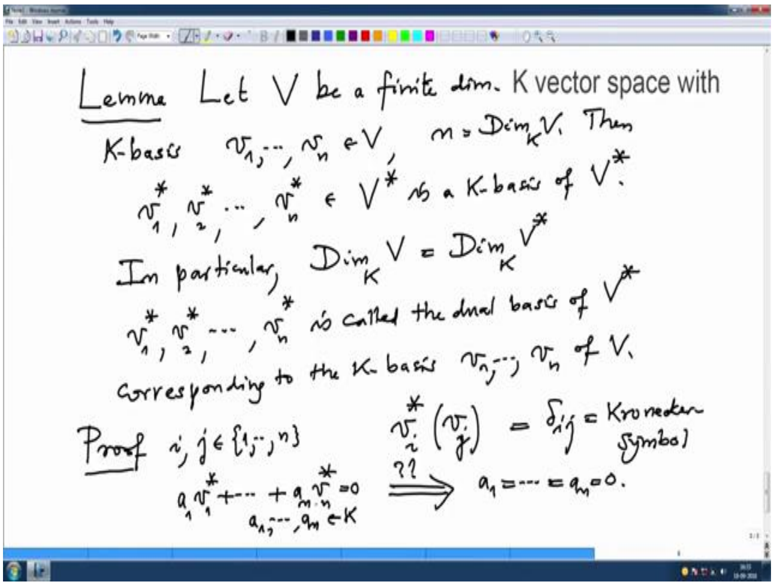

So, we will come back to this second half of this lecture and first thing I want to check that is. So, let us write it as a lemma this is I am considering a finite dimensional case. So, let V be a finite dimensional K vector space with basis K-basis v 1 to v n so; that means, n is the dimension of V, then the coordinate function we have just now defined v 1 star v 2 star v 1 star which are elements of the dual space V star is equal to dual space of V is a K-basis of V star.

In particular the dimensions are equal dimension V equal to dimension V star and these basis v 1 star v 2 star v n star this is called is called the dual basis of V star corresponding to the K-basis v 1 to v n of V. So, in particular part is clear we only have to show that is coordinate functions form a basis of V star and for that we need to check that they are linearly independent and they generate V star.

So, proof and the most important thing that we will use in the proof is if you take, if you fix j in j; a j in 1 to n and any i and j both are n to 1 then by definition v i star evaluated at v i, v j this is delta i j Kronecker symbol, is clear? We call if I want to evaluate v i star on v j then I have to write v j as the unique linear combination between v 1 to v n, but it is already one of the vector. So, it is already in that form and then the coefficient of that is 1, so it is 1 when i j it is 0 when i is not j.

So, now first I will proof that linearly independent so; that means, if you linear combination a 1 v 1 star etcetera etcetera a n v n star this is 0 where a 1 to a n are scalars then from here I want to conclude that all a i's are 0. Well, you fix j and so these are this is a linear form this 0 linear form.

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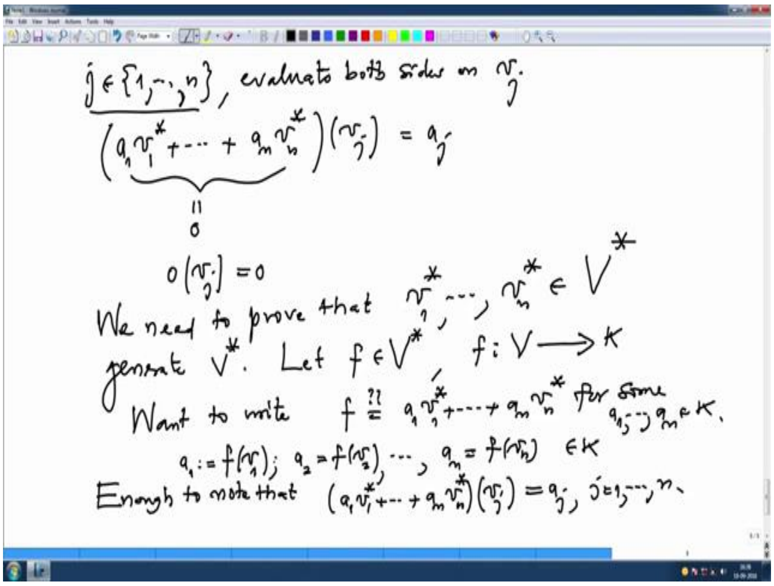

So, both evaluate on the vector v j. So, take any j in 1 to n and evaluate both sides on the vector v j this side you will get a 1 v 1 star etcetera etcetera a n v n star evaluated at v j, but the only term which will survive here is the jth term and that as coefficient a j. So, this is a j.

On the other side this sum we were assuming 0. So, it is 0 linear form evaluated at v j is 0, so we proved a j equal to 0 for all j from 1 to n. So, that proves linear dependence or linear independence of v 1 star to v n star. Now we want to prove that we generate V star. So, we need to prove now prove that v 1 star v n star which are elements of V star generate V star in other words it is a generating star.

So, for this start with let f be any linear form V star, that is f is a linear map from V to K and I want to write this. So, want to write f as a combination a 1 v 1 star plus plus plus

plus plus a n v n star for some a 1 to a n scalars in K. So, we are looking for a 1 to a n so that this equality happens, but it is clear what do I take. So, for example, how do I take a 1? How do I find out a 1? Just evaluate f 1 v 1 and define it a 1.

So, a 1 is defined to be f of v i note that this is an element in the field similarly a 2 f of v 2 and so on, a n f of v n. So, I got the scalars and I want to check only these equality and to check the equality I have to evaluate both sides on arbitrary vector v and check that they are equal or they are 2 linear maps from v to somewhere and if I could check we agree on a basis then they are equal. So, I will just have to agree I have to check that on v 1, v 2 up to v n they agree both sides agree, but you see this a 1 to a n I have defined, so that this side, this side agree this side is. So, when I evaluate the right side on v 1 I should get a 1, but that is also clear if I evaluated the left side on a vector v 1 then the only term which will contribute this and that will be v 1 star evaluated at v 1 which is 1, so a 1.

So, that will, so enough to check that enough to note that right hand right hand side that is a 1 v 1 star etcetera etcetera etcetera, a n v n star on arbitrary v j is a j - j is on 1 to n this is clear because our properties of the v i star and on the (Refer Time: 09:01) f of v j which is we have called it each. So, therefore, it prove that it is a generating system for the vector space V star. So, in this case they are equal.

Now, I want to write down few examples to get a feeling for this dual space.

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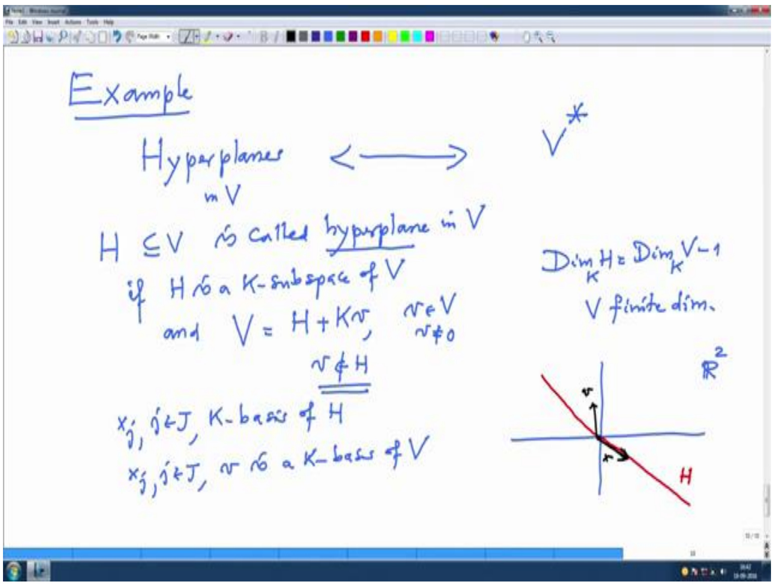

So, let us first write down first example this example shows a close connection between hyper planes in V, hyper planes in V and the linear forms and V star nonzero linear forms to be precised. So, let us recall what is a hyper plane H in V is called hyper plane, hyper plane in V, if H is a subspace K-subspace of V and dimension of H is 1 less than the dimension V, but this means and we should be able to write V as H or V generated by H and one more vector K v or some v in V, v nonzero and; obviously, you will need v should not in H.

If v is not in H then definitely you need in a generating system and it along with that v it generates, so that is exactly one dimension more. So, thing to think about it is because H is a subspace you take a basis of H x j, x j, j in j K-basis of H and I need one more element in this basis extra. So, that you get a basis of V. So, so that x j j in j along with v is a K-basis of V, in a finite dimensional case which is one. In a finite dimensional case we could simple say that dimension of H is 1 less than the dimension V this make sense only when V is finite dimension otherwise this equity then make sense, (Refer Time: 12:09) minus (Refer Time: 00:10) to (Refer Time: 12:10). So, it does not make sense.

So, such a such a thing is called a hyper plane and we have lots if examples for examples if you take usually Euclidean plane R 2 then these lines are for example, which passes through origin this is a hyper plane, this is H this is hyper plane. So, for examples this is 1 vector in H which is a basis in this picture H is one dimensional, but if I take along with these vector some vector which is like this. So, if you call this as what did I call it x and this as v, x together with v will generate R 2 and x is a basis of H and nothing special about R 2. You can take a plane in R 3 passing through origin that will be 2 dimensional and take a vector which does not lie in this plane that along with the basis of the plane will give you a basis of V.

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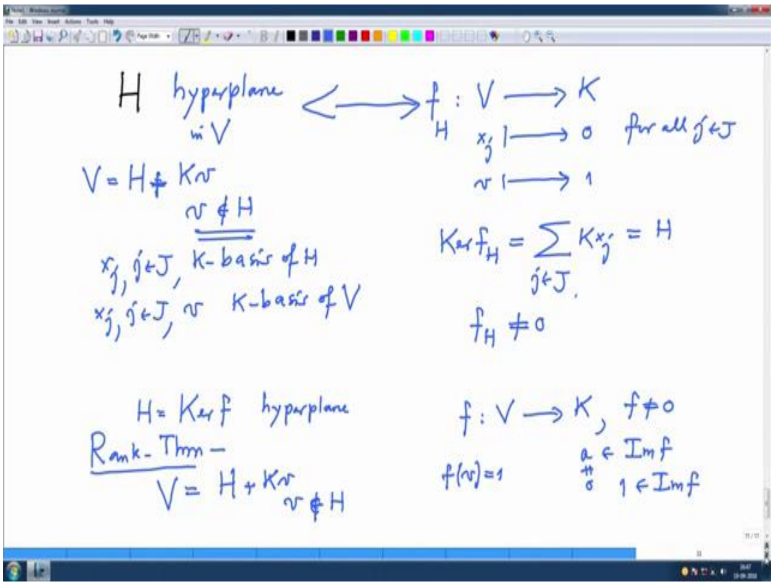

So, these are hyper planes and now from each hyper plane we can get a linear form, nonzero linear form how do we get a nonzero linear form is. So, suppose H is given, H hyper plane in V that is given; that means, they have given H equal to, V equal to H plus K v where v is not image, v not image we cannot be 0 also. From here I want to define a linear from f which will depend on this H. So, linear form is V to K. So, if I have to define a linear form I have to define its values on a given basis.

So, take a basis of H. So, x j, j in J this is a K-basis of H and then I know what is a basis of V, x j is along with v is a basis of V, along with v this is a K-basis of V and to give this linear form I just have to define what f of where do x j goes and where do v go. So, I will map v to 1 and all x j as to be 0 and then I get a linear form from v to k. So, this f H first of all what is kernel of f H? It contains all the x j's and it cannot contain any more because v is not in the kernel. So, kernel of H, f H is precisely the subspace of v generated by x j's.

But this is nothing, but H in fact, it is a basis of H, but this H and this f H is also nonzero. So, from my given hyper plane we got a nonzero linear form, conversely given a nonzero linear form f, f is nonzero it is not a 0 map then I want to get a hyper plane and who could that be the kernel of f, kernel of f I claim it is a hyper plane because this linear form is nonzero it has to be surjective because some nonzero elements a in the image because its nonzero all values cannot be 0. So, some nonzero value will be there. So, in the image, but then 1 will also be in the image because this K is a vector space and if 1 is 0 then everybody will go to 0. So, 1 is also in the image, so 1 will come from somebody.

So, suppose 1 is coming from v f of v is 1 then the proof of the rank theorem if you observe this v along with a basis of the kernel will generate will give a basis of v. So, that shows that this is hyper plane. So, rank theorem proof will show that V is H plus K v and this v is not in H. So, that is meaning that H is a hyper plane. So, to study hyper planes in a vector space is equivalent to study nonzero linear forms.

Alright actually 1 also need to study this I will not do in this course, but one will also need to study hyper plane subspace which are not subspace, but which are parallel they are translation of the subspace.

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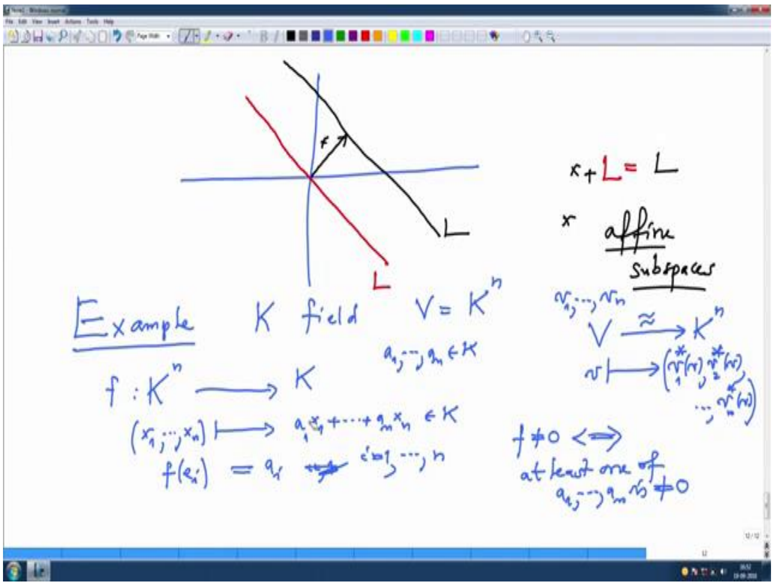

For example I will just draw one picture suppose in R 2 these are the these are the hyper planes hyper planes in R 2 are lines hyper plane in R 3 are planes and so on and they are all passing through the origin because we are saying their subspace, but suppose one want to study a line which is not necessarily passing through origin. So, it will be parallel to somebody which is passing through origin and this line is close properties with this line which is the only difference this is passing through origin, this is not passing through origin. So, this is not a subspace, but it is not too bad as a subspace. So, if for examples if you take this vector and add it to this line all the points in this line if you call this vector as x and x plus this red line x plus this L, this is L is precisely this black L, for x in the picture.

So, to study lines one would like to study these are called affine subspaces which are translation of the subspace. So, similarly one can study in general hyper planes which are not necessarily subspace, but they will be translated from the hyper plane which are passing through the origin.

So, next example I want to study, now let us take a finite dimensional vector space. So, K field and let us take the vector space V is K power n actually we have seen any finite dimensional vector space over field is isomorphic to this, if we choose a basis and then this will this is if we choose a basis of V if v 1 to v n is a basis there is a isomorphic there is a very natural isomorphism here which uses this basis namely any v map it to v 1 star of v, v 2 star of v etcetera v n star of v this is n tuple and these are the coordinates.

So, basis this is a coordinates and this map is clearly linear map and it is a bijective map. So, it is an isomorphism. So, in principle any vector space is a isomorphic to this not only that given basis goes to the standard basis. So, to do this, if you take a linear form on v v that is K power n any linear form f to K, so this form is where will the tuple go x 1 to x n go to some elements here. So, that is that the coordinate already given. So, it is a 1 x 1 plus plus plus plus plus a n x n, this is clearly a element in K because these guys are coordinates. So, elements in K and this sum is in K, so its element in K and it is surjective because if I want to get one in the image what do I do? I just take, so this is for a fixed a 1 to a n. So, f of f of e i is a i this is true for i 1 to 1 to n.

So, f is nonzero, f nonzero means at least one of the a should be nonzero equivalent to saying at least 1 of a 1 to a n is not equal to 0, and it will be a nonzero linear form and then what will it, just now we have seen it will corresponds to hyper plane which hyper plane: that the kernel one.

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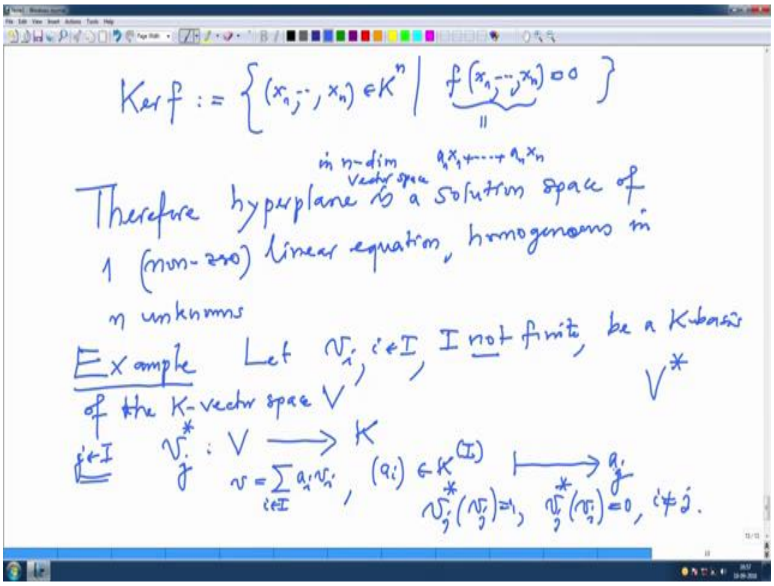

So, what is a kernel in this case? So, kernel of f is precisely by definition all those tuples x 1 to x n in K n such that f of x 1 to x n is 0 this is precisely a definition of kernel, but what is this f of x 1 to x n our definition it is a 1 x 1 plus plus plus plus plus a n x n.

So, therefore, hyper plane is nothing, but solution space of one linear equation in n unknowns. So, therefore, what we noted is therefore, hyper plane is a solution space of 1 nonzero equation, linear equation homogeneous, homogeneous in n unknowns where this n is the dimension of. So, if you want the hyper plane in n dimensional vector space it is 1 linear equation solution space of one homogeneously equation in n unknowns, alright

So, now one example I want to discuss with infinite dimension. So, that we see the picture that in infinite dimension it can be very different. So, let v i, i in I, I not finite, be a K-basis of a vector space, of the vector space, of the K vector space V then I want to show you in this case V star is also not finite dimensional and it is bigger in fact, we will see what I mean by that. So, first of all note that if I have a basis then I definitely have the coordinate functions v i star, v i star are defined they are linear forms on V for each i in I. The definition is the same, so I have to define it on each v that each v is a finite linear combination of this basis v a I v i i in I and this tuple a i belongs to K power round bracket I.

So, I will map this v to a i, i is fixed. So, this may be, definitely some guys will go to 0 almost all vectors will go to 0, let me call for j otherwise it will get confused with this,

this is j. So, this is a j. So, only finitely many elements on these basis under v j will go to so nonzero guys in fact, v j star of v j goes to 1 and all other goes to 0, v i this is 0 if I is not j. So, by the same proof by the above we can check that this family of coordinate function v j, j varies in i this is K-linearly independent and then elements in V star.

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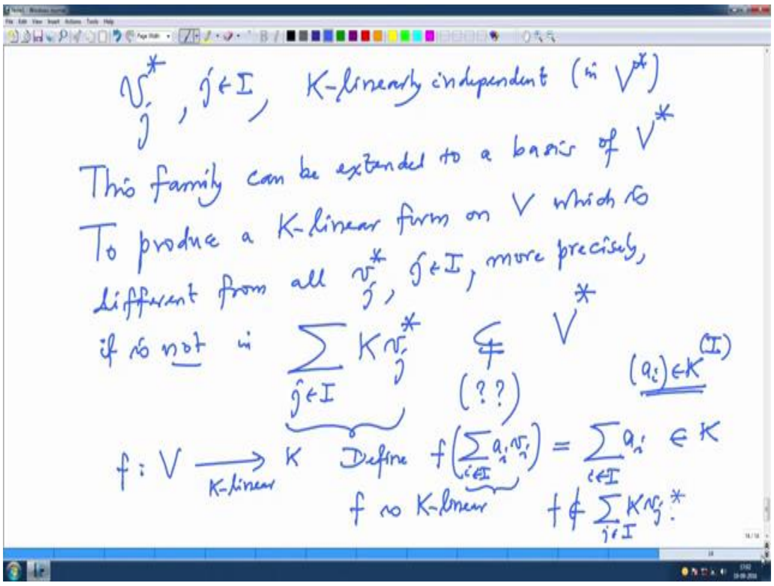

So, in vector space in the dual space V star I have this family which is linearly independent. Now it may be basis it may not be basis so, but out theorem abstract theorem says that any linearly independent family of vectors I can extend it to a basis. So, this family can be extended to a basis of V star. So, therefore, in the basis of V star there will be at least as many elements as that of a basis in V because this is a same index in set, but I want to show more there will be more elements in this; that means, this alone need not a basis we have to extend it by so, so I have to produce a linear form, so to produce a linear form, K-linear form on V which is none of this guys, which is different from all v j stars j in I.

Not only different from all this guys or different from which is more precisely it is not in the subspace generated by v j star, this is a subspace of V star generated by v j star; that means, it is a smallest subspace which contain all this v j star and because we have noted that v j star are linearly independent these vector space this subspace definitely has a v j star as a basis because it is generating set we are made it a generating set and we are already check that v j star are linearly independent and we want to produce a linear form;

that means, an element here which is not here. So, I want to check this, this is what I want to check.

So, this is what I mean by saying that in general in infinite dimensional case or not finite dimensional case V star is a bigger space. So, our problem is simple we want to produce linear from f V t o K, K-linear form which is not here. So, define, to define f I just have to give what are its values on arbitrary elements. So, I want to define f of summation a i v i where this a i tuple belongs to K power round bracket I, so that this make sense and if I define it on arbitrary element what is this and I need a scalar. So, what can it be? It is very simple because this tuple is here let us look at the summation a i, this make sense this is again an element in K.

So, this defines and so this is, this defines a linear form it is you one as to check that f is K-linear, but that is a more or less obvious, if you have 2 sums like this finite sums the sum and apply f this same thing as taken the sum of the coefficients and also when as to check now that this f is different f does not belong to the subspace generated by v j, f does not belong this way I have to check. But that is also not so difficult let us check that.

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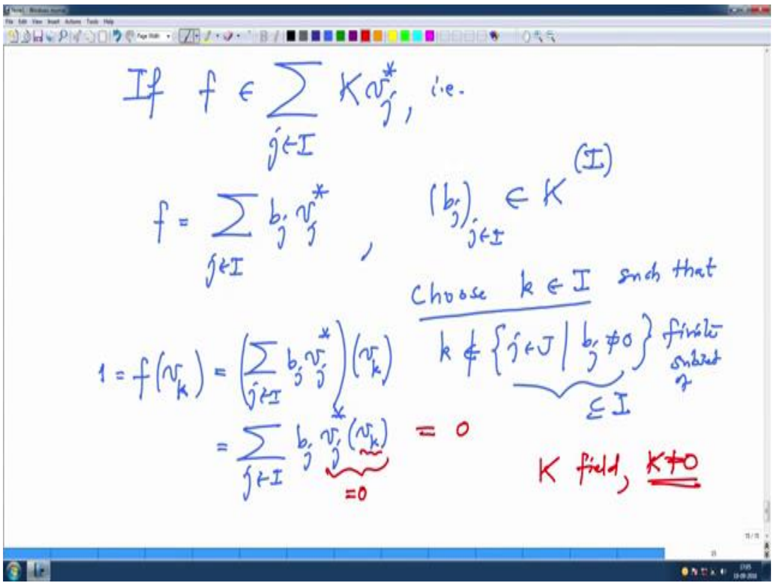

So, if f is that, if suppose f belongs, if f belongs to this subspace, what does that mean? That means f has a expression v i star, would I write here this is v i star. What does that means? f we can write as the combination summation b j, b j star, j in I with the property that this b j tuple which is I tuple and this tuple is in K power round bracket I. So, it as suppose it is this then we should get a contradiction. So, only finitely b j are nonzero and I is our infinite set.

So, choose and index small k in I which is different from such that this index k does not belong to all though j's for which b j are nonzero, this is a finite subset of I, this is finite subset of I is contained in that and definitely I is infinite therefore, I can always choose K in I which is not in this. And now what will be f o f v k? Remember our definition of f of v k says the sum of the coefficients, but here is only one coefficient namely 1, v k has coefficient 1 and all other v as coefficient 0, so this is one.

On the other side if it is equality here then it will be this j in I, b j evaluated this evaluated at v k, but this evaluated in v k I will take it inside because this is a finite sum all these are linear form, so I will take it inside. So, this is all those j in I, b j, b j star, v k, but is it j does not appear here where ever b j is nonzero this k does not appear there, does not appear when k is the index j where b j is 0 then this is 0 that is contradict, when k is none of the b j, none of the index j. So, that b j is nonzero, this has to be 0. So, altogether this side is 0.

So, 1 equal to 0 that is a contradiction because k is a field fields are never 0, field by definition, field, k is a field, field is never 0 that is part of the definition of a field, one should not forget this . So, therefore, in case of v is not finite dimensional then the study is more complicated to study dual spaces and also looking for duality etcetera it will become more and more complicated. So, we will confine ourself to finite dimensional cases.

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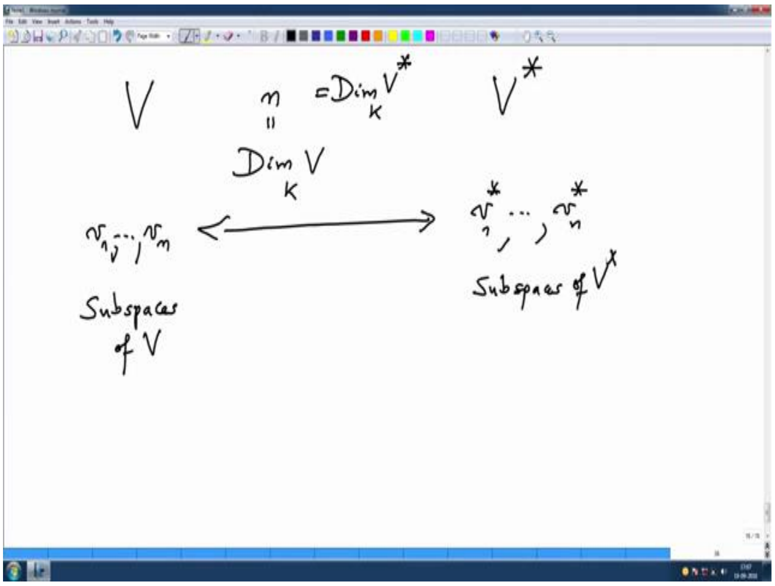

And the next task is now, next what we are going to invoke the study is. So, V is here V star is here and we are assuming the V is finite dimensional n is the dimension of V and we have also check that this is also dimension of V star. Nice, we have a basis here then we have a corresponding basis v 1 star v n star or dual basis.

Next thing I want to study subspace here, subspace of V and subspaces of V star. This I will do it in a next lecture, but what will prove that is if somebody is a subspace then what is the corresponding subspace here, what is the relation between the dimensions and so on. So, that is going to establish the duality. We will continue next time.

Thank you.