Linear Algebra
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Lecture – 33
Dual spaces (continued)

Let us continue with the proof of this theorem which I stated before first half of the lecture.

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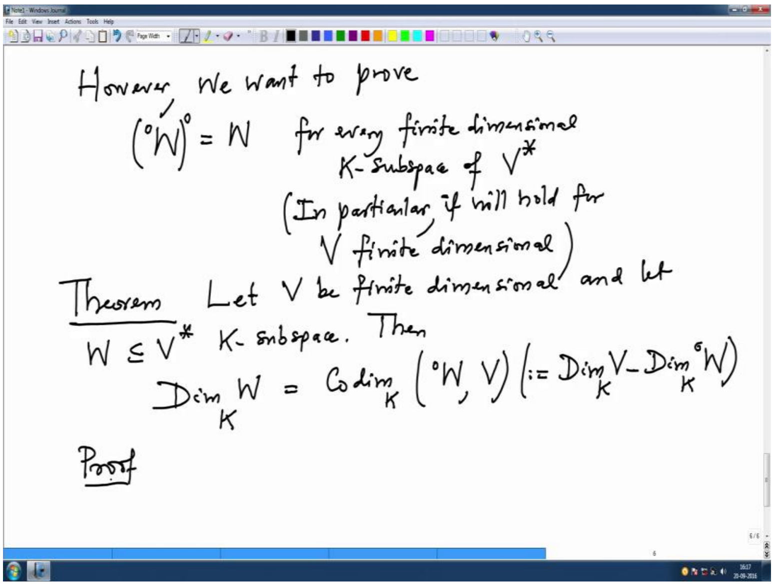

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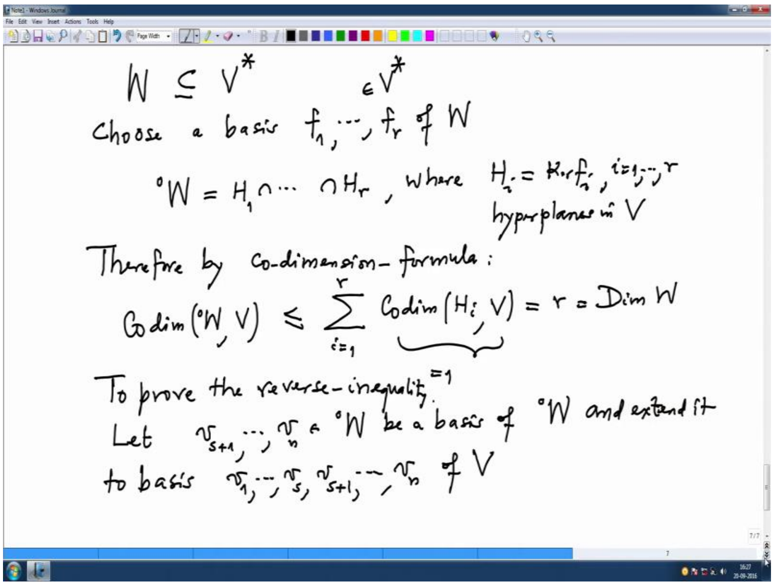

So, we have given W V is subspace of V star. And V star is finite dimensional, so W is also finite dimensional; so we can choose a basis of W. So, choose a basis. Remember W elements of W are linear forms, so I will call the basis to be f 1 to f r of W. Remember these are elements in V star. And by dimension of W (Refer Time: 01:09) W then not (Refer Time: 01:11) W left circle these are all those vectors in V which vanish in all linear form in W.

So therefore, this is nothing but the intersection of H 1 intersection intersection intersection H r where H i if the hyper plane correspondent to this linear from f I that is hi is kernel of f i i is from 1 to r these are hyper planes; hyper planes in V. Therefore, we have a co-dimension formula. Therefore, by co-dimensional formula, co-dimension of codim left circle W in V is smaller equal to the some of the co-dimensions i is from 1 to r codim H i in V, but this is co-dimension of each one of them is 1 because H i are the hyper planes.

So therefore, this number is r which is dimension W. So, that was a statement; no, that was. I will just want to show you the statement what we want to do. So, we wanted to prove was u dimension W equal to co-dimension this. And what we proved is co-dimension of left circle W is smaller equal to co-dimension W. So, I have to prove the reverse in equality. So, to prove the reverse in equality; now what your do, now let us choose the basis of W left circle.

So, let I will call it v s plus 1 to v n in f circle W be a basis of left circle W this is a subspace of V, so I choose a basis and I number it like this. And what will I do, there is one way you can do you can choose a basis and exchange to a basis. So, I will exchange these two basis of V; and extended it to basis which I will not call it v 1, v s I added and this gets s plus 1 v n of V.

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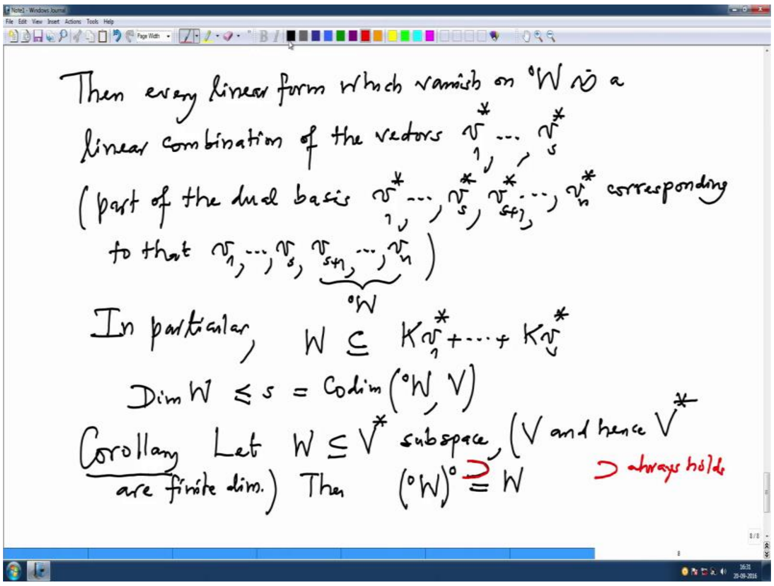

Then, what we just proved about; every linear form which vanish on left circle W is a linear combination of the vectors v 1 star v s star. You see all together for this just note that v 1 star v s star put to v n star this is a basis of v star. And later forms definitely vanish on the basis of left circle W. So therefore, they will not contribute anything in the linear combination. Therefore, every linear form which vanish on left circle W linear combination of this vectors.

This is the part of the dual basis v 1 star v s star v s plus 1 star v n star corresponding to the basis v 1 to v n, v s, v s plus 1, v n, this is a basis of W left circle. And therefore, star full vanish obviously there and namely linear from which vanish on W star; therefore, will have coefficient only from first s coordinate functions. So, that will means that W in particular; the W has to be contained in the subspace span by v 1 star plus plus plus plus plus K v s star. But that will mean that the dimension of W cannot exceed more than s, because this are linearly independent, but W the subspace here. And therefore, W h 2 have dimensional then equal to dimension of this space which is s. And this s is nothing but co-dimension of left circle W in V; so this two the other inclusion; other reverse in equality.

Therefore, immediate consequence of this is the following what we where are looking for; that is let me call it corollary. So, let W contain in dual subspace, and we are assuming V and hence V star are finite dimension. Then when you take left circle W;

that means you coming back to a subspace of V and then take the right circle you get back W. And what is the always two is this inclusion this is always, this inclusion this always holds without any finite dimensional assumption. So, let us indicate the proof.

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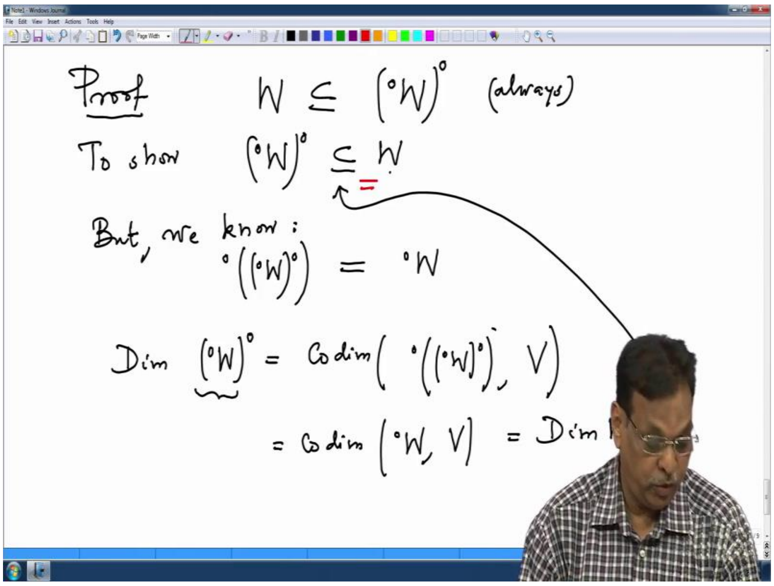

Proof as I said W contained in left circle W this is always without finite dimension assumption. Therefore, I want to show to show the equality I will show the other inclusion. So, to show left circle W and then right circle W is contain in w. But definitely I know. If I take; but we know from the basic property that I listed above, that if I take left circles is then they equal this we know as we approved earlier. So therefore, I know the co-dimensions. So, let us look at the co-dimension because of this equality codim; codim or dim first let us take dim. Dim of left circle W and then right circle W this is equal to codim of there, because I want to apply what just we proved; W equal to this codim I want to apply this equality. So, this equal to codim 0 and 0 W this is in V, V equal to codim remove this. So, this is left circle W in V which is dim W.

You see this guy also in V so I am applying the earlier theorem to once to this and once to this, this is a subspace. So, I am applying 1 to this and 1 to this. So, I get codim of this equal to dim of this; no, dim dimension of this equal to dimension of this. But then see one is a subspace of other and the dimension are equal I should. So therefore, equality holds here, this implies equality here. That is what we wanted to.

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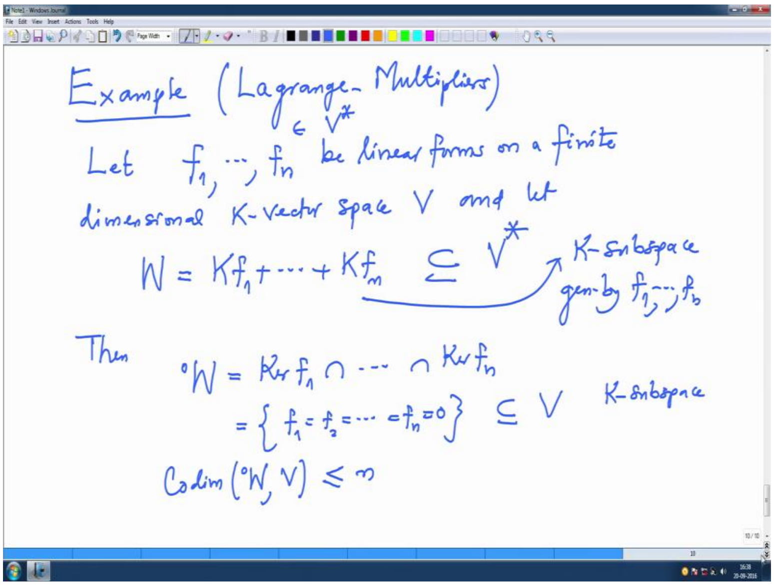

I want to note one very important example which is use by (Refer Time: 13:38). This is called also Lagrange multipliers. What is it? Supposed we have n linear forms f 1 to f n be linear forms on of finite dimensional K vector space V that is element of V star. And let us take W; and let W be a subspace generated by this linear form K f 1 plus plus plus plus K f n. This subspace of V star K-subspace generated by f 1 to f n this one here.

Then by definition of this left circle W this is nothing but the intersection there kernels; kernel f 1 intersection intersection kernel f n. So, which is also written some times as this is the common solutions of this equation f 1 think of linear forms and equation as equations. So, this is f 1 equal to f 2 equal to equal to equal to f n equal to 0. This is subspace of V K-subspace. Then therefore, what is the co-dimension? This one is generated by n elements. So, the co-dimension codim left circle W in V is less equal to n.

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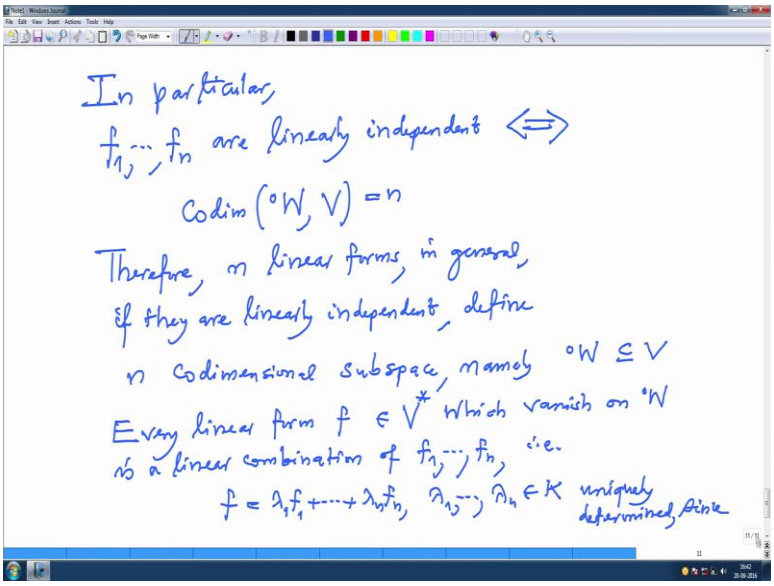

In particular, when are the linearly independent f 1 to f n are linearly independent if and if only the equality holds above; that means, codim of left circle W in V equal to n. Therefore, n linear forms in general define if there are linearly independent; if they are linearly independent define, what I should rub it define n co-dimensional subspace. Namely, left circle W this are subspace of V.

And what we just prove above that every linear form f that is in V star which vanish on this left circle W should therefore be a combination; is a linear combination of f 1 to f n that is every linear form f we should able to write this is lambda 1 f 1 plus plus plus plus lambda n f n, where lambda 1 to lambda n are scalars. And there are uniquely determine, because we are assuming this f 1 to f n are linearly independent; uniquely determine.

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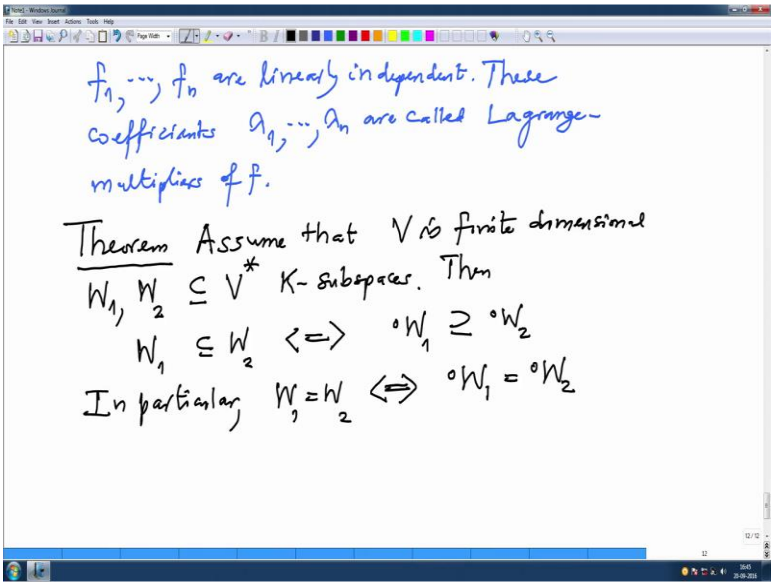

Since, f 1 to f n are linearly independent. These coefficient lambda 1 to lambda n are called Lagrange multipliers; multipliers of f because there are uniquely determine by f. So, with this I want to shade the following statement. So, theorem; again I will assume V finite dimensional.

So, assume that V is finite dimensional. And W 1, W 2 are two subspaces of V star K-subspaces. As I said in the beginning last lecture that normally these correspondents work very well when we assume V is finite dimensional, because then V star is also finite dimensional. In case we not finite dimensional then V star becomes bigger than V and therefore this correspondent will not have a good behaviour, but if you restrict to finite dimensional subspaces then it will still give some partial results.

Then W 1 containing W 2 if and only if left circle W 1 contains left circle W 2. This is the analog; you see the first theorem today I proved in the beginning was such a theorem for subspace of v, but when you go to V star that statement is no more true. So, we have to assume a finite dimensional. So, in particular there equal if and only if the left circle are equal. So, let us let me also consider double one. Let us go to the next page.

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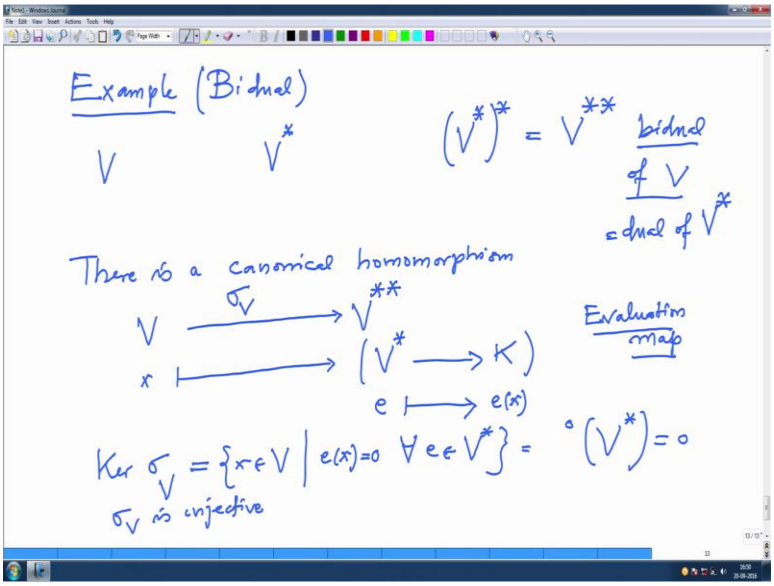

So, now let us write it as a example. This is a bi dual; remember this process by the vector space V we have a new vector space V star. If V finite dimensional this also finite dimensional. Then we can do the same process again that is V star star which I will denote by V double star. This is called bi dual of V, which is a dual of the dual spaces star. And there is a canonical map there is a canonical homomorphism of vector spaces from V to V double star. There will no map from V 2 to V star canonically, but V to V double star there is a canonical map what is a map that map and I going to denote by sigma V sigma space V.

So, what does it do? For a given vector x I want a double dual; double dual means I want a linear form on V star that means, I want where this should go to e and e should be; not e this x should go to somebody which maps V star to k. So, what is that map? That means, I need to know where do go arbitrary e go. So, e and this x you can do this e e of it e evaluate e it x. So, this is like evaluation map; this called evaluation map. It evaluates a linear form on the given vector.

So, let us write down what is a kernel of this. So, the kernel of sigma V is by definition all those vector x in V such that it go to 0. That means, e every linear form should vanish on x. So, that mean e of x should be 0 for every e in V star, but that is precisely the definition of V star left circle we call. If W space of V star W left circle was all those elements all those vectors in V which vanish on the linear form on W. So, this is nothing but left circle of V star.

But what is this? This is definitely 0, because given any vector space we can always find given any x if are non zero we can always find a linear form which is non zero and x by extending a x y basis and mapping x to 1 and everybody x to 0. So, defiantly this is 0. Therefore, the first consequence sigma V is injective, because its kernel is 0.

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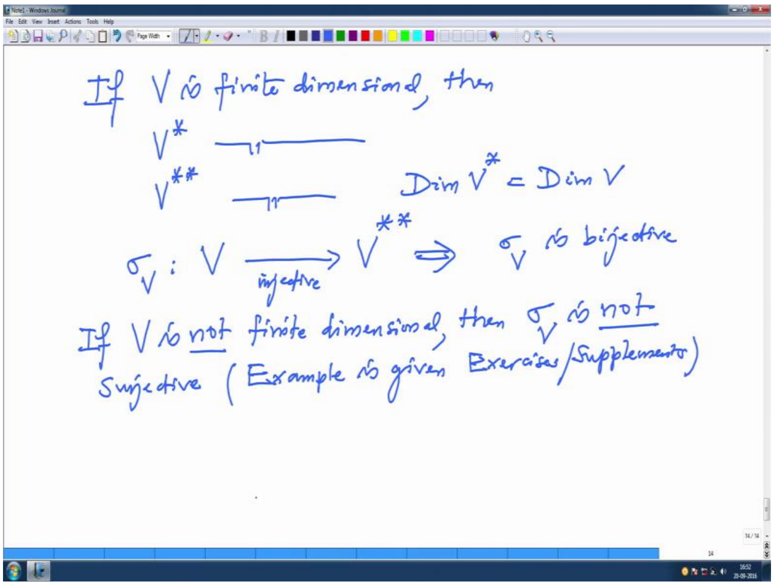

Moreover, if V finite dimensional then V star is also finite dimensional and the dimensions are equal. Now, once again applying that V double star also finite dimensional and dimension is equal to dimension V star which is also dimension V. So, the linear map sigma V was V to V double star both has the same dimension and this map was we have checked that is injective. Therefore, by pigeonhole principle this is sigma V is bijective.

So, it is give an isomorphism between V and V double star. This is canonical isomorphism. If you remember even if V to V star there is an isomorphism, but the isomorphism there is not canonical. These are the finite dimension case: if V is not finite dimensional then definitely sigma V is not surjective. This example I would write in is given in exercises must supplements.

We will stop here and continue for the next time.

Thank you.