Linear Algebra
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Lecture – 32
Dual Spaces

Welcome to this course on Linear Algebra. Today lecture, as I announce in the last lecture at the end we will study the subspaces of a vector space V and it is dual.

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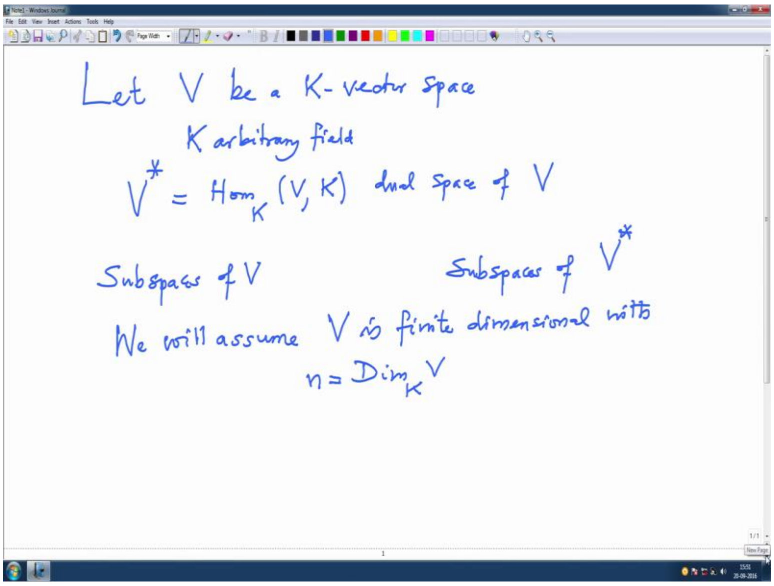

So, let V be a K vector space K as arbitrary field and V star that is linear forms on V Hom K V K the dual space of V. And I want to study the relation between the subspaces of V on one end, on the other hand subspaces of V star. And most often we will assume V is finite dimensional with n is the dimensional, but this is really not necessary. I will try to indicate when it is not necessary or statement hold in general.

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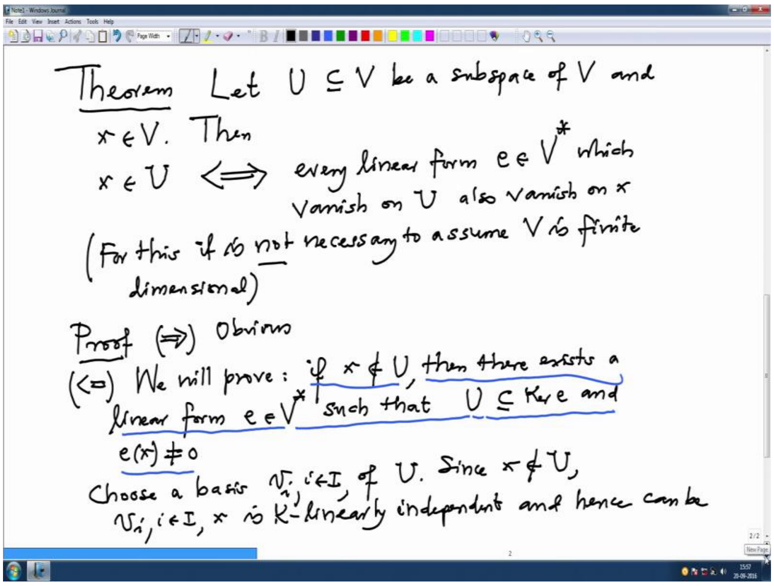

So, let us start with very important observation which I will call it a theorem. Let U be a subspace of V and x be a given in vector V, then I want to give a criterion how do we decide this x belongs to U or not. Then x belong to U if and only if every linear form e which vanish on U also vanish on x. This is very very important criterion to test a given vector is a given subspace or not by using linear forms.

And has you can see for this statement it is not necessary to assume V is finite dimension; so this work for arbitrary vector space. Proof: one implication is obvious this implication is obvious; if x belong to U the obviously your linear form which vanish on U also will vanish on x. Conversely, I need to prove. So, what I will prove that if I will prove (Refer Time: 04:54). So, prove this we will prove following statement: if x is not a U then there exist a linear form e such that e vanish on U means U is contend in the kernel of e, and e should not vanish on x and e of x should not be 0. This is what I need to prove.

Now, I need to prove this statement I will underline the statement, if x is not U then I want to produce a linear form e where U is contend in the kernel of e and e x is non 0. This is very simply. So, what is that we can always; we have only have to do we choose a basis. So, choose a basis V i i in I of U every vector spaces as basis, so in particular subspace as basis that I denoting by V i. And, so the fact that x is we are assuming x is not a U; since x is not a U, V i is along with x is K linearly independent. That is clear because x is not a U. And we have also check that we have also proved earlier that any linearly independent sheet can be extended to a basis of a vector space.

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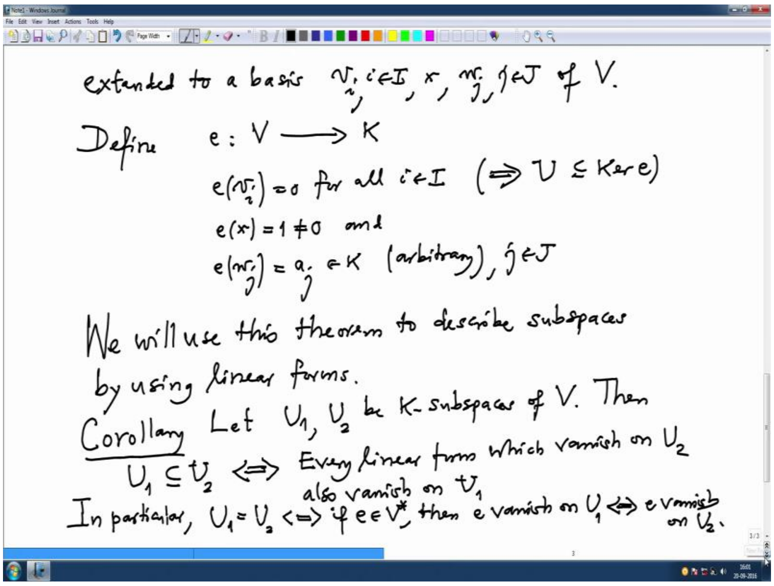

And hence can be extended to a basis. Now, I will write it is not really necessary to write, but let decided this is the basis of U and x here and then I will had few more element let us call them W j j in j of v. So, we have exchanged this along with this to a basis of e. Now to define a linear form e on V it is enough to give which values on the basis. And we want it 0 on U; so obviously, e of V i define it to 0 for all i in I. So, that we will make sure that U is containing the kernel e. Define on x to be non 0; for example, 1 you can define which is non 0. And on W j we can define anything like, and W j define some a j this are in K you can take arbitrary for any j in j. So obviously, this e is a required form that what we wanted a form which is 0 on U non 0 on x and so on.

So, I will use this to described subspace using linear form. So, we will use this theorem to describe subspaces by using linear forms. So, let us write one immediate corollary to this above theorem. So let us write corollary: let U 1 and U 2 be K subspaces of V, then one is contend in the other U 1 is contend U 2 if and only if every linear form which vanish on U 2 also vanish on U 1. In particular U 1 equal to U 2 if and only if e is a linear form, then e vanish on U 1 if and only if e vanish on U 2. This is immediately from the above theorem, because you can apply to you take element x in U 1 and when does belong to U 2 is precisely earlier theorem; that will to this containment.

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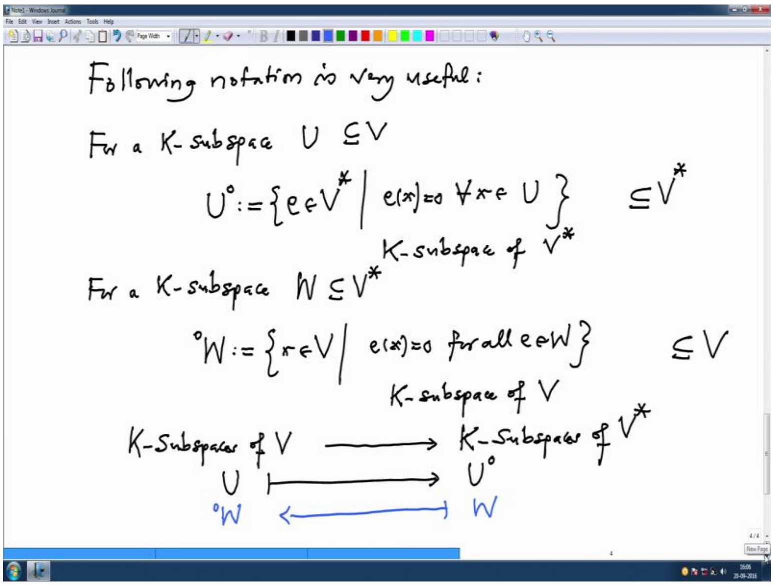

Now set up a notation. So, following notation is very useful: so for a subspace K subspace U contains a V i, am denoting U and this side left circle this will be subspace of V star what is it; this by definition all those linear forms which vanish on U. So, e of x is 0 for all x in U. Note that this is a K subspace of V star. So, we need to check that if one linear form vanish an all U another vanish on all U, then there sum also vanish on all U but that is obvious, that how we define the some of the linear form, similarly the K linear combination.

So therefore, it is a subspace. I do the sorry; I want to use this circle on the other side. Similarly, for a K subspace W of V star I want to define W circle on this side let us say that by definition and this will be a subspace of V. Now all those element x and V such that all linear forms in W should vanish on this x: e x equal to 0 for linear forms e in W. So, again it is clear that this left circle W is a K subspace of V. Now we are going to study this correspondent see we define that means we have define this maps. So, think of this right circle and left circles there maps from, map from K subspaces of V to K subspaces of V star. This direction map is U going to right circle U and this direction map is W going to W left circle. Now we are going to study the properties of this map. In particular I would like to know when this is bijective map so that will give us correspondent between a bijective correspondent between K subspace of V and K subspace of V star.

So, let us write down first obvious properties which are obvious in arbitrary cases. So, the first rule easy properties which almost need not proof.

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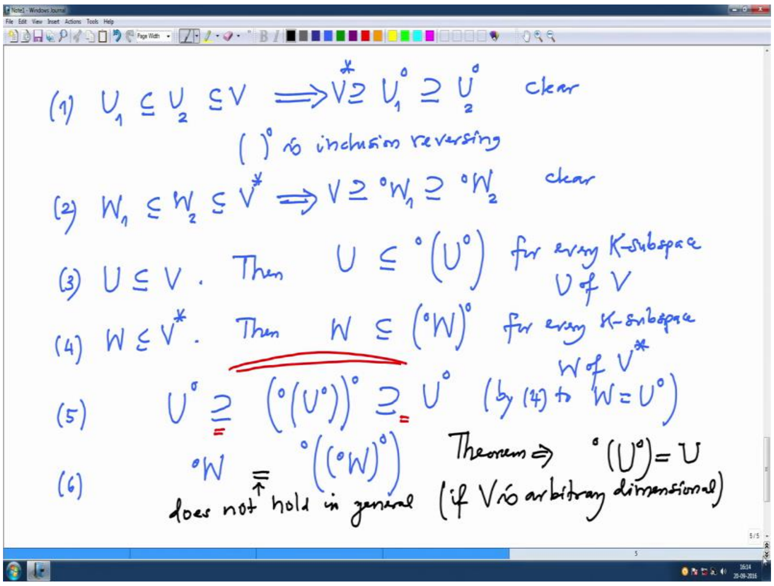

So, if U 1 containing U 2 the subspace of V, then U 1 right circle contains U 2 right circles. This is obvious because if some linear form vanish on U 2 and vanish on one also. So, this is I would say just clear. So, it inclusion that what we seen; the first right circle map is this map is inclusion reversing.

Similarly, for the subspace of V star; if W 1 containing W 2 containing V star subspace, then if I take left circles W 1 this will contend left circle W 2; this is also clear. Because, by definition this circle is all this linear forms in this are all elements in the vectors space V which vanish on every linear form. So, if they vanish on every linear form here. So, this are all subspace of V, and this is all subspace of V star. Further these are all obvious.

So now, the little bit thing to take into the following: for any subspace U of V. Let me number this 1 2 and 3. If V is case of subspace of V then what is relation between what can, we do U right circle then you get a subspace of V star and then take a left circle then you get back to subspace of V. And what is asserted U is contend here. This is for every subspace K subspace U of V. And similarly for W if W is the subspace of V star then what can be do W left circle and then right circle: this we contend W for every K subspace W of V star. These are simply they follow immediately from the definition.

So 5th, now I have the two subspaces: one is this U and other is which I obtained from U which is also subspace of e. So, you take the right circle and left circle. Now when I apply circle to this right circle to this, the inclusion will get reverse. That means, I want note that; that means U right circle is contends this and then I would to apply right circle. But this already contend U right circle. So, this is immediate 6th one which is not immediate. So, this one is you see the only think to know is this, but that is immediate because you apply four to you right circle here then this and then this is by applying by 4 2 W equal to U this, and noting that this left circle also in inclusion reversing.

6 is, to this I apply left circle. So, it will no; so for this we apply left circle then what do you get, we will get; no, I do not want to apply. So, I wanted to write that W circle and left W circle and then right W circle this and then this. So, you will only get one is continue in other I will write this later. So, what did the earlier statement, earlier theorem what we proved what does it say that. So, this one is because this is contend here equality holds here. In particular here this are equal, this are equal therefore all equality.

And theorem even says- let me be little bit more precise. The theorem implies: if I take U right circle and then left circle that is U. This is precise, this is bigger one the subspaces bigger one and we have decided when a bigger subspace and this also contend here, so it is equal the analog. The analog of the statement this one the equality here, so I will just right here equality does not hold in general. In general means, when if V is arbitrary dimension, but I want to prove this equality in a finite dimensional case, or even more generally even W is finite dimensional subspace then.

So, let me right in the next page.

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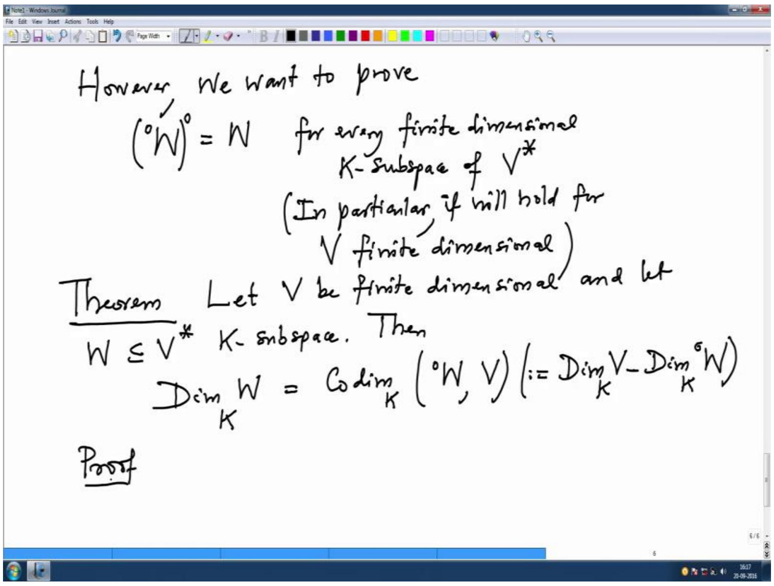

However, we want to prove W equal to; W this left circle right circle equal to W for every finite dimensional case of space of V star. In particular it will hold for; in particular it will hold for V finite dimensional. Because e is finite dimensional and V star also finite dimensional we have seen, and therefore it will hold for that.

So, to prove this I will first prove the following assertion. So, let us assume V finite dimensional; let V finite dimensional and let W V subspace of V star K subspace. Then I want to prove; then V star is finite V star also finite dimensional and W is also finite dimensional so I want say something of the dimension of W then dimension W equal to co dimension of left circle W V, which is by definition dimension of V minus dimension of the left circle and left circle W.

This is what I want to prove. So proof: I will complete this proof after the break.